

## SOLUTIONS TO TEST 2–MATH 1102

1. Suppose the matrix

$$\begin{bmatrix} 1 & 4 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 & 2 & -6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is the RREF of the augmented matrix of a system of linear equations over  $\mathbb{C}$ .

(a) Compute the solution set to this system in *vector form*.

**Solution:** The indices of the columns without leading ones and corresponding to free variables are 2, 5 and 6. (The final column corresponds to the vector of constants, not a variable.) We must solve for the dependent variables  $x_1$ ,  $x_3$  and  $x_4$ :

$$\begin{aligned} x_4 &= -2x_5 + 6x_6 \\ x_3 &= -x_5 + 3x_6 \\ x_1 &= -4x_2 - 2x_5 - x_6. \end{aligned}$$

The solution set is

$$\begin{aligned} & \left\{ \begin{bmatrix} -4c_2 - 2c_5 - c_6 \\ c_2 \\ -c_5 + 3c_6 \\ -2c_5 + 6c_6 \\ c_5 \\ c_6 \end{bmatrix} : c_2, c_5, c_6 \in \mathbb{C} \right\} \\ &= \left\{ c_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_5 \begin{bmatrix} -2 \\ 0 \\ -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c_6 \begin{bmatrix} -1 \\ 0 \\ 3 \\ 6 \\ 0 \\ 1 \end{bmatrix} : c_2, c_5, c_6 \in \mathbb{C} \right\}. \end{aligned}$$

**Grading:** 2 for identifying free/dependent variables. 2 for solution set. 2 for vector form.

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(b) Write the solution set above as the span of some vectors.

**Solution:**

$$\text{span} \left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \\ 6 \\ 0 \\ 1 \end{bmatrix} \right\}$$

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2. Determine a linearly independent subset of

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -8 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$$

whose span is equal to  $\text{span}S$ .

**Solution:** Following Theorem BS, we write the matrix whose columns are given by the above vectors in the order that they appear

$$\begin{bmatrix} 1 & -1 & -3 & 1 & -1 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 2 & -8 \end{bmatrix}.$$

Row-reducing, we obtain

$$\begin{array}{l} R_1 \xrightarrow{+R_2} \\ (1/2)R_3 \\ R_1 \xrightarrow{-R_3} \end{array} \begin{bmatrix} 1 & 0 & -2 & 1 & -3 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix}.$$

The indices of the columns with leading ones are 1, 2 and 4. Therefore

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

is a linearly independent subset of the above set with the same span.

**Grading:** 2 for matrix. 3 for RREF. 3 for identifying linearly independent subset.

3. The set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}, \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix} \right\} \subseteq \mathbb{C}^3$$

is linearly dependent. Compute and display a non-trivial linear dependence relation by first solving the appropriate homogeneous system of linear equations.

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**Solution:** Any linear dependence relation is a solution to the homogeneous system of linear equations whose augmented matrix is

$$\begin{bmatrix} 1 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{-iR_2} \begin{bmatrix} 1 & 1 & i & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & 2i & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The RREF on the right allows us to read off the solution  $x_3 = 1$ ,  $x_2 = i$  and  $x_1 = -2i$ . The corresponding non-trivial linear dependence relation is

$$-2i \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + i \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} + 1 \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

**Grading:** 2 for augmented matrix. 2 for non-trivial solution. 2 for non-trivial dependence relation.

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4. Suppose  $F$  is a field and  $v, w \in F^n$ . Define the vector  $v + w$ .

**Solution:**  $[v + w]_j = [v]_j + [w]_j$  for all  $1 \leq j \leq n$ .

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5. Suppose  $F$  is a field and  $w, v \in F^n$ . Prove that  $v + w = w + v$ .

**Solution:** Suppose  $1 \leq j \leq n$ . Then, by the commutativity of addition in  $F$

$$[v + w]_j = [v]_j + [w]_j = [w]_j + [v]_j = [w + v]_j.$$

Therefore  $v + w = w + v$ .

**Grading:** 2 for expressing the entries of  $v + w$ . 2 for using commutativity in  $F$ .

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6. What is the property in the previous exercise called?

**Solution:** The commutativity of vector addition.

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7. Suppose  $F$  is a field. Define the identity matrix  $I_n \in M_{nn}(F)$ .

**Solution:** For  $1 \leq j, k \leq n$

$$[I_n]_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}.$$

8. Suppose  $F$  is a field and  $w, v_1, \dots, v_k \in F^n$ .

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(a) Prove that

$$\text{span}\{v_1, \dots, v_k\} \subseteq \text{span}\{w, v_1, \dots, v_k\}.$$

**Solution:** Suppose  $\alpha_1 v_1 + \dots + \alpha_k v_k \in \text{span}\{v_1, \dots, v_k\}$  for some  $\alpha_1, \dots, \alpha_k \in F$ . Then

$$\alpha_1 v_1 + \dots + \alpha_k v_k = 0w + \alpha_1 v_1 + \dots + \alpha_k v_k \in \text{span}\{w, v_1, \dots, v_k\}.$$

**Grading:** 2 for style of proof. 2 for correct linear combination. 2 for equation.

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(b) Suppose further that  $w \in \text{span}\{v_1, \dots, v_k\}$ . Prove that

$$\text{span}\{v_1, \dots, v_k\} = \text{span}\{w, v_1, \dots, v_k\}.$$

**Solution:** In view of 8 (a), it suffices to prove the containment

$$\text{span}\{v_1, \dots, v_k\} \supseteq \text{span}\{w, v_1, \dots, v_k\}.$$

Suppose  $\alpha w + \alpha_1 v_1 + \dots + \alpha_k v_k \in \text{span}\{w, v_1, \dots, v_k\}$  for some  $\alpha, \alpha_1, \dots, \alpha_k \in F$ . Since  $w \in \text{span}\{v_1, \dots, v_k\}$  there exists  $\beta_1, \dots, \beta_k \in F$  such that  $w = \beta_1 v_1 + \dots + \beta_k v_k$ . Substituting this linear combination in for  $w$  above we obtain

$$\begin{aligned} \alpha w + \alpha_1 v_1 + \dots + \alpha_k v_k &= \alpha(\beta_1 v_1 + \dots + \beta_k v_k) + \alpha_1 v_1 + \dots + \alpha_k v_k \\ &= (\alpha\beta_1 + \alpha_1)v_1 + \dots + (\alpha\beta_k + \alpha_k)v_k \end{aligned}$$

and this last linear combination belongs to  $\text{span}\{v_1, \dots, v_k\}$ . This proves the desired containment.

**Grading:** 2 correct containment. 2 for substituting linear combination in for  $w$ . 2 for showing the substitution lies in  $\text{span}\{v_1, \dots, v_k\}$ .

9. Indicate whether each of the two statements below is true or false. If you think a statement is true, provide some justification. If you think a statement is false, provide a counterexample to it.

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(a) Suppose  $F$  is a field,  $\{v_1, \dots, v_k\} \subset F^n$  is linearly independent, and  $1 \leq j \leq k$ . Then  $\{v_1, \dots, v_j\}$  is linearly independent.

**Solution:** True. Suppose  $\alpha_1, \dots, \alpha_j \in F$  such that

$$\alpha_1 v_1 + \dots + \alpha_j v_j = 0.$$

Then

$$\alpha_1 v_1 + \dots + \alpha_j v_j + 0v_{j+1} + \dots + 0v_k = 0.$$

The linear independence of  $\{v_1, \dots, v_k\}$  now implies that  $\alpha_1 = \dots = \alpha_k = 0$ . In conclusion, we have proven that

$$\alpha_1 v_1 + \dots + \alpha_j v_j = 0 \Rightarrow \alpha_1 = \dots = \alpha_k = 0.$$

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(b) Suppose  $F$  is a field. Every square matrix with entries in  $F$  is non-singular.

**Solution:** False. The matrix

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in M_{22}(F)$$

is square but the solution set to  $\mathcal{LS}(A, 0)$  is  $F^2$ . In particular, there are non-trivial (non-zero) solutions. Therefore  $A$  is (very) singular.