# Tangent Planes (§14.4)

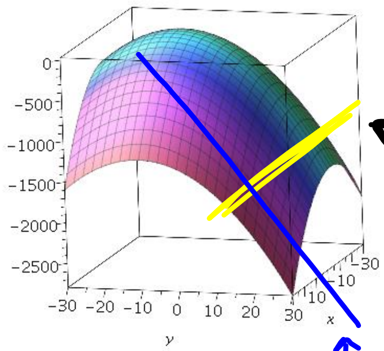
recall: last week
function $f(x,y)$ in 2 var, partial derivatives
$f_x$ or $\frac{\partial f}{\partial x}$, $f_y$ or $\frac{\partial f}{\partial y}$

Ex: $f(x,y) = 3x^2y^3 + 7x^3y$

$f_x = 3\cdot(2x)\cdot y^3 + 7\cdot(3x^2)\cdot y$ simplify....

$f_y = 3x^2\cdot(3y^2) + 7x^3\cdot 1$ —"—

geometric interpretation




tangent line along $x$

tangent line along $y$

Today: combine two tangent lines to get a tangent plane!

Tangent line equation $f$ has continous first partial derivatives. An equation of the tangent plane to the surface (graph of fct) $z = f(x,y)$ at the point $p(x_0, y_0, z_0)$ ($z_0$ is given by $z_0 = f(x_0, y_0)$) is:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Ex $f(x,y) = 2x^2 + y^2$ tangent plane to paraboloid at $(\underbrace{1}_{x_0}, \underbrace{1}_{y_0}, \underbrace{3}_{z_0})$

$$f_x = 2 \cdot 2x + 0 = 4x$$

$$f_y = 0 + 2y = 2y$$

check: $2 \cdot x_0^2 + y_0^2 = z_0$

$2 \cdot 1^2 + 1^2 = 3$ ✓

(has to match, point lies on surface)

now, evaluate $f_x$, $f_y$ at $(x_0, y_0) = (1,1)$

$$f_x(1,1) = 4 \cdot 1 = 4$$

$$f_y(1,1) = 2 \cdot 1 = 2$$

tangent line equation:

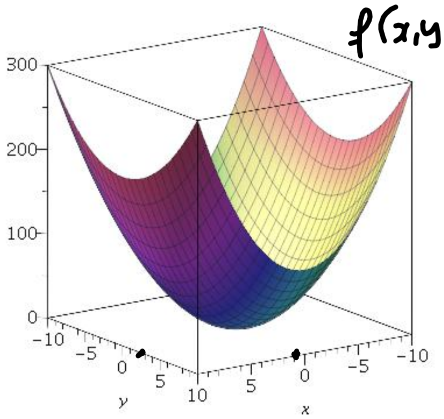
$$z - z_0 = \underbrace{f_x(x_0,y_0)}_{=4} \cdot (x - x_0) + \underbrace{f_y(x_0,y_0)}_{=2}(y - y_0)$$

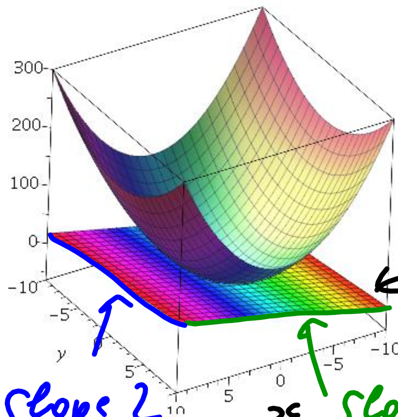
$$z - 3 = 4(x-1) + 2 \cdot (y-1) = 4x - 4 + 2y - 2 = 4x + 2y - 6$$

$$z - 3 = 4x + 2y - 6$$

$$\boxed{z = 4x + 2y - 3}$$ ← tangent plane

f(x,y)




tangent plane at (1,1,3)

slope 2
$f_y$

x

slope 4
$f_x$

(office hours: $2-3^{30}$ Wed MAR 30)

## Linear approximations

in 1-d: $y=f(x)$, Taylor polynomials, series:

$$T_f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

base point

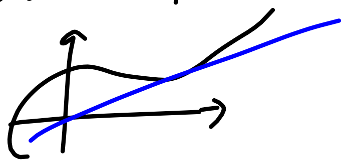
special case: $a=0$: MacLaurin series

if only a linear approximation:

$$L_f(x) = \sum_{n=0}^{1} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + f'(a)(x-a)$$

tangent line equation at $(a, f(a))$



Same in 3d: tangent plane is a linear approximation

**Def** $z=f(x,y)$ is differentiable at $(a,b)$ if $\Delta z$ can be expressed in the form

$$\Delta z = f_x(a,b)\cdot\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \cdot \Delta x + \varepsilon_2 \cdot \Delta y$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

" $y-b$

distance along $x$ from $a$

$\Delta x = (x-a)$

**Thm** If the partial derivatives $f_x, f_y$ exist near $(a,b)$ and are continous at $(a,b)$, then $f$ is differentiable at $(a,b)$. (easier to check for differentiability)

(not exclusive)

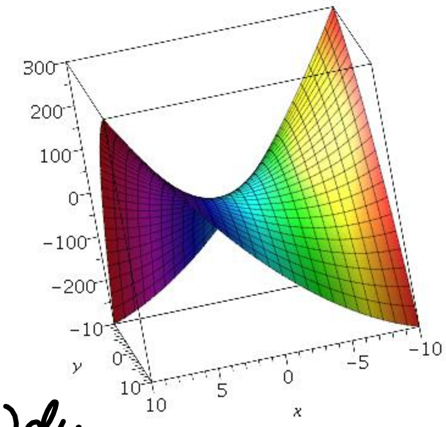
## Total differentials / total derivative:

A total differential of a function $f(x,y)$ is defined as

$$dz = f_x(x,y)\,\underline{dx} + f_y(x,y)\,\underline{dy}$$

symbols

need: $f$ is differentiable




Ex $f(x,y) = x^2 + 3xy - y^2$

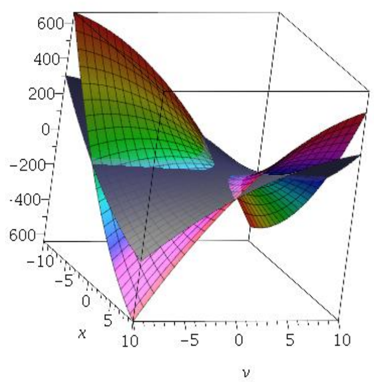
$f_x = 2x + 3\cdot 1\cdot y - 0 = 2x + 3y$

$f_y = 0 + 3x - 2y = 3x - 2y$

$dz = (2x+3y)\,dx + (3x-2y)\,dy$

if we replace $dx$ by $\Delta x$ and $dy$ by $\Delta y$: get tangent plane back.

eg: $(x,y) = (1,1)$, so $\Delta x = x-1$, $\Delta y = y-1$:



## Chain rule (§ 14.5)

Ex: $f(x) = (\sin(3x^2))^3$

$f'(x) = 3 \cdot (\sin(3x^2))^2 \cdot \cos(3x^2) \cdot (3 \cdot 2x)$

Set up: $z = f(x(t), y(t))$

$x(t), y(t)$ are both differentiable

$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}}$$

Ex $z = x^2y + 3xy^4$, $x(t) = \sin(2t)$, $y(t) = \cos(t)$

$\frac{\partial f}{\partial x} = 2xy + 3y^4$ $\qquad \frac{dx}{dt} = \cos(2t) \cdot 2 = 2\cos(2t)$

$\frac{\partial f}{\partial y} = x^2 + 3x \cdot 4 \cdot y^3$ $\qquad \frac{dy}{dt} = -\sin(t)$

$\frac{dz}{dt} = (2xy + 3y^4)(2\cos(2t)) + (x^2 + 12xy^3)(-\sin(t))$

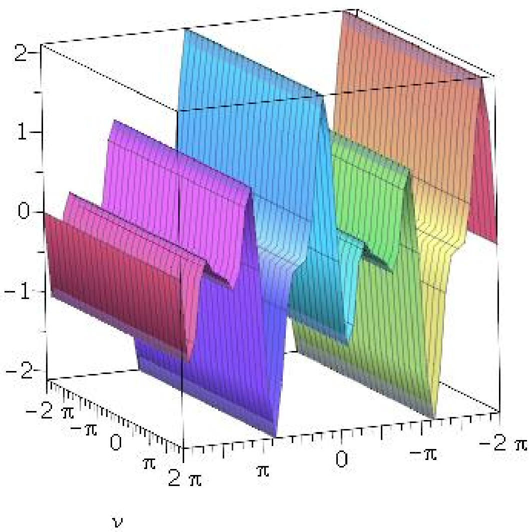
has 3 variables, substitute $x(t) = \sin(2t)$, ...

$= (2 \cdot \sin(2t) \cdot \cos(t) + 3 \cdot \cos^4(t)) \cdot 2 \cdot \cos(2t)$

$+ (\sin^2(2t) + 12 \cdot \sin(2t) \cdot \cos^4(t))(-\sin(t))$

one variable

remark: can also substitute $x, y$ in the beginning to get a function in one variable and then use CALC knowledge!



## Chain rule CASE II:

$z = f(x,y)$ where $x = x(s,t)$, $y = y(s,t)$

now if you substitute, then $z$ becomes a function in $s,t$.
So again: have 2 partial derivatives and one total derivative.

$$\boxed{\begin{aligned} \frac{dz}{ds} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} \\ \frac{dz}{dt} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} \end{aligned}}$$

Example: $z = e^x \cdot \sin(y)$, where $x(s,t) = st^2$
$y(s,t) = s^2 t$

find $\frac{\partial z}{\partial s}$, $\frac{\partial z}{\partial t}$

note: $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ are used in $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$!
→ just compute once!

$\frac{\partial z}{\partial x} = e^x \cdot \sin(y)$, $\frac{\partial x}{\partial s} = t^2$, $\frac{\partial x}{\partial t} = 2st$

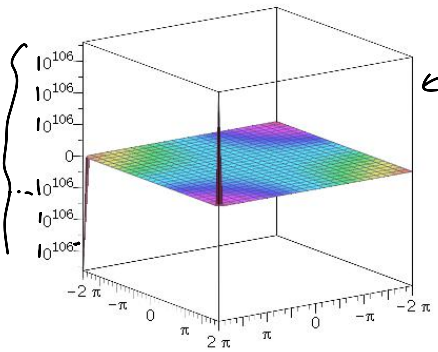
$\frac{\partial z}{\partial y} = e^x \cdot (\cos(y))$, $\frac{\partial y}{\partial s} = 2st$, $\frac{\partial y}{\partial t} = s^2$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$
$$= (e^x \cdot \sin(y)) \cdot t^2 + (e^x \cdot \cos(y)) \cdot (2st)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$
$$= (e^x \cdot \sin(y)) \cdot (2st) + (e^x \cdot \cos(y)) \cdot (s^2)$$

Can again sustitute before calculating:
$\left(z = e^{st^2} \cdot \sin(s^2 t)\right)$ and take partial der.
Chain rule tells you, this is allowed!!
(use chain rule 'implicitly')

out of scale




← just looks flat due to scale!!