Solution to Midterm Test 1

MAT 1322X, Summer 2018
Total = 20 marks

Version A

I. Multiple-Choice Questions ($2 \times 4 = 8$ marks)

Answers: CDAC

1. The area of the region under the graph of $y = 3 + 3x - 2x^2$ and above the graph of $y = x^2 + x + 2$ **in the first quadrant** is

(A) $\frac{4}{3}$; (B) $\frac{3}{4}$; (C) 1; (D) $\frac{6}{5}$; (E) $\frac{2}{3}$; (F) 2.

Solution. (C) The intersections of the graphs of these two functions is obtained by

$3 + 3x - 2x^2 = x^2 + x + 2$, $3x^2 - 2x - 1 = 0$. Then $x = 1$, $x = -\frac{1}{3}$. The area of this region is

$$\int_0^1 ((3+3x-2x^2)-(x^2+x+2))dx = \int_0^1 (-3x^2+2x+1)dx = \left[-x^3+x^2+x\right]_{x=0}^1 = 1.$$

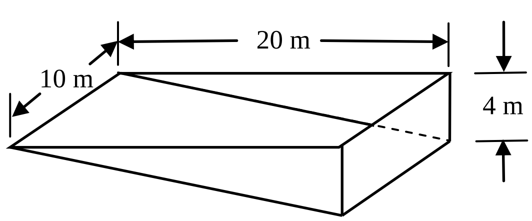
2. Let R be the region under the graph of $y = \sin x$ and above the x-axis, $0 \le x \le \pi$. Solid S has R as the base, and the cross sections perpendicular to the x-axis are squares. Then the volume of this solid is calculated by the integral

(A) $\int_0^\pi \sin x dx$; (B) $\int_0^\pi (1-\sin^2 x)dx$; (C) $\int_0^\pi (1-\sin x)dx$;
(D) $\int_0^\pi \sin^2 x dx$; (E) $\pi \int_0^\pi \sin^2 x dx$; (F) $\pi \int_0^\pi (1-\sin x)dx$.

Solution. (D) The area of the cross section $A(x) = \sin^2 x$. The volume of solid S is calculated by the integral

$$V = \int_0^\pi \sin^2 x dx.$$

3. A pool has the form of a prism as shown in the following figure:



The top of the pool is a rectangle of dimensions 20 m × 10 m. The depth of the deep end is 4 m. The pool is filled with water of density ρ kg/m^3. We want to find the work needed to pump all the water in the pool to a nozzle 3 meters above the top of the pool. Let x be the distance between a horizontal layer of water in the pool and the height of the nozzle, i.e., 3 meters above the top of the pool. Then the total work is calculated by the integral

(A) $50\rho g\int_3^7 x(7-x)dx$; (B) $50\rho g\int_3^7 (x+3)(4-x)dx$;

(C) $50\rho g\int_0^7 x(7-x)dx$; (D) $50\rho g\int_0^7 (x+3)(4-x)dx$;

(E) $50\rho g\int_0^4 x(7-x)dx$; (F) $50\rho g\int_0^4 (x+3)(4-x)dx$.

Solution. (A) The area of a horizontal layer of water x meters under the nozzle is $A(x) = 10\times\dfrac{20}{4}(7-x) = 50(7-x)$. The volume of this layer with thickness dx is $V(x) = A(x)dx = 50(7-x)dx$. The weight of this layer of water is $w(x) = \rho g V(x) = 50\rho g(7-x)dx$. The work needed to pump this layer of water to the nozzle is $W(x) = xw(x) = 50\rho g x(7-x)dx$. The total work needed to pump all water in the pool to the nozzle is

$$W = 50\rho g\int_3^7 x(7-x)dx.$$

4. Consider improper integral $\int_0^1 \frac{1}{\sqrt{x^2+x}}dx$. Which one of the following statement is true?

(A) Because $\frac{1}{\sqrt{x^2+x}} > \frac{1}{\sqrt{2x}}$ when $0 < x < 1$, and $\int_0^1 \frac{1}{\sqrt{2x}}dx$ is divergent, this improper integral is divergent.

(B) Because $\frac{1}{\sqrt{x^2+x}} > \frac{1}{\sqrt{2x}}$ when $0 < x < 1$, and $\int_0^1 \frac{1}{\sqrt{2x}}dx$ is convergent, this improper integral is convergent.

(C) Because $\frac{1}{\sqrt{x^2+x}} < \frac{1}{\sqrt{x}}$ when $0 < x < 1$, and $\int_0^1 \frac{1}{\sqrt{x}}dx$ is convergent, this improper integral is convergent.

(D) Because $\frac{1}{\sqrt{x^2+x}} < \frac{1}{\sqrt{x}}$ when $0 < x < 1$, and $\int_0^1 \frac{1}{\sqrt{x}}dx$ is divergent, this improper integral is divergent.

(E) Because $\frac{1}{\sqrt{x^2+x}} > \frac{1}{\sqrt{2x^2}} = \frac{1}{\sqrt{2}x}$ when $0 < x < 1$, and $\int_0^1 \frac{1}{\sqrt{2}x}dx$ is divergent, this improper integral is divergent.

(F) Because $\frac{1}{\sqrt{x^2+x}} < \frac{1}{\sqrt{x^2}} = \frac{1}{x}$ when $0 < x < 1$, and $\int_0^1 \frac{1}{x}\,dx$ is divergent, this improper integral is divergent.

Solution. (C) When x is close to 0, x^2, compared with x, can be ignored. Hence, the behavior of the function $\frac{1}{\sqrt{x^2+x}}$ is similar to function $\frac{1}{\sqrt{x}}$. Since $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is convergent, this integral should be convergent. Then (A), (D) and (F) are false. (B) is false because the integral of the smaller function is convergent does not imply that the integral of the bigger function is convergent. (E) is false because the inequality $\frac{1}{\sqrt{x^2+x}} > \frac{1}{\sqrt{2x^2}}$ is not true when x is close to 0. This can easily be verified by, say, $x = 0.1$. Then $x^2 + x = 0.11$, and $2x^2 = 0.02$.

II. Detailed Answer Questions (12 marks)

1. (3 marks) Consider the region R under the graph of $f(x) = 4x - x^3$ and above the x-axis in the first quadrant. Find the volume of the solid obtained by revolving R about the line $x = 3$.

Solution. Use the cylindrical shell method. The graph of $f(x)$ intersects the x-axis at $x = 2$. The distance between the line $x = 3$ and a vertical line between the graph of $f(x)$ the x-axis is $r(x) = 3 - x$. The volume of the solid is

$$V = 2\pi\int_0^2 (3-x)(4x-x^3)dx = 2\pi\int_0^2 (12x-4x^2-3x^3+x^4)dx$$

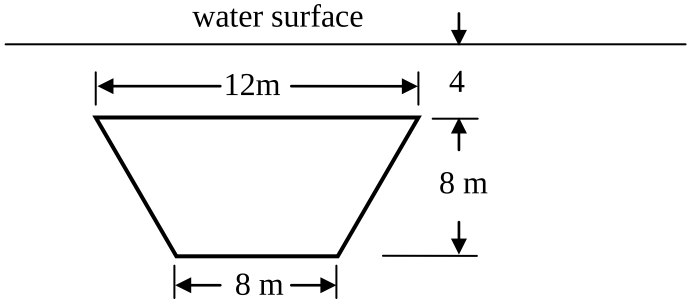
$$= 2\pi\left[6x^2 - \frac{4}{3}x^3 - \frac{3}{4}x^4 + \frac{1}{5}x^5\right]_{x=0}^{2} = \frac{232}{15}\pi .$$

2. (4 marks) **Use the definition of improper integral** to find the value of $\int_0^1 \frac{1}{x+\sqrt{x}}\,dx$.

Solution. $\int_0^1 \frac{1}{x+\sqrt{x}}\,dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{x+\sqrt{x}}\,dx = \lim_{a\to 0^+}\int_{\sqrt{a}}^1 \frac{1}{x+\sqrt{x}}(2\sqrt{x})du = 2\lim_{a\to 0^+}\int_{\sqrt{a}}^1 \frac{1}{u+1}\,du$

$$= 2\lim_{a\to 0^+}\left[\ln(u+1)\right]_{u=\sqrt{a}}^{1} = 2\lim_{a\to 0^+}(\ln 2 - \ln(\sqrt{a}+1)) = 2\ln 2 .$$

3. (5 marks) Suppose a surface of the shape of a trapezoid as shown in the following figure is submerged vertically into water so that the top is 4 meters under the water surface. Find the force, in Newtons, acting on this face. The density of water is 1000 kg/m^3, and the acceleration of gravity is $g = 9.81$ m/sec^2.



Solution. Let x be the distance between a horizontal stripe of the surface and the bottom of the surface. Then the length of this stripe is $l(x) = 8 + \frac{x}{2}$.

Let the height of this stripe be dx. Then the area of this stripe is $A(x) = l(x)dx = \left(8+\frac{x}{2}\right)dx$.

The depth of this stripe is $D(x) = (4 + 8) - x = 12 - x$. The pressure on this stripe is $P(x) = \rho g D(x) = \rho g(12 - x)$.

The force acting on this stripe is $F(x) = A(x)P(x) = \rho g(12 - x)\left(8+\frac{x}{2}\right)dx$.

The total force acting on this surface is

$$F = \rho g\int_0^8 (12-x)\left(8+\frac{x}{2}\right)dx = 6.07 \times 10^6 \text{ Newton.}$$

Alternative solutions:

A. If you let x be the distance between a horizontal stripe of the surface and the water surface, then the integral is $F = \rho g\int_4^{12} x\left(14-\frac{x}{2}\right)dx$.

B. If you let x be the distance between a horizontal stripe of the surface and the top of the trapezoid, the integral is $\rho g\int_0^8 (x+4)\left(12-\frac{x}{2}\right)dx$.

Version B

I. Multiple-Choice Questions ($2 \times 4 = 8$ marks)

Answers: AAFA

1. The area of the region under the graph of $y = 3 + 3x - 2x^2$ and above the graph of $y = x^2 + x + 2$ **in the first quadrant** is

(A) 1; (B) $\frac{6}{5}$; (C) $\frac{2}{3}$; (D) 2; (E) $\frac{4}{3}$; (F) $\frac{3}{4}$.

Solution. (A) The intersections of the graphs of these two functions is obtained by

$3 + 3x - 2x^2 = x^2 + x + 2$, $3x^2 - 2x - 1 = 0$. Then $x = 1$, $x = -\frac{1}{3}$. The area of this region is

$$\int_0^1 ((x^2 + x + 2) - (3 + 3x - 2x^2))dx = \int_0^1 (3x^2 - 2x - 1)dx = \left[x^3 - x^2 - x\right]_{x=0}^1 = 1.$$

2. Let R be the region under the graph of $y = \sin x$ and above the x-axis, $0 \le x \le \pi$. Solid S has R as the base, and the cross sections perpendicular to the x-axis are squares. Then the volume of this solid is calculated by the integral

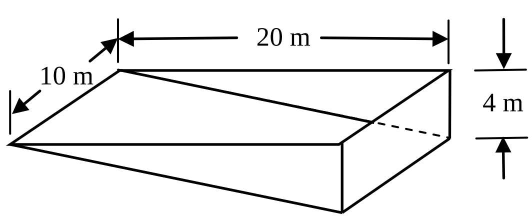
(A) $\int_0^\pi \sin^2 x dx$; (B) $\pi \int_0^\pi \sin^2 x dx$; (C) $\pi \int_0^\pi (1 - \sin x) dx$.

(D) $\int_0^\pi \sin x dx$; (E) $\int_0^\pi (1 - \sin^2 x) dx$; (F) $\int_0^\pi (1 - \sin x) dx$;

Solution. (A) The area of the cross section $A(x) = \sin^2 x$. The volume of solid S is calculated by the integral

$$V = \int_0^\pi \sin^2 x dx.$$

3. A pool has the form of a prism as shown in the following figure:



The top of the pool is a rectangle of dimensions 20 m × 10 m. The depth of the deep end is 4 m. The pool is filled with water of density ρ kg/m^3. We want to find the work needed to pump all the water in the pool to a nozzle 3 meters above the top of the pool. Let x be the distance between a horizontal layer of water in the pool and the top of the pool. Then the total work is calculated by the integral

(A) $50\rho g\int_3^7 x(7-x)dx$; (B) $50\rho g\int_3^7 (x+3)(4-x)dx$;

(C) $50\rho g\int_0^7 x(7-x)dx$; (D) $50\rho g\int_0^7 (x+3)(4-x)dx$;

(E) $50\rho g\int_0^4 x(7-x)dx$; (F) $50\rho g\int_0^4 (x+3)(4-x)dx$.

Solution. (F) The area of a horizontal layer of water x meters under the nozzle is $A(x) = 10 \times 5(4-x) = 50(4-x)$. The volume of this layer with thickness dx is $V(x) = A(x)dx = 50(4-x)dx$. The weight of this layer of water is $w(x) = \rho g V(x) = 50\rho g(4-x)dx$. The work need to pump this layer of water to the nozzle is $W(x) = (x+3)w(x) = 50\rho g(x+3)(7-x)dx$. The total work needed to pump all water in the pool to the nozzle is

$$W = 50\rho g\int_0^4 (x+3)(4-x)dx.$$

4. Consider improper integral $\int_0^1 \frac{1}{\sqrt{x^2+x}}dx$. Which one of the following statement is true?

(A) Because $\frac{1}{\sqrt{x^2+x}} < \frac{1}{\sqrt{x}}$ when $0 < x < 1$, and $\int_0^1 \frac{1}{\sqrt{x}}dx$ is convergent, this improper integral is convergent.

(B) Because $\frac{1}{\sqrt{x^2+x}} < \frac{1}{\sqrt{x}}$ when $0 < x < 1$, and $\int_0^1 \frac{1}{\sqrt{x}}dx$ is divergent, this improper integral is divergent.

(C) Because $\frac{1}{\sqrt{x^2+x}} > \frac{1}{\sqrt{2x}}$ when $0 < x < 1$, and $\int_0^1 \frac{1}{\sqrt{2x}}dx$ is divergent, this improper integral is divergent.

(D) Because $\frac{1}{\sqrt{x^2+x}} > \frac{1}{\sqrt{2x}}$ when $0 < x < 1$, and $\int_0^1 \frac{1}{\sqrt{2x}}dx$ is convergent, this improper integral is convergent.

(E) Because $\frac{1}{\sqrt{x^2+x}} > \frac{1}{\sqrt{2x^2}} = \frac{1}{\sqrt{2}x}$ when $0 < x < 1$, and $\int_0^1 \frac{1}{\sqrt{2}x}dx$ is divergent, this improper integral is divergent.

(F) Because $\frac{1}{\sqrt{x^2+x}} < \frac{1}{\sqrt{x^2}} = \frac{1}{x}$ when $0 < x < 1$, and $\int_0^1 \frac{1}{x}dx$ is divergent, this improper integral is divergent.

Solution. (A) When x is close to 0, x^2, compared with x, can be ignored. Hence, the behavior of the function $\dfrac{1}{\sqrt{x^2+x}}$ is similar to function $\dfrac{1}{\sqrt{x}}$. Since $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ is convergent, this integral should be convergent. Then (B), (C) and (F) are false. (D) is false because the integral of the smaller function is convergent does not imply that the integral of the bigger function is convergent. (E) is false because the inequality $\dfrac{1}{\sqrt{x^2+x}} > \dfrac{1}{\sqrt{2x^2}}$ is not true when x is close to 0. This can easily be verified by, say, $x = 0.1$. Then $x^2 + x = 0.11$, and $2x^2 = 0.02$.

II. Detailed Answer Questions (12 marks)

1. (3 marks) Consider the region R under the graph of $f(x) = 3x^2 - x^3$ and above the x-axis in the first quadrant. Find the volume of the solid obtained by revolving R about the line $x = 4$.

Solution. Use the cylindrical shell method. The graph of $f(x)$ intersects the x-axis at $x = 3$. The distance between the line $x = 4$ and a vertical line between the graph of $f(x)$ the x-axis is $r(x) = 4 - x$. The volume of the solid is

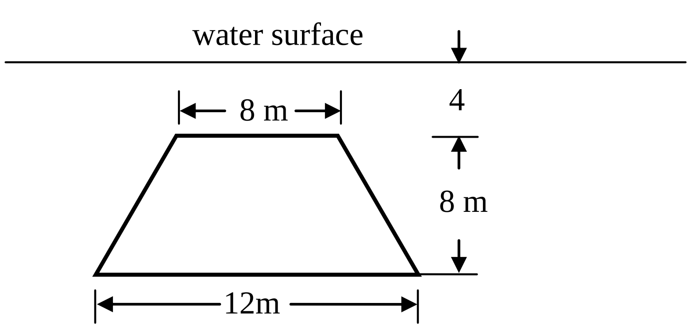
$$V = 2\pi\int_0^3 (4-x)(3x^2-x^3)dx = 2\pi\int_0^3 (12x^2-7x^3+x^4)dx = 2\pi\left[4x^3-\frac{7}{4}x^4+\frac{1}{5}x^5\right]_{x=0}^{5} = \frac{297}{10}\pi\,.$$

2. (4 marks) **Use the definition of improper integral** to find the value of $\displaystyle\int_0^1 \frac{1}{\sqrt{x}(x+1)}dx$.

Solution. $$\int_0^1 \frac{1}{\sqrt{x}(x+1)}dx = \lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}(x+1)}dx = \lim_{a\to 0^+}\int_{\sqrt{a}}^1 \frac{1}{\sqrt{x}(x+1)}(2\sqrt{x})du = 2\lim_{a\to 0^+}\int_{\sqrt{a}}^1 \frac{1}{u^2+1}du$$

$$= 2\lim_{a\to 0^+}\left[\arctan u\right]_{u=\sqrt{a}}^1 = 2\lim_{a\to 0^+}(\arctan 1 - \arctan\sqrt{a}) = \frac{\pi}{2}.$$

3. (5 marks) Suppose a surface of the shape of a trapezoid as shown in the following figure is submerged vertically into water so that the top is 4 meters under the water surface. Find the force, in Newtons, acting on this face. The density of water is 1000 kg/m^3, and the acceleration of gravity is $g = 9.81$ m/sec^2.



Solution. Let x be the distance between a horizontal stripe of the surface and the bottom of the surface. Then the length of this stripe is $l(x) = 12 - \frac{x}{2}$.

Let the height of this stripe be dx. Then the area of this stripe is $A(x) = l(x)dx = \left(12 - \frac{x}{2}\right)dx$.

The depth of this stripe is $D(x) = (4 + 8) - x = 12 - x$. The pressure on this stripe is $P(x) = \rho g D(x) = \rho g(12 - x)$.

The force acting on this stripe is $F(x) = A(x)P(x) = \rho g(12 - x)\left(12 - \frac{x}{2}\right)dx$.

The total force acting on this surface is

$$F = \rho g \int_0^8 (12 - x)\left(12 - \frac{x}{2}\right)dx = 6.49 \times 10^6 \text{ Newton.}$$

Alternative solutions:

A. If you let x be the distance between a horizontal stripe of the surface and the water surface, then the integral is $F = \rho g \int_4^{12} x\left(6 + \frac{x}{2}\right)dx$.

B. If you let x be the distance between a horizontal stripe of the surface and the top of the trapezoid, the integral is $\rho g \int_0^8 (x + 4)\left(8 + \frac{x}{2}\right)dx$.