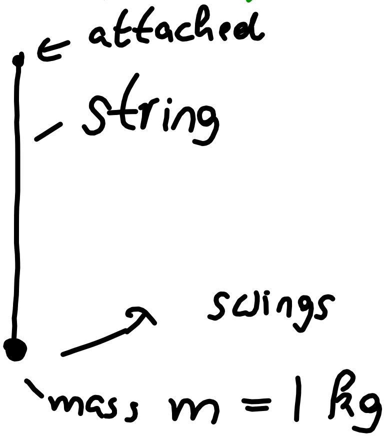
Differential equations (§ 9.1)

springs / pendulum



know from physics:
y describes position of mass at time t

$$\underline{y''} \cdot m = \underline{-k} \cdot \underline{y}, \quad k \text{ a coefficient } (= \text{constant})$$

for example: $y(t) = \underline{\sin(t \cdot \sqrt{k})}$ fits into equation

$$y'(t) = \sqrt{k} \cdot \cos(t \cdot \sqrt{k})$$

$$y''(t) = \underbrace{\sqrt{k} \cdot \sqrt{k}}_{=k} \cdot (-\sin(t \cdot \sqrt{k})) = \underline{-k \cdot \sin(t \cdot \sqrt{k})}$$

Check equation: $\underline{-k \cdot \sin(t \cdot \sqrt{k})} \cdot \underbrace{m}_{=1} = \underline{-k} \cdot (\underline{\sin(t \cdot \sqrt{k})})$

We have verified that $y(t)$ satisfies the equation.
→ more in textbook

Population model: (textbook)

$$\frac{dP}{dt} = k \cdot P \qquad \text{where } \frac{dP}{dt} = P'(t) \quad (\text{calc I})$$

Check that $P(t) = e^{kt}$ is a solution to the eq.

$$P'(t) = e^{kt} \cdot k = k \cdot e^{kt}$$

chain rule

$$\frac{dP}{dt} = k \cdot e^{kt} = k \cdot (e^{kt}) \checkmark$$

But: what about $P(t) = 5 \cdot e^{kt}$?

$$P'(t) = 5 \cdot e^{kt} \cdot k = 5k\, e^{kt}$$

$$5k \cdot e^{kt} = k \cdot (5 \cdot e^{kt}) = 5k \cdot e^{kt}$$

So $P(t) = 5 \cdot e^{kt}$ also satisfies equation!

Turns out, all $P(t) = C \cdot e^{kt}$, $C \in \mathbb{R}$, satisfy it!

(later today)

What do we mean by a solution of a differential eq.?

→ A function that satisfies the equation.

A differential equation relates an unknown function to its derivative(s) and / or a variable.

e.g. $\frac{dy}{dx} = -5x$

Here, integrate! first: separate x and y:

$dy = -5x\,dx$ (multiply dx over)

now $\int$: $\int dy = \int -5x\,dx$

don't have to write C_2, C_1

$y + C_1 = -\frac{5x^2}{2} + C_2$

$y = -\frac{5x^2}{2} + (C_2 - C_1)$ then define $C = C_2 - C_1$

$$\underline{\underline{y = -\frac{5x^2}{2} + C}}, \quad C \in \mathbb{R}$$

Ex2 $\frac{dy}{dx} = \frac{x^2}{y^2}$ (§9.3)

again, separate x and y: $\underline{y^2 dy = x^2 dx}$ multiply by y^2 and dx

$$\int y^2 dy = \int x^2 dx$$

$$\frac{y^3}{3} = \frac{x^3}{3} + C$$

$$y^3 = x^3 + 3C$$

or $y = \sqrt[3]{x^3 + 3C}$ can't take C out of root!!

Ex 3 $\frac{dy}{dx} = \frac{x+7}{\sqrt{y}}$ bring y to one side, x to the other

would do: $/ : dy$ and then $/ : (x+7)$

$$\frac{1}{(x+7)dx} = \frac{1}{\sqrt{y}\,dy}$$ separate, but can't integrate yet!

→ need dx, dy in numerator!

Better: multiply by dx and $\sqrt{y}$!!

$\rightsquigarrow \sqrt{y}\,dy = (x+7)dx$ ($\rightsquigarrow$ means: we get as result)

$$\int \underbrace{\sqrt{y}}_{y^{1/2}}\,dy = \int (x+7)\,dx$$

exercise

$$\frac{2}{3} \cdot y^{3/2} = \frac{x^2}{2} + 7x + C$$

Now: get y out!

first: multiply by 3, divide by 2.

$$y^{3/2} = \frac{3}{2}\left(\frac{x^2}{2} + 7x + C\right)$$

then: raise to power $\frac{2}{3}$

square everything → always pos.

$$y = \left(\frac{3}{2}\left(\frac{x^2}{2} + 7x + C\right)\right)^{2/3}$$

3rd root → no worries about sign

→ so no ± here!!

In general: separate x to one side (dx on top!!)
and y to the other side (dy on top!)
this is called
Method of separating variables (first solution strategy)
→ compare to textbook

Ex3 using method of sep. var. again

$$\frac{dy}{dx} = x^2 \cdot y$$

first, y's to the left, x's to right

$$\frac{dy}{y} = x^2 \cdot dx$$

$$\int \frac{dy}{y} = \int x^2 dx$$

don't forget C!!

$$\ln|y| = \frac{x^3}{3} + C$$

don't forget abs. value!!

Solve for y:

(remember: $e^{(\ln x)} = x$ → inverse functions)

$$|y| = \underbrace{e^{\frac{x^3}{3} + C}}_{\text{always positive}}$$

get rid of |.|: In general:

$$\boxed{e^{a+b} = e^a \cdot e^b}$$

$$y = \pm e^{\frac{x^3}{3} + C} = \pm e^{\frac{x^3}{3}} \cdot \underbrace{e^C}_{\text{pos. constant for all } C \in \mathbb{R}}$$

$$= \pm K \cdot e^{\frac{x^3}{3}} \quad \text{for a positive constant } K = e^C$$

Instead of a pos. K, allow all $K \in \mathbb{R}$ where $K \neq 0$. Then we can forget about $\pm$.

$$y = K \cdot e^{\frac{x^3}{3}} \quad \text{for } K \in \underbrace{\mathbb{R} \setminus \{0\}}_{(-\infty, 0) \cup (0, \infty)}$$

What about all these constants?

Always had many solutions so far, one for each choice of C or K ("family of solutions")

Real life problems want <u>one</u> solution.

How do we get a specific solution?

In reality, $y(0)$ is given to us.

$y(0)$ is called an initial value (initial value problem)
(p 583 in textbook)

<u>Ex</u> find y where $y' = 7x^3$ and $y(0) = 3$

first, solve equation:

$y' = 7x^3$ so antiderivatives:

$$y = \int 7x^3\,dx = \frac{7x^4}{4} + C$$

Now use $y(0) = 3$ to get a value for C:

$$\underbrace{y(0)}_{=3} = \frac{7 \cdot 0^4}{4} + C$$

$$3 = \underbrace{\frac{7 \cdot 0^4}{4}}_{=0} + C = C \leftarrow$$ (solving linear equations for C)

So $\boxed{C = 3}$ and $\boxed{y = \frac{7x^4}{4} + 3}$

Sometimes the initial value determines behaviour.

back to population equation (beginning of class)

$$\frac{dy}{dt} = k \cdot y$$ now we can solve this ourselves!

k: constant

use method of sep. variables: y's one side, t's other side

$$\frac{dy}{y} = k \cdot dt$$

$$\int \frac{dy}{y} = \int k \cdot dt$$

$$\ln|y| = k \cdot t + C$$

as before

$$y(t) = A \cdot e^{kt}$$ for $A = \pm e^C$, so $A \in \mathbb{R} \setminus \{0\}$

Consider two scenarios: $y(0)=3$, 2nd: $y(0)=-3$

for $y(0)=3$:

$$y(0) = 3 = A \cdot \underbrace{e^{k \cdot 0}}_{e^0 = 1}$$ wanna find A

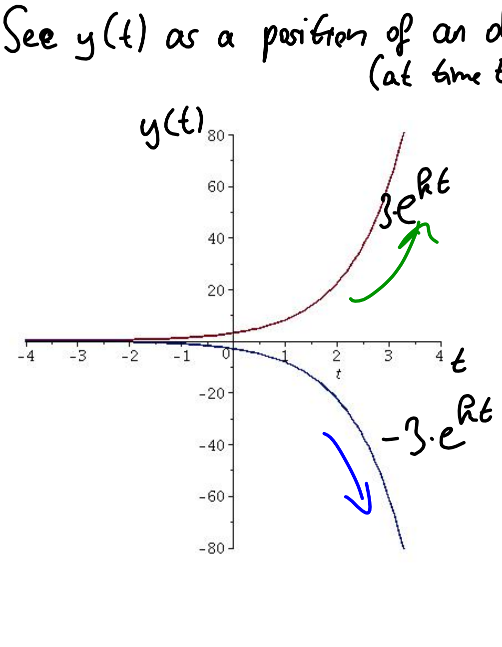
$$\underline{3 = A} \quad \text{So} \quad \underline{y_1(t) = 3 \cdot e^{k \cdot t}}$$

for $y(0) = -3$: we get $A = -3$ and $\underline{y_2(t) = -3 \cdot e^{kt}}$

$y_1(t)$ and $y_2(t)$ are two different solutions!

both satisfy equation.

See $y(t)$ as a position of an object (at time t), $y_1(t)$ and $y_2(t)$ are completely different paths!



depending on $y(0)$, the behaviour changes drastically

(for ex: move up to moon or crash into ocean)

⇝ We need to discuss different solutions in more detail!

p583 $\quad y = \dfrac{1 + C\cdot e^t}{1 - C\cdot e^t}$ given

turns out, it satisfies $y' = \frac{1}{2}(y^2 - 1)$ (check with book)

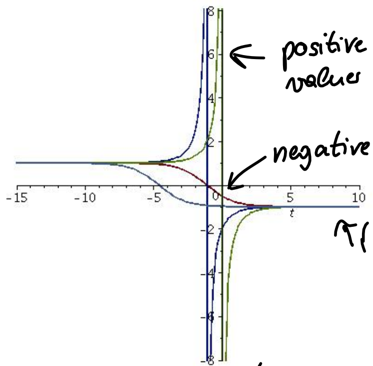
Now if $y(-1) = 3$

we get: $y(-1) = 3 = \dfrac{1 + C\cdot e^{-1}}{1 - C\cdot e^{-1}}$

⇝ solve for C: $C = \frac{e}{2}$

not in textbook

or: if $y(-1) = 0$, then we get $\underline{C = -e}$.

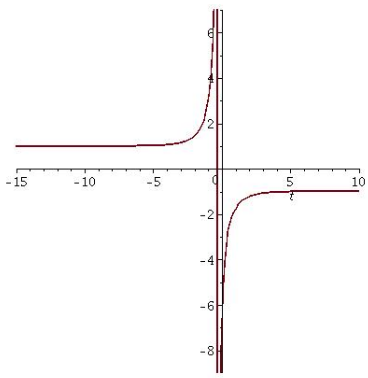


← positive values for C

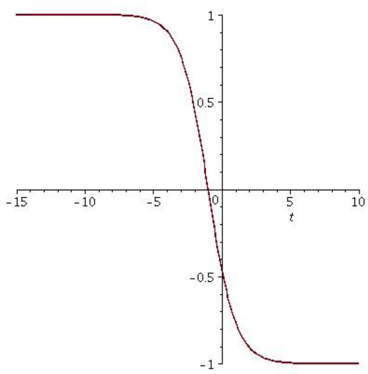
negative values for C

(horizontal asymptote Calc I)

$y(t) = \dfrac{1 + \frac{e}{2}\cdot e^t}{1 - \frac{e}{2}e^t}$ ← $y(-1) = 3$



In contrast, $y(t) = \dfrac{1 - e\cdot e^t}{1 + e\cdot e^t}$ $\quad y(-1) = 0$



Next time:

determine behaviour from equation.