Second Midterm Practice Sheet
MAT1322C

Applications of differential equations

1. A turkey with a temperature of $-15°$ is put in a $245°$ oven. We denote by $h(t)$ the temperature of the turkey t minutes after being placed in the oven. After 20 minutes the temperature of the turkey is 85°C.

   Give a problem with initial value satisfied by $h(t)$, then solve the problem to find $h(t)$. The problem may depend on some parameter that you will need to determine using the data given.

   At what time does the temperature of the turkey reaches 150°?

   Solution : By Newton's heating/cooling law, we have

   $$\frac{dT}{dt} = k(T - T_s) = k(T - 245).$$

   Let $y = T - 245$, then $\frac{dy}{dt} = \frac{dT}{dt}$ and $y(0) = T(0) - 245 = -15 - 245 = -260$. So,

   $$\frac{dy}{dt} = ky \iff y(t) = y(0)e^{kt} = -260e^{kt}.$$

   We are given $T(20) = 85$ so $y(20) = T(20) - 245 = 85 - 245 = -160$ and

   $$-160 = -260e^{20k} \Rightarrow e^{20k} = \frac{-160}{-260} \Rightarrow 20k = \ln(\frac{8}{13}) \Rightarrow k = \frac{1}{20}\ln(\frac{8}{13}) \approx -0.024.$$

   Therefore,

   $$T(t) = y(t) + 245 = -260e^{-0.024t} + 245.$$

   We are looking for t such that $T(t) = 150$. Thus,

   $$150 = -260e^{-0.024t} + 245 \Rightarrow 260e^{-0.024t} = 95 \Rightarrow e^{-0.024t} = \frac{95}{260}.$$

   $$-0.024t = \ln\left(\frac{19}{52}\right) \Rightarrow t = \frac{\ln\left(\frac{19}{52}\right)}{-0.024} = 41.95\text{mins}$$

2. A vat contains 500 gallons of beer at 4% of alcohol. Beer with 6% of alcohol is pumped into the vat a rate of 5 gallons per minute, and the mixture is pumped out of the vat at the same rate. We denote by $Q(t)$ the quantity of alcohol (in gallons) in the vat at time t (in minutes).

   Give a problem with initial value satisfied by $Q(t)$, then solve the problem to find $Q(t)$.

   What is the quantity of alcohol in the vat after 30 minutes?

   Solution : Let $Q(t)$ denote the quantity of alcohol (in gallons) in the vat as a function of time (measured in minutes). Initially, we have $Q(0) = \frac{4}{100} \times 500 = 20$ gallons, and we know that

   $$\frac{dQ(t)}{dt} = (\text{rate entering}) - (\text{rate leaving})$$

Each minute, 5 gallons enters at a concentration of 6% alcohol, so the rate of alcohol entering is $5 \times \frac{6}{100} = \frac{3}{10}$. Each minute, 5 gallons of the mixture leaves the vat, with an alcohol concentration of $\frac{Q(t)}{500}$, thus, the rate of alcohol leaving is $5 \times \frac{Q(t)}{500} = \frac{Q(t)}{100}$. Thus, $Q(t)$ satisfies the following differential equation:

$$\frac{dQ(t)}{dt} = \frac{3}{10} - \frac{Q(t)}{100}$$

We solve this separable DE by separating the variables, then integrating both sides:

$$\begin{aligned} \int \frac{dQ}{30-Q} &= \frac{1}{100}\int dt \\ -\ln|30-Q| &= \frac{t}{100} + C \\ 30 - Q &= Ae^{-\frac{t}{100}} \\ \implies \quad Q(t) &= 30 - Ae^{-\frac{t}{100}} \end{aligned}$$

where A is a constant ($A = \pm e^C$). Using the initial condition $Q(0) = 20$, we solve for the constant A:

$$\begin{aligned} 20 &= 30 - A \\ A &= 10 \end{aligned}$$

Thus, our solution for the quantity of alcohol in the vat at time t is $Q(t) = 30 - 10e^{-0.01t}$.

After 30 minutes, the vat will contain $Q(30) = 30 - 10e^{(-0.01)(30)} \simeq 22.6$ gallons of alcohol.

Value of the sum of a series

1. Compute the sum of the series $\sum_{n=0}^{\infty} \frac{2^{3n} - (-1)^n 5^{n+1}}{3^{2n}}$.

   Solution : We see that this series is comprised of geometric series:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{2^{3n} - (-1)^n 5^{n+1}}{3^{2n}} &= \sum_{n=0}^{\infty} \frac{2^{3n}}{3^{2n}} - \frac{(-1)^n 5^{n+1}}{3^{2n}} \\ &= \sum_{n=0}^{\infty} \left(\frac{2^3}{3^2}\right)^n - \sum_{n=0}^{\infty} 5\left(\frac{-5}{3^2}\right)^n. \end{aligned}$$

   Thus, we have a difference of two geometric series. The first series' first term is $a = 1$ and its common ratio is $r = \frac{8}{9}$ (clearly $\left|\frac{8}{9}\right| < 1$). The second series' first term is $a = 5$ and its common ratio is $r = \frac{-5}{9}$ (again, we have $\left|\frac{-5}{9}\right| < 1$). Thus, each of these two geometric series is convergent. Now, we can evalauate the sum of the original series as follows:

$$\sum_{n=0}^{\infty} \frac{2^{3n} - (-1)^n 5^{n+1}}{3^{2n}} = \frac{1}{1-\frac{8}{9}} - \frac{5}{1+\frac{5}{9}}$$
$$= \frac{9}{9-8} - \frac{45}{9+5} = \frac{9 \times 14 - 45}{14} = \frac{81}{14}$$

2. Show that the following series is convergent $S = \sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^3}$.

   Give an upper bound on the error if we use the 100-th partial sum S_{100} to approximate its sum.

   Solution : Let $f(x) = \frac{1}{x(\ln(x))^3}$. Then our series looks like $S = \sum_{n=2}^{\infty} f(n)$, where $f(x)$ is a positive, continuous function that approaches 0 as $x \to +\infty$. Moreover, $f(x)$ is decreasing: for $x > 1$

$$f'(x) = -\frac{(\ln(x))^3 + x\frac{3(\ln(x))^2}{x}}{x^2(\ln(x))^6} = -\frac{\ln(x)+3}{x^2(\ln(x))^4} < 0.$$

   Thus, we can use the Integral Test! We have

$$\int_2^{\infty} \frac{1}{x(\ln(x))^3}dx = \lim_{t \to +\infty} \left[\frac{-1}{2(\ln x)^2}\right]_2^t$$
$$= \frac{1}{2(\ln 2)^2} \simeq 1.04$$

   Since the above improper integral is convergent, so too is the series, by virtue of the Integral Test.

   If we use the 100th partial sum S_{100} to estimate the sum of this series, then the Remainder Estimate for the Integral Test tells us that the error $R_n = S - S_n$ is at most

$$\text{error} = S - S_{100} \leq \int_{100}^{\infty} \frac{1}{x(\ln(x))^3}dx$$
$$= \lim_{t \to +\infty} \left[\frac{-1}{2(\ln x)^2}\right]_{100}^t$$
$$= \frac{1}{2(\ln 100)^2} \simeq 0.02358$$

   Therefore, the error in estimating the sum of this series by its 100th partial sum is at most 0.02358.

3. Show that the following series is convergent $S = \sum_{n=0}^{\infty} (-1)^n \frac{\sqrt{n+1}}{3n+1}$.

Give an upper bound on the absolute value of the error if we use the first 10 terms of the series to approximate its sum.

Solution : This is an alternating series. Let's use the alternating series test. The absolute value of the nth term of the series is of the form $b_n = f(n)$, where $f(x) = \frac{\sqrt{x+1}}{3x+1}$. First, note that

$$f'(x) = \frac{\frac{3x+1}{2\sqrt{x+1}} - 3\sqrt{x+1}}{(3x+1)^2} = \frac{3x+1-6(x+1)}{2\sqrt{x+1}(3x+1)^2} = \frac{-3x-5}{2\sqrt{x+1}(3x+1)^2} < 0$$

Thus, $f(x)$ is a decreasing function. Since $b_n = f(n)$, it follows that $b_{n+1} \leq b_n$ for all $n \geq 0$. Moreover,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\sqrt{n+1}}{3n+1} = 0.$$

Thus, the series is convergent by virtue of the Alternating Series Test.

By the Alternating Series Estimation Theorem, if we use the 10th partial sum S_{10} to estimate S, then the absolute value of the error is at most

$$|\text{error}| = |S - S_{10}| \leq b_{11} = \frac{\sqrt{11+1}}{3(11)+1} \simeq 0.1019$$

Tests for convergence:

(a) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$: Integral test (substitute $u = e^{-x^3}$), converges.

(b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$: Integral test (substitute $u = (\ln(x))$), converges.

(c) $\sum_{k=1}^{\infty} \frac{1}{k^2+k^3}$: This sum is less than $\sum \frac{1}{k^3}$, which converges, so the series converges by comparison test.

(d) $\sum_{i=1}^{\infty} \frac{i}{\sqrt{i^5+1}}$: exponent of the a_i is $1-5/2 = 3/2$. Compare to $\sum \frac{i}{\sqrt{i^5}} = \sum \frac{1}{\sqrt{i^3}}$ which converges.

(e) $\sum_{s=1}^{\infty} \frac{s^2-5s}{s^3+s+1}$: exponent is $2-3=1$, compare to the harmonic series to show it diverges.

(f) $\sum_{i=1}^{\infty} \frac{ln(i)}{i}$: Compare to the harmonic series as for $i \geq 2$ we have $\frac{ln(i)}{i} \geq \frac{1}{i}$. Diverges.

(g) $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n!}$: Ratio test, converges.

(h) $\sum_{n=1}^{\infty} (-1)^{n+3} \frac{n^2}{n^3+4}$: Alternating series test shows it converges. Hint: by using the derivative of an appropriate function you can show faster that the sequence is decreasing.

(i) $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{2n+3}$: Alternating series test, converges. Use again the derivative of a function to show the sequence is decreasing for at least $n \geq 2$ (in fact $a_1 < a_2$ here, but as stated in class it is enough if it decreases for all $n > N$ for some fixed N).

(j) $\sum_{n=1}^{\infty} \frac{n!}{100^n}$: Ratio test shows it diverges. Faster: observe that $\lim_{n\to\infty} \frac{n!}{100^n} \neq 0$, which immediately gives divergence.

(k) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$: Classic root test case, converges.

(l) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+4}$: Alternating series test, converges conditionally. It does not converge absolutely as $\sum_{n=1}^{\infty} \frac{n}{n^2+4}$ can be shown to be bigger than $\sum_{n=1}^{\infty} \frac{1}{5n}$ which diverges.

Do the following series **converge absolutely**?

(a) $\sum_{n=1}^{\infty} (-1)^{n+3} \frac{n^2}{n^3+4}$. It does not converge absolutely as $\sum_{n=1}^{\infty} \frac{n^2}{n^3+4}$ can be shown to be bigger than $\sum_{n=1}^{\infty} \frac{1}{5n}$ which diverges.

(b) $\sum_{n=1}^{\infty} (-1)^{n} \frac{n}{n^3+4}$. It converges absolutely as $\sum_{n=1}^{\infty} \frac{n}{n^3+4}$ can be shown to be smaller than $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges. (you can also use the limit comparison test).

(c) $\sum_{n=1}^{\infty} (-1)^{n} \frac{\cos^2 n+1}{n^{7/8}}$. It does not converge absolutely as $\sum_{n=1}^{\infty} \frac{\cos^2 n+1}{n^{7/8}}$ can be shown to be larger than $\sum_{n=1}^{\infty} \frac{1}{n^{7/8}}$ which diverges by p-test (note that $0 \leq \cos^2 n \leq 1$).

(d) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n+2}$. It does not converge absolutely as $\sum_{n=1}^{\infty} \frac{n}{n+2}$ can be shown to be divergent by test for divergence.

(e) $\sum_{n=0}^{\infty} (-1)^{n-1} \frac{\sqrt{n+2}}{\sqrt{n^2+7}}$. It does not converge absolutely as $\sum_{n=1}^{\infty} \frac{\sqrt{n+2}}{\sqrt{n^2+7}}$ behaves like $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ where by limit comparison test is divergent since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent.

Find radius and Interval of Convergence:

1. $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n \sqrt{n}}$:

   For the radius we need to observe for which x the series converges. The form of the terms in the series suggests that the root test might be successful:

$$L = \lim_{n\to\infty} \sqrt[n]{\frac{|(2x-1)^n|}{|5^n \cdot \sqrt{n}|}} = \lim_{n\to\infty} \frac{|2x-1|}{|5 \cdot \sqrt[n]{n}|}.$$

   Now $|2x-1|/5$ is independent of n, so we can put it in front of the limit:

$$L = \frac{|2x-1|}{5} \cdot \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}}.$$

   The latter limit now is 1. So we get that $L = \frac{|2x-1|}{5} = \frac{2}{5}|x - \frac{1}{2}|$. The root test says that this converges if $L < 1$ and diverges if $L > 1$. So we check:

$$\begin{aligned}\frac{|2x-1|}{5} < 1 \quad &\implies \quad |2x-1| < 5 \\ &\implies -5 < 2x-1 < 5 \\ &\implies -5+1 < 2x < 5+1 \\ &\implies -4 < 2x < 6 \\ &\implies -2 < x < 3\end{aligned}$$

Now we need to check the endpoints of this interval, in other words does the series converge if $x = 3$ or if $x = -2$?

For $x = 3$ the series is rewritten as $\sum_{n=1}^{\infty} \frac{(2\cdot 3-1)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. We can see that

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \geq \sum_{n=1}^{\infty} \frac{1}{n},$$

and we know that the latter series (the harmonic series) diverges, so the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges too. Hence $x = 3$ is NOT included in the interval of convergence.

For $x = -2$ we get $\sum_{n=1}^{\infty} \frac{(2\cdot(-2)-1)^n}{5^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. We can use the alternating series test to show that this series converges. So $x = -2$ is included in the interval of converges.

To sum up, the series $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}}$ converges for $-2 \leq x < 3$. For the radius, we calculate the length of the interval: $3 - (-2) = 5$. The the radius is the length divided by 2, so here we get $\frac{5}{2}$ as radius of convergence.

Don't forget to check endpoints of intervals; it'd be a deduction of points otherwise!

2. $\sum_{n=1}^{\infty} \frac{x^n}{n\cdot 3^n}$: root test gives the inequality $|\frac{x}{3}| < 1$ from which we get the two cases $x < 3$ and $x > -3$. For $x = -3$ the series becomes the alternating harmonic series, so it converges by what we learned in class. For $x = 3$ we have the harmonic series, which diverges. Hence the interval of convergence is $-3 \leq x < 3$ and the radius is $(3-(-3))/2 = 6/2 = 3$.

3. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2+1}$: Use the ratio test, for $L = \lim_{n\to\infty} \frac{\frac{|(x-2)^{n+1}|}{|(n+1)^2+1|}}{\frac{|(x-2)^n|}{|n^2+1|}} = |x-2|$. Then $|x-2| < 1$ gives $x < 3$ and $x > 1$. For the edges, for $x = 1$ the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$ and this converges and for $x = 3$ the series converges too. So the interval of convergence is $1 \leq x \leq 3$. The radius is $(3-(1))/2 = 2/2 = 1$.

4. $\sum_{n=1}^{\infty} \frac{n\cdot(x+1)^n}{4^n}$: Root test gives the inequality $|\frac{x+1}{4}| < 1$ from which we get the two cases $x < 3$ and $x > -5$. For both edges $x = -5$ and $x = 3$ the series diverges, so the interval of convergence is $-5 < x < 3$ and the radius is $(3-(-5))/2 = 8/2 = 4$.

Power series and functions

Consider the function $f(x) = \displaystyle\int_0^x \frac{1}{1+2t^4}dt$.

1. Determine the representation as a power series centered at 0 of $f(x)$.
2. What is the interval of convergence?
3. Use your result to express $\int_0^{0.1} \frac{1}{1+2t^4}dt = f(0.1)$ as a series.
4. Approximate $f(0.1)$ with 8 correct decimals.

Solution : We know that $\displaystyle\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, for $|x| < 1$. Substituting x by $(-2t^4)$ in the above expression we get:

$$\frac{1}{1+2t^4} = \sum_{n=0}^{\infty}(-2t^4)^n = \sum_{n=0}^{\infty}(-2)^n t^{4n}$$

which will converge for $|2t^4| < 1$, that is, for $|t| < \frac{1}{\sqrt[4]{2}}$. Using the theorem regarding integration of power series (which tells us that the integral of the series is the series of term-wise integrals), we know that, for $|x| < 2^{-1/4}$, we have

$$\begin{aligned} f(x) = \int_0^x \sum_{n=0}^{\infty}(-2)^n t^{4n} dt &= \sum_{n=0}^{\infty}(-2)^n \int_0^x t^{4n} dt \\ &= \sum_{n=0}^{\infty} \frac{(-2)^n}{4n+1} x^{4n+1} \end{aligned}$$

By the theorem regarding integration of power series, the radius of convergence for the integrated series is the same as the radius of convergence of the original series. Hence the power series $\sum_{n=0}^{\infty} \frac{(-2)^n}{4n+1} x^{4n+1}$ is absolutely convergent for $|x| < 2^{-1/4}$, and divergent for $|x| > 2^{-1/4}$. To determine the interval of convergence we need to determine what happens at the boundaries of the interval that is $x = 2^{-1/4}$ and $x = -2^{-1/4}$.
For $x = 2^{-1/4}$, the series is

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{4n+1}(2^{-1/4})^{4n+1} = \sum_{n=0}^{\infty} \frac{(-2)^n}{(4n+1)2^{1/4}2^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+1)2^{1/4}}$$

This is an alternating series, we can show it is convergent by the Alternating series test.
For $x = -2^{-1/4}$, the series is

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{4n+1}(-2^{-1/4})^{4n+1} = \sum_{n=0}^{\infty} \frac{(-2)^n(-1)^{4n+1}}{(4n+1)2^{1/4}2^n} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(4n+1)2^{1/4}}$$

This is an alternating series, we can show it is convergent by the Alternating series test.
Hence the interval of convergence is $[-2^{-1/4}, 2^{-1/4}]$.

$0.1 \in [-2^{-1/4}, 2^{-1/4}]$, hence

$$f(0.1) = \sum_{n=0}^{\infty} \frac{(-2)^n}{4n+1}(0.1)^{4n+1} = \sum_{n=0}^{\infty} \frac{(-2)^n}{(4n+1)10^{4n+1}}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{(4n+1)10^{4n+1}}$$

This is an alternating series $\sum_{n=0}^{\infty}(-1)^n b_n$ with $b_n = \frac{2^n}{(4n+1)10^{4n+1}}$
By the theorem on the error of approximation of alternating series,

$$|S - s_k| \leq b_{k+1} = \frac{2^{k+1}}{(4k+5)10^{4k+5}}$$

For $k = 1$ (if we use only the first term) the error is at most $b_2 = \frac{4}{9\times 10^9} \simeq 4.4 \times 10^1 0$
Hence

$$f(0.1) \simeq b_0 - b_1 \pm 4.4 \times 10^1 0$$
$$= \frac{1}{10} - \frac{2}{5 \times 10^5} \pm 4.4 \times 10^1 0$$
$$= 0.09999600 \pm 4.4 \times 10^1 0$$

Do you remember these facts?

1. What is the series $\sum_{n=1}^{\infty} \frac{1}{n}$ called? Does it converge?

   This is the *harmonic series* and it diverges.

2. What is the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ called? Does it converge?

   Alternating harmonic series, converges to $\ln(2)$.

3. What is the series $\sum_{n=0}^{\infty} q^n$ called? Does it converge? If so, what is its sum?

   Geometric series, converges if $|q| < 1$ to $\frac{1}{1-q}$.

4. What is absolute convergence? Give an example of a series that converges but does not converge absolutely.

   A series $\sum_{n=0}^{\infty} a_n$ converges absolutely if $\sum_{n=0}^{\infty} |a_n|$ converges. The series $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$ converges, but $\sum_{n=1}^{\infty} |(-1)^n \frac{1}{n}| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

5. If a series converges absolutely, does it converge?

   Yes, as this means $\sum_{n=0}^{\infty} |a_n|$ converges, and $\sum_{n=0}^{\infty} |a_n| \geq \sum_{n=0}^{\infty} a_n$, so by the comparison theorem the series converges.

6. If a series converges, does it converge absolutely?

No. See the example of the alternating harmonic series.

Good luck with studying!