Power series (§11.8)

so far: series, alternating series $\left(\sum (-1)^n \cdot a_n\right)$
test for convergence (absolute convergence: $\sum |a_n|$ converges)

Power series are special series, which contain a variable x.

general form: $\boxed{\sum_{n=0}^{\infty} c_n \cdot x^n} = c_0 + c_1 x + c_2 \cdot x^2 + \dots$

variable

coefficient

Now. when does a power series converge or diverge?
for which x.

$\left(\sum_{n=1}^{\infty} q^n \left(\text{eg: } \sum_{n=1}^{\infty} \frac{1}{2^n} \right) \right)$

geometric series converges. $|q| < 1$

So: power series look at bit like geometric series.

Wanna test for convergence:

Ex $\sum_{n=1}^{\infty} \frac{2^n}{n!} (4x-8)^n$ converges for which values of x?

ratio test (or: root test because of powers n)

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\left| \frac{2^{n+1}(4x-8)^{n+1}}{(n+1)!} \right|}{\left| \frac{2^n}{n!}(4x-8)^n \right|}$$

double fraction → numerator, denominator

$$\lim_{n\to\infty} \left| \frac{\cancel{2^{n+1}}(4x-8)^{\cancel{n+1}} \cdot n!}{\cancel{2^n} \cdot \cancel{(4x-8)^n} \cdot (n+1)!} \right| = \lim_{n\to\infty} \left| \frac{2 \cdot (4x-8) \cdot \cancel{(1 \cdot 2 \cdot 3 \cdot \dots \cdot n)}}{(\cancel{1 \cdot 2 \cdot 3 \cdot \dots \cdot n} \cdot (n+1))} \right|$$

(recall: $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$)

$$= \lim_{n\to\infty} \left| \frac{2 \cdot (4x-8)}{n+1} \right| = |2(4x-8)| \cdot \underbrace{\lim_{n\to\infty} \frac{1}{n+1}}_{=0} = 0 < 1$$

so series converges for all value of x!!

Usually we only get an interval

for all value of x!!

Usually, we only get an interval for x on which the power converges.

Ex Almost same: $\sum_{n=1}^{\infty} \frac{2^n}{n}(4x-8)^n$

root test: $L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n}|(4x-8)|^n}$

$= \lim_{n\to\infty}\left(\frac{\cancel{\sqrt[n]{2^n}}}{\sqrt[n]{n}} \cdot \cancel{\sqrt[n]{(4x-8)^n}}\right) = \lim_{n\to\infty}\left(\underbrace{(2\cdot|4x-8|)}_{\text{independent of } n\,!!} \cdot \underbrace{\frac{1}{\sqrt[n]{n}}}_{\to 1}\right)$

$= 2\cdot|4x-8|\cdot 1 = |8x-16|$

converges if $|L|<1$ Diverges for $|L|>1$.

$|8x-16|<1$: need two cases!

case 1. $8x-16$ is positive, so ignore $|\cdot|$

case 2: $8x-16$ is negative, so multiply by -1.

Case 1: $8x-16<1 \quad |+16$

$8x<17 \quad |:8$

$x<\frac{17}{8}$

case 2:

$(-1)(8x-16)<1$

$-8x+16<1 \quad |-16$

$-8x<-15 \quad |\cdot(-8)$ (flip ineq. sign!!!)

$x>\frac{-15}{-8} = \frac{15}{8}$

Get: $\frac{15}{8}<x<\frac{17}{8}$ Series converges for those x.

If $L=1$: no conclusion yet. → Check edges of interval, so $x=\frac{15}{8}$ and $x=\frac{17}{8}$.

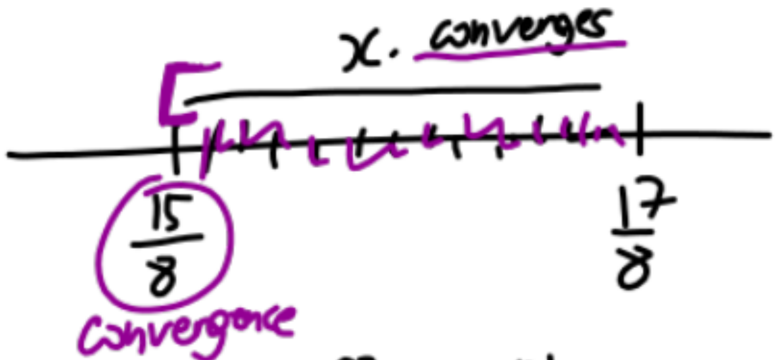
(1) $x=\frac{15}{8}$: $\displaystyle\sum_{n=0}^{\infty}\frac{2^n}{n}(4x-8)^n = \underbrace{\sum_{n=0}^{\infty}\frac{2^n}{n}\left(4\cdot\frac{15}{8}-8\right)^n}_{\text{no power series anymore!!}}$

Simplify first:

$$\sum_{n=0}^{\infty}\frac{2^n}{n}\left(\frac{60-64}{8}\right)^n = \sum_{n=0}^{\infty}\frac{2^n}{n}\left(\frac{-4}{8}\right)^n = \sum_{n=0}^{\infty}\frac{2^n}{n}\cdot\left(-\frac{1}{2}\right)^n$$

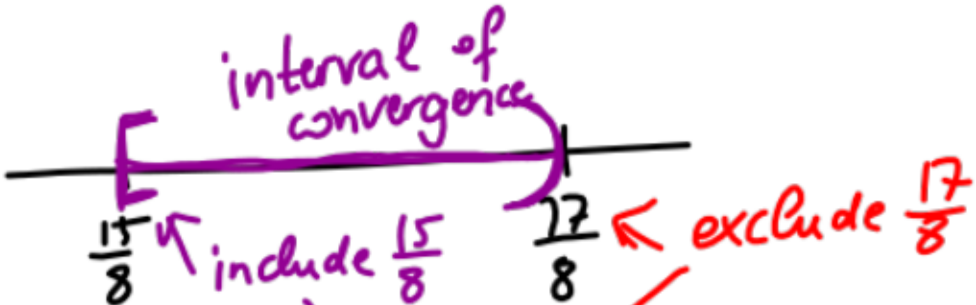
$$= \sum_{n=1}^{\infty}\frac{\cancel{2^n}}{n}\cdot\frac{(-1)^n}{\cancel{2^n}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n} < \infty.$$

converges alternating harmonic series



case 2. $x=\frac{17}{8}$: $\displaystyle\sum_{n=0}^{\infty}\frac{(2^n)}{n}\left(4\cdot\frac{17}{8}-8\right)^n = \sum\frac{2^n}{n}\left(\frac{68-64}{8}\right)^n$

$$= \sum_{n=1}^{\infty}\frac{2^n}{n}\left(\frac{4}{8}\right)^n = \sum\frac{\cancel{2^n}}{n}\cdot\frac{1}{\cancel{2^n}} = \underbrace{\sum_{n=1}^{\infty}\frac{1}{n}}_{\text{diverges: harmonic series}}$$



interval of convergence: $\frac{15}{8} \le x < \frac{17}{8}$

Thm (p 743) Three cases for $\sum_{n=0}^{\infty} c_n \cdot (x-a)^n$ ← constant

(i) converges only if $x=a$.

(ii) converges for all x (first example)

(iii) converges if $|x-a| < R$ and diverges if $|x-a| > R$ (edges: $|x-a| = R$: check separately) (second example)

R is called the radius of convergence !!

Ex: $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$, find radius of convergence

either root or ratio test.

root test: $L = \lim_{n\to\infty} \sqrt[n]{\left|\frac{n\cdot(x+2)^n}{3^{n+1}}\right|} = \lim_{n\to\infty} \frac{\sqrt[n]{n}\cdot\sqrt[n]{|(x+2)^n|}}{\sqrt[n]{3^n\,3}}$

$= \lim_{n\to\infty} \frac{\sqrt[n]{n}\,|(x+2)|}{3\cdot\sqrt[n]{3}} = \frac{|x+2|}{3} \underbrace{\lim_{n\to\infty} \sqrt[n]{\frac{n}{3}}}_{=1} = \frac{|x+2|}{3}$.

Converges if $L<1$: $\frac{|x+2|}{3} < 1 \rightsquigarrow$ again two cases:

case 1: ignore $|.|$

$\frac{x+2}{3} < 1$

$x+2 < 3$

$x < 1$

case 2: $(-1)\left(\frac{x+2}{3}\right) < 1 \quad | \cdot 3$

$-x-2 < 3 \quad | +2$

$-x < 5 \quad | \cdot(-1)$ switch sign!

$x > -5$

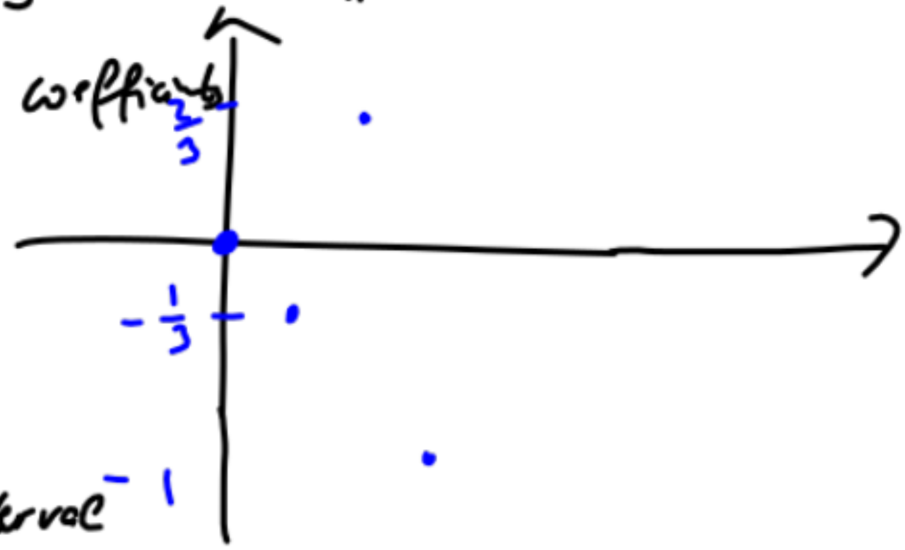
convergence interval:

$-5 < x < 1$ Check edges! $x=-5, x=1$

$\underline{x=-5.}$ $\displaystyle\sum_{n=0}^{\infty} \frac{n\cdot(-5+2)^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{n\cdot(-3)^n}{3^{n+1}} = \sum \frac{n\cdot(-1)^n \cancel{3^n}}{\cancel{3^n}\, 3}$

$\displaystyle = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{n}{3}$

diverges, limit DNE



exclude $x=-5$ from interval of convergence.

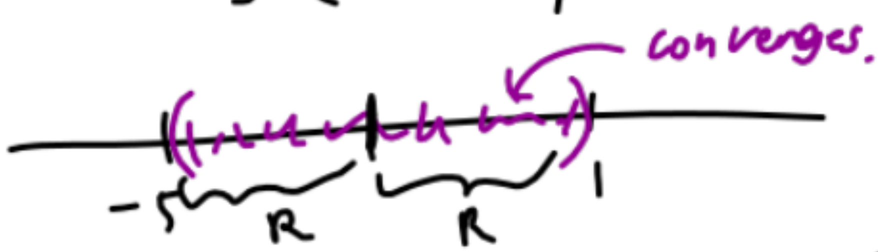
$\underline{\text{check } x=1:}$ $\displaystyle\sum_{n=0}^{\infty} \frac{n\,(1+2)^n}{3^{n+1}} = \sum \frac{n\cdot\cancel{3^n}}{\cancel{3^n}\cdot 3} = \sum_{n=0}^{\infty} \frac{n}{3}$ diverges.

reason both cases diverge: $\lim_{n\to\infty} a_n \neq 0$

Here. $\lim_{n\to\infty} \frac{n}{3} = \infty \underline{\underline{\neq 0}}$.

Both edges gives divergent series, so interval of convergence is

$-5 < x < 1$, radius of convergence:



How do we find R? Wanna split the interval in the middle. Then the distance between midpoint and the edges is the radius of convergence.

Here: find midpoint: add both edges, divide by 2:

$\underline{\underline{m}} = \frac{-5+1}{2} = \frac{-4}{2} = \underline{\underline{-2}}$

radius: distance from -5 to -2 (or -2 to 1)

$\Rightarrow$ $\underline{\underline{\text{Radius is } 3}}$.

Power Series as functions (§11.9)

Example $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ if $|x|<1$

geometric series

(recall)

this is just a function in x.

need power series to approach functions we can't compute otherwise (eg. $\sin(x)$, $\cos(x)$, $\ln(x)$, ...)

Sometimes I can't integrate a function. In that case, we can integrate its power series!! $\int \frac{\ln(x)}{\sin(x)\, e^{x^{\cos}}}$

Example: Integrate $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

integrate $\sum_{n=0}^{\infty} x^n$ instead of $\frac{1}{1-x}$.

$$\int \left(\sum_{n=0}^{\infty} x^n\right) dx = \int (1 + x + x^2 + x^3 + \dots)\, dx$$

$$= \int 1\,dx + \int x\,dx + \int x^2\,dx + \int x^3\,dx + \dots = \sum_{n=0}^{\infty} \left(\int x^n\,dx\right)$$

$$= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C$$