Midterm 3 March Winter 2018, questions and answers

Fundamental Mathematics II (Concordia University)

MATH 209/4 all sections except EC: - Fundamental Mathematics II

Midterm - March 3, 2018 (1h30min)
Only approved calculators are permitted.

MARKS

[5] **1.** Let $g(x) = 2 - 13x^3$. Work out the following in detail:

$$\lim_{a \to 0} \frac{[g(x+a) - g(x)]}{a}$$

[10] **2.** (a) If $f(x) = -7x^{25} - 8$, find $f'(x)$. Do not simplify.

(b) If $f(x) = (5x^8 - 2)[ln(x^2) + 5]$, find $f'(x)$. Do not simplify.

(c) Find $g'(x)$ if $g(x) = \frac{2x^8 - e^x}{[e^{8x} + ln(x)]}$. Do not simplify.

(d) Find the value of dy if $y = x^4 + 3, x = 3$ and the change in x is 0.2.

[7] **3.** A sum of seven thousand dollars is invested for eight years. Assume that interest is compounded continuously and determine the annual rate of return in the following three cases:

(a) the value after 8 years is nineteen thousand dollars.

(b) the value after 8 years is seven thousand dollars.

(c) the value after 8 years is six thousand dollars.

[8] **4.** The cost of printing x books is given by the function $C(x) = 10,000 + 15x$.

(a) Find the average cost per unit if 100 books are printed.

(b) Find the marginal average cost when 100 books are printed and interpret the results.

(c) Use (a) and (b) to estimate the average cost per book if 101 books are printed.

[7] **5.** Sales of x units of a product are found to be given by the function $S(x) = 3x^4 + x^2 - 5$. At what rate are sales changing when $x = 4$?

[8] **6.** A point is moving along the negative x axis at a constant rate of 5 units per second. Find the rate of change of its distance from $(1, 1)$ when $x = -3$.

[15] **7.** (a) Give an example of a function f whose derivative equals 1 when evaluated at $x = 2$ and equals 4 when evaluated at $x = 3$.

(b) Prove or disprove the following statement. There is a polynomial of degree 2 which has a derivative and is a derivative.

(c) Without using the quotient rule find the derivative of the function $\ln(\frac{x+3}{x-7})$

1.

$$g(x) = 2 - 13x^3$$

$$\lim_{a \to 0} \frac{[g(x+a) - g(x)]}{a}$$

a) $g(x+a) = 2 - 13(x+a)^3$

$= 2 - 13(x^3 + a^3 + 3a^2x + 3x^2a)$

$= 2 - 13x^3 - 13a^3 - 39a^2x - 39x^2a$

b) $g(x+a) - g(x) = (2 - 13x^3 - 13a^3 - 39a^2x - 39x^2a) - (2 - 13x^3)$

$= \cancel{2} - \cancel{13x^3} - 13a^3 - 39a^2x - 39x^2a - \cancel{2} + \cancel{13x^3}$

$= -13a^3 - 39a^2x - 39x^2a$

c) $$\frac{g(x+a) - g(x)}{a} = \frac{-13a^3 - 39a^2x - 39x^2a}{a}$$

$$= \frac{\cancel{a}(-13a^2 - 39ax - 39x^2)}{\cancel{a}}$$

$= -13a^2 - 39ax - 39x^2$

d) $$\lim_{a \to 0} \frac{g(x+a) - g(x)}{a} = \lim_{a \to 0} (-13a^2 - 39ax - 39x^2)$$

$= -39x^2$

2. (a) $f(x) = -7x^{25} - 8$

$f'(x) = -7 \times 25 \cdot x^{24} - 0 = -175x^{24}$

(b) $f(x) = (5x^3-2)[\ln(x^2)+5]$

$f'(x) = (5x^3-2)'[\ln(x^2)+5] + (5x^3-2)[\ln(x^2)+5]'$

$= 15x^2(\ln(x^2)+5) + (5x^3-2)\left(\frac{2}{x}\right)$

(c) $g(x) = \dfrac{2x^3 - e^x}{[e^{3x}+\ln(x)]}$

$$g'(x) = \frac{(e^{3x}+\ln(x))(2x^3-e^x)' - (e^{3x}+\ln(x))' \cdot (2x^3-e^x)}{[e^{3x}+\ln(x)]^2}$$

$$= \frac{(e^{3x}+\ln(x))(6x^2-e^x) - (3 \cdot e^{3x}+\frac{1}{x})(2x^3-e^x)}{[e^{3x}+\ln(x)]^2}$$

(d) $y = x^4 + 3$, $x = 3$, $dx = 0.2$.

$dy = y'(x) \cdot dx$

$= 4x^3 \cdot dx$

$dy\Big|_{\substack{x=3 \\ dx=0.2}} = 4(3)^3 \cdot (0.2)$

$= 4 \times 27 \cdot (0.2)$

$= 21.6.$

3. $P = 7000, \quad A = Pe^{rt}$

a) $A = 19{,}000, \quad t = 8$ years

$19{,}000 = 7000\, e^{r \cdot 8}$

$\frac{19\cancel{000}}{7\cancel{000}} = e^{r \cdot 8}$

$\ln\left(\frac{19}{7}\right) = \ln\left(e^{r \cdot 8}\right)$

$\ln(2.714) = r \cdot 8 \Rightarrow r = 0.998/8 = 0.1248$

$\Rightarrow r = 12.48\%$

b) $A = 7000, \quad t = 8$ years.

$7\cancel{000} = 7\cancel{000}\, e^{r \cdot 8}$

$1 = e^{r \cdot 8}$

$\Rightarrow \ln(1) = r \cdot 8$ or $r = \frac{\ln(1)}{8} = 0\%$

c) $A = 6{,}000, \quad t = 8$ years

$6\cancel{000} = 7\cancel{000}\, e^{r \cdot 8}$

$\Rightarrow \frac{\cancel{6000}}{\cancel{7000}} = e^{r \cdot 8}$

$0.857 = e^{r \cdot 8} \Rightarrow \ln(0.857) = r \cdot 8$

or $r = -\frac{0.154}{8} = -0.0192$ or $r = -1.92\%$

4. $C(x) = 10{,}000 + 15x$

a) Average cost at 100 books

Average cost : $\overline{C}(x) = \dfrac{C(x)}{x}$

$$\overline{C}(x) = \frac{10{,}000 + 15x}{x} = \frac{10{,}000}{x} + 15$$

$$\overline{C}(100) = \frac{10{,}000}{100} + 15 = 100 + 15 = 115$$

b) Marginal average cost : $\overline{C}'(x) = \dfrac{d}{dx}\left(\overline{C}(x)\right)$

$$\left(\overline{C}(x)\right)' = \frac{d}{dx}\left(\frac{10{,}000}{x} + 15\right) = -\frac{10{,}000}{x^2}$$

$$\left(\overline{C}(100)\right)' = \frac{-10{,}000}{(100)^2} = -1.$$

Interpretation: At a production level of 100 books the average cost is \$115. This cost is decreasing at the rate of \$-1 per book.

c) Average cost of 101 books from (a) and (b)

$$\overline{C}(101) = \overline{C}(100) + \overline{C}'(100)$$

$$= 115 + (-1) = \$114$$

5. $S(x) = 3x^4 + x^2 - 5$

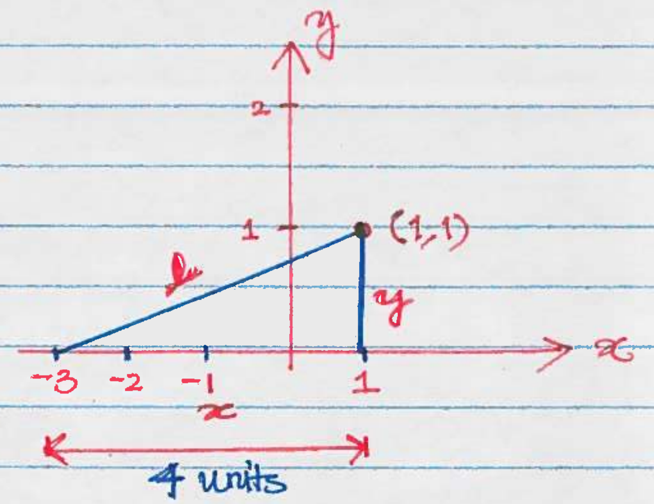
Rate of change of sales : $\frac{d}{dx} S(x) = S'(x)$

$S'(x) = 12x^3 + 2x$

$$S'(x)\Big|_{x=4} = 12(4)^3 + 2(4)$$

$$= 768 + 8 = 776$$

6. From the diagram



$l^2 = x^2 + y^2$ ①

$l^2 = (4)^2 + (1)^2$

$= 16 + 1 = 17$

$\Rightarrow l = \sqrt{17}$

We have $x = 4$ units, $\frac{dx}{dt} = 5$. To find $\frac{dl}{dt}$, we differentiate with respect to time the equation ①

$\frac{d}{dt}\{l^2\} = \frac{d}{dt}\{x^2 + y^2\}$

$= \cancel{2} \cdot l \cdot \frac{dl}{dt} = \cancel{2} \cdot x \cdot \frac{dx}{dt} + 0$ y-coordinate is not changing with respect to time.

$\frac{dl}{dt} = \frac{x}{l} \cdot \frac{dx}{dt} = \frac{4}{\sqrt{17}} \cdot 5 = \frac{20}{\sqrt{17}}$

7. (a) Given $f'(x)$ is such that

$$f'(x)\Big|_{x=2} = 1 \quad \text{and} \quad f'(x)\Big|_{x=3} = 4$$

To find $f(x)$ satisfying the above conditions, let us take $f(x)$ of the form $f(x) = Ax^2 + Bx + C$

Then, $f'(x) = 2A.x + B$

Now, $f'(x)\Big|_{x=2} = 2.A.(2) + B = 1$

$f'(x)\Big|_{x=3} = 2A.(3) + B = 4$

We have two equation for two unknowns A and B.

$$6A + B = 4 \quad ①$$
$$4A + B = 1 \quad ②$$

① − ② = $2A + 0 = 3 \implies 2A = 3$ or $A = 3/2$

Using ② $4A + B = 1$ or $\overset{2}{\cancel{4}}\left(\frac{3}{\cancel{2}}\right) + B = 1$

$\implies 6 + B = 1$ or $B = 1 - 6 = -5$

$\therefore \quad f(x) = 3/2\, x^2 - 5x$

(b) We PROVE the statement by giving example of a function $f(x)$ of degree 2 which has a derivative and is a derivative

Let $f(x) = x^2$

$f(x)$ has a derivative $f'(x) = 2 \cdot x$

and $f(x) = x^2$ is the derivative of $g(x) = \frac{x^3}{3}$

i.e $g'(x) = f(x) = x^2$.

(c) $\ln\left(\frac{x+3}{x-7}\right) = \ln(x+3) - \ln(x-7)$

$$\frac{d}{dx}\left(\ln\left(\frac{x+3}{x-7}\right)\right) = \frac{d}{dx}\left(\ln(x+3)\right) - \frac{d}{dx}\left(\ln(x-7)\right)$$

$$= \frac{1}{x+3} - \frac{1}{x-7}$$