Partial Derivative (§14.3)

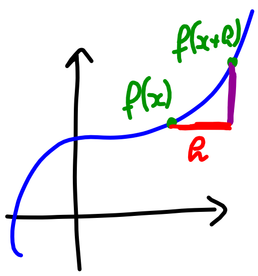
Monday: limits for functions in 2 var
- many ways to approach a pt in the (x,y)-plane (vs 2 ways on a line in CALC I)
- only way to show what the limit is: (δ, ε)-criterion (definition of limit)

Today: extend derivatives to functions in 2 variables (or 3, 4, 5, ... variables)

recall from CALC I: $f(x)$ then

definition of derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



interpretation: $f'(x)$ is the slope of f at the point x.

- had a set of rules for computing derivatives.

Now: partial derivatives of functions in 2 var.

formal definition:

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a,b)}{h}$$

partial derivative of f with respect to x at (a,b)

$$f_y(a,b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a,b)}{h}$$

partial der. of f wrt y at (a,b)

Notation:

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial f(x,y)}{\partial x}$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial f(x,y)}{\partial y}$$

Ex 1 (16) $f(x,y) = x^4 \cdot y^3 + 8x^2 y$

$f_x(x,y) = (x^4)' \cdot y^3 + (8x^2)' \cdot y$

rule: treat only x as a variable, all others as constants!

$= 4x^3 \cdot y^3 + 8 \cdot 2 \cdot x^1 \cdot y = 4x^3y^3 + 16xy$

$f_y(x,y) = x^4 \cdot 3y^2 + 8x^2 \cdot 1$

(now x is treated as a constant)

Ex 2 (like 23 from textbook)

$$f(x,y) = \frac{3x+7y}{4x-2y}$$

quotient rule: (recall) $\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$

$$f_x(x,y) = \frac{(3x+7y)' \cdot (4x-2y) - (3x+7y)(4x-2y)'}{(4x-2y)^2}$$

quotient rule

$$= \frac{3 \cdot (4x-2y) - (3x+7y) \cdot 4}{(4x-2y)^2}$$

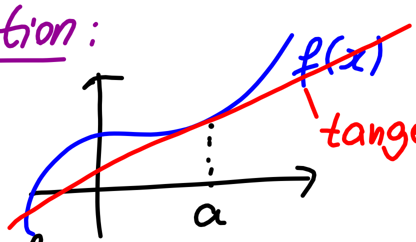
$$= \frac{\cancel{12x} - 6y - \cancel{12x} - 28y}{(4x-2y)^2} = \frac{-34y}{(4x-2y)^2}$$

$$f_y(x,y) = \left(\frac{3x+7y}{4x-2y}\right)'^{y} = \frac{7 \cdot (4x-2y) - (3x+7y)(-2)}{(4x-2y)^2}$$

$$= \frac{28x - \cancel{14y} + 6x + \cancel{14y}}{(4x-2y)^2} = \frac{34x}{(4x-2y)^2}$$

Geometric interpretation:

recall: $y = f(x)$



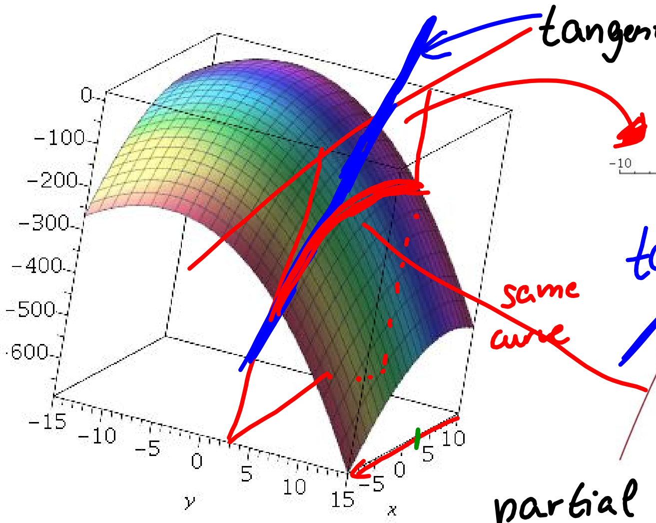
$f'(a)$ slope of function (tangent line) at a.

<u>here</u>: 2 directions, approach along x or y.

$z = f(x,y)$ then $f_x(x,y)$ slope in x-direction at a point

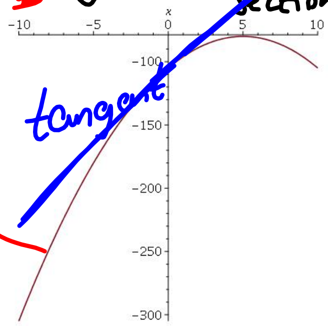
$f_y(x,y)$: slope along y

tangent line parallel to x axis



same curve

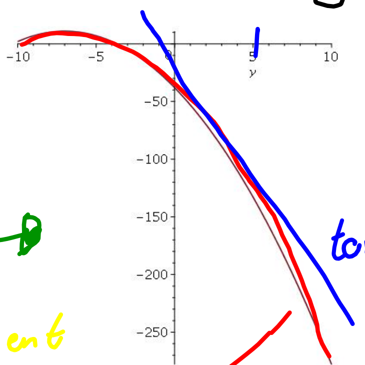
graph of cross section function ($y=3$)



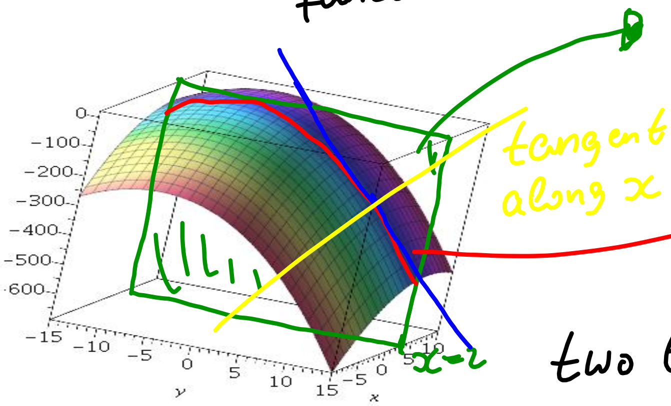
↖ a fct in x

partial der. $f_x(x,y)$ gives the slope of tangent line of cross section function at y.

f_y: slope of f along y:



cross section function: $x=2$



tangent along x

same curve

two lines in a 3d-space span a plane.

Higher derivatives:

"derivatives of derivatives"

$$f_{xx} = (f_x)_x = \frac{\partial^2 f}{\partial^2 x} \quad \Big| \quad f_{xy} = (f_y)_x = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{yy} = \frac{\partial^2 f}{\partial^2 y}, \quad f_{yx} = \frac{\partial^2 f}{\partial x \partial y} \quad \text{(Notation)}$$

Ex 1: $f(x,y) = x^3 + x^2y^3 - 2y^2$, find all 2nd derivatives

$f_{xx}(x,y) = ?$

need: $f_x(x,y) = 3x^2 + 2xy^3 - 0$

now can find

$$f_{xx}(x,y) = (3x^2 + 2xy^3)'^x = 3\cdot 2\cdot x + 2y^3 = \underline{6x + 2y^3}$$

$$f_{yx}(x,y) = (f_x)_y = (3x^2 + 2xy^3)'^y = 0 + 2x\cdot 3y^2 = \underline{6xy^2}$$

$$f_y(x,y) = x^2\cdot 3y^2 - 4y$$

$$f_{xy}(x,y) = (3x^2y^2 - 4y)'^x = 3\cdot 2\cdot x\cdot y^2 - 0 = \underline{6xy^2}$$

$$f_{yy}(x,y) = x^2\cdot 3\cdot 2\cdot y - 4 = \underline{6x^2y - 4}$$

Here: $f_{xy} = f_{yx}$, not a coincidence!

Clairaut's Theorem: f defined on at least a disk D that contains (a,b). If f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

(can't do this at edge of the domain)

Use this to check your computations (if $f_{xy} \neq f_{yx}$ then probably a mistake)

Ex $f(x,y) = 3x^4y^2 + \frac{x}{y^2}$, $f_{xxyx} = ?$

$$f_x(x,y) = 3\cdot 4\cdot x^3\cdot y^2 + \frac{1}{y^2}\cdot 1$$
$$= 12x^3y^2 + \frac{1}{y^2}$$

$$f_{yx}(x,y) = 12x^3\cdot 2y + \frac{-2}{y^3} = 24x^3y - \frac{2}{y^3}$$

$$f_{xyx}(x,y) = 24\cdot 3\cdot x^2\cdot y - 0 = 72x^2y$$

$$f_{xxyx}(x,y) = 72\cdot 2x\cdot y = 144xy$$

Ex $f_{xyx} = ?$ $f(x) = 3\sin(x^2+2y)$

derivative

$$f_x(x,y) \underset{\text{chain rule}}{=} 3\cdot\cos(x^2+2y)\cdot 2x$$
$$= 6x\cdot\cos(x^2+2y)$$

$$f_{yx}(x,y) = 6x\cdot(-\sin(x^2+2y))\cdot 2$$
$$= -12x\cdot\sin(x^2+2y)$$

$$f_{xyx}(x,y) \underset{\text{product rule}}{=} (-12x)^{\prime x}\cdot\sin(x^2+2y) + (-12x)\cdot(\sin(x^2+2y))'$$

$$= -12\sin(x^2+2y) - 12x\cos(x^2+2y)\cdot 2x$$
$$= -12\sin(x^2+2y) - 24x^2\cdot\cos(x^2+2y)$$