Power series (§11.9)

$\sum_{n=0}^{\infty} c_n \cdot (x-a)^n$ — radius of convergence (convergence depends on x)

functions as power series: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ ← converges $|x|<1$

↑ geometric series

so radius is 1.

integrate and differentiate:

Ex: $\left(\frac{1}{1-x}\right)' = \frac{-1}{(1-x)^2}$ want to express as a series:

$$\left(\sum_{n=0}^{\infty} x^n\right)' = \left(1 + x + x^2 + x^3 + x^4 + \ldots\right)'$$

$$= 0 + 1 + 2x + 3x^2 + 4x^3 + \ldots = \sum_{n=1}^{\infty} n \cdot x^{n-1}$$

now $f(x) = -\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n \cdot x^{n-1}$

Thm p 758: $\sum_{n=0}^{\infty} c_n (x-a)^n$ ← radius of conv. is >0 then

(i) $\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right)' = \sum_{n=0}^{\infty} n c_n \cdot (x-a)^{n-1}$

(ii) $\int \left(\sum_{n=0}^{\infty} c_n \cdot (x-a)^n\right) dx = \sum_{n=0}^{\infty} \left(\int c_n \cdot (x-a)^n \, dx\right)$

$$= \sum_{n=0}^{\infty} c_n \cdot \frac{(x-a)^{n+1}}{n+1} + C$$

Taylor & MacLaurin series (§11.10)

If f has a power series representation at a point $x=a$:
then its coefficients are given by its derivatives:

$x=a$: constant!! (given)

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}$$

Taylor series at $x=a$.

Ex TS of e^{2x} at $a=1$:

$f'(x) = (e^{2x})' = 2\cdot e^{2x}$

$f^{(2)}(x) = 2\cdot 2\cdot e^{2x}$

$f^{(3)}(x) = 2\cdot 2\cdot(2\cdot e^{2x})$ ... $f^{(n)}(x) = 2^n\cdot e^{2x}$

put into formula with $a=1$:

$$e^{2x} = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n \overset{a=1}{=} \sum_{n=0}^{\infty}\frac{f^{(n)}(1)}{n!}(x-1)^n$$

$$= \sum_{n=0}^{\infty}\frac{2^n e^{2\cdot 1}}{n!}(x-1)^n = \sum_{n=0}^{\infty}\frac{2^n e^2}{n!}(x-1)^n$$

remark: radius of convergence ....

MacLaurin series: is a Taylor series with base point $a=0$

$$\boxed{f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n}$$

Ex MacLaurin series for $\sin(x)$:

$f'(x) = \cos(x)$ $\quad f^{(3)}(x) = -\cos(x)$

$f''(x) = -\sin(x)$ $\quad f^{(4)}(x) = -(-\sin(x)) = \sin(x)$

evaluate $f^{(n)}(0)$:

$f'(x) = \cos(0) = 1$ $\quad f^{(3)}(x) = -\cos(0) = -1$

$f''(x) = -\sin(0) = 0$ $\quad f^{(4)}(x) = 0$

sum up:
$$f^{(n)}(0) = \begin{cases} 1 & \text{for } n = 4k+1, \quad k \geq 0 \\ -1 & \text{for } n = 4k+3 \\ 0 & n \text{ even, also } n=0! \end{cases}$$

$4\cdot 0+1 = 1$
$4\cdot 1+1 = 5$
$4\cdot 2+1 = 9$
$\vdots$

put into formula:

$$\sin(x) = \sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{2n+1}}{(2n+1)!}$$

← skips all even numbers $f^{(2n)}(0)=0$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \dots\dots$$

p 762 textbook important series: $\frac{1}{1-x}$, e^x, $\sin(x)$, $\cos(x)$

Ex MacLaurin series for $\sin(x^2)$

$$f'(x) = \cos(x^2)\cdot 2x$$

$$f^{(2)}(x) = (-\sin(x^2))\cdot 2x\cdot 2x + \cos(x^2)\cdot 2$$

$\therefore$ much more complicated!!

Use $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}$

just substitute:

$$\sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{4n+2}}{(2n+1)!}$$

simplify a bit

allowed to quote from textbook that $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}$ and then substitute!!

And also: Macl. series for $x\cdot\sin(x)$:

$$x\cdot\sin(x) = x\cdot\left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{x\cdot x^{2n+1}}{(2n+1)!}$$

simplify $\nearrow$ $= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n+1)!}$

Ex to demonstrate use of power series:

$\int \sin(x^2)\,dx \leftarrow$ can't compute that with integral rules.

Use power series: $\sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{4n+2}}{(2n+1)!}$

$\uparrow$ before

recall thm about integration:

$$\int \left(\sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{4n+2}}{(2n+1)!}\right) dx = \sum_{n=0}^{\infty} \left(\int \underbrace{(-1)^n \cdot \frac{1}{(2n+1)!}}_{} x^{4n+2}\, dx \right)$$

indep. of x!
take them out of integral

$$= \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{(2n+1)!} \cdot \int x^{4n+2}\, dx \right) = \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{(2n+1)!} \cdot \frac{x^{4n+2+1}}{(4n+2+1)} \right) + C$$

simplify

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{4n+3}}{(2n+1)!\,(4n+3)} + C = \left(\int \sin(x^2)\, dx \right)$$

most integrals can only be expressed as a power series, not as a closed function (no infinite sum)

<u>Ex</u> Find power series of $\ln(1+x)$ and its radius of convergence:

inner derivative chain rule

$$f'(x) = (\ln(1+x))' = \frac{-1}{1+x}\cdot(+1) = \frac{1}{1+x}$$

$$f''(x) = \frac{-1}{(1+x)^2} \quad \ldots \quad f^{(n)}(x) = \frac{(-1)^{n+1}}{(1+x)^n}$$

$$f^{(3)}(x) = \left(\frac{1}{(1+x)^2}\right)' = \frac{-1}{(1+x)^{4}}\cdot(-2)\cancel{(1+x)} = \frac{2}{(1+x)^3}$$

chain rule!!

$$f^{(4)}(x) = \left(\frac{2}{(1+x)^3}\right)' = \frac{2}{(1+x)^{3\cdot 2}}\cdot(-3)(1+x)^2 = \frac{-6}{(1+x)^4}$$

in general: (proof by induction) $f^{(n)}(x) = \frac{(n-1)!(-1)^{n+1}}{(1+x)^n}$

put into formula for base point $a=0$: (MacLaurin series)

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum \frac{\frac{(n-1)!(-1)^{n+1}}{(1+0)^n}}{n!}x^n = \sum_{n=1}^{\infty} \frac{-(-x)^n}{n}$$

$$n! = 1\cdot 2\cdot 3\cdot \ldots \cdot n$$

$$\text{so } \frac{(n-1)!}{n!} = \frac{\cancel{1\cdot 2\cdot 3\cdot \ldots \cdot (n-2)(n-1)}}{\cancel{1\cdot 2\cdot 3\cdot \ldots \cdot (n-1)}\cdot n} = \frac{1}{n}$$

coefficients!!

this was the long way!

Shorter:

$\ln(1+x)$ is the integral of $\frac{1}{1+x} = \frac{1}{1-(-x)}$

geometric series

know $\frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n$

$$\text{So } \ln(1+x) = \int \frac{1}{1+x}\,dx = \int \frac{1}{1-(-x)}\,dx$$

$$= \int \sum_{n=0}^{\infty} (-x)^n\,dx = \sum_{n=0}^{\infty}\left(\int (-x)^n\,dx\right)$$

$$= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}(-1)^n + C$$

remembering some power series and substituting often faster!!

<u>radius of convergence</u>: for which x does $\sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}(-1)^n$ converge?

Use that absolute convergence give convergence!!
if can show, that series converges absolutely, it converges.

ratio test: $$\underline{\underline{L}} = \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim \frac{\left|\frac{x^{n+1+1}}{n+1+1}\right|}{\left|\frac{x^{n+1}}{n+1}\right|} = \lim \left|\frac{x^{n+2}\cdot(n+1)}{(n+2)\cdot x^{n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^{\cancel{n+2}}}{\cancel{x^{n+1}}}\cdot\frac{n+1}{n+2}\right| = \underset{\text{indep. of } n}{|x|}\cdot\underbrace{\left|\lim_{n\to\infty}\frac{n+1}{n+2}\right|}_{=1} = \underline{\underline{|x|}}$$

ratio test: if $L<1$: converges
$L>1$: diverges.

So the series converges <u>if $|x|<1$</u>

diverges if $|x|>1$.

need to check $|x|=1$: so check $x=-1$, $x=1$

does $\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1} \underset{x=1}{=} \sum(-1)^n\cdot\frac{1}{n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$

alternating harmonic series

at $x=1$: converges!!

at $x=-1$: $\sum_{n=0}^{\infty}(-1)^n\frac{(-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty}\frac{1}{n+1}$

it converges!!
(remember this!!)

Harmonic series diverges!!

(alternating Harmonic series is conditionally conv, but NOT absolutely)

radius of convergence: $-1 < x \leq 1$

divergent; converges at <u>+1</u>.