last time: alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \underbrace{a_n}_{\geq 0}$

$\uparrow$ sign

- convergence, divergence
- alternating series test for convergence

estimating sums: $\sum_{n=k}^{\infty} a_n$ want to estimate this.
(pos. series)

$n=k$ ← start at $k>0$

used integral

Today: estimate $\underbrace{\sum_{n=k}^{\infty} (-1)^n \cdot a_n}_{\text{tail end}}$

if $a_{n+1} \leq a_n$ (monotonically decreasing)

then $\boxed{\sum_{n=k}^{\infty} (-1)^n \cdot a_n \leq a_{n-1}}$

Absolute convergence §11.6

normal series $\sum_{n=1}^{\infty} a_n$, alternating: $\sum_{n=1}^{\infty} (-1)^n \cdot a_n$

know convergence (last few classes)

Instead of $\sum_{n=1}^{\infty} (-1)^n \cdot a_n$, look at $\sum_{n=1}^{\infty} |a_n|$

study convergence of (non-neg: ≥ 0 / positive: > 0)

A series $\sum a_n$ is called **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges.

Ex harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

can show

but: $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n}$ converges!! $(= \ln 2)$

Convergent, but not absolutely convergent.
(so convergence $\neq$ abs. convergent)

Ex $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n^2}$

test if abs. convergent: look at $\sum_{n=1}^{\infty} \frac{1}{n^2}$

converges!

So $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n^2}$ converges absolutely!

A series $\sum a_n$ is called **conditionally convergent** if $\sum a_n$ converges but $\sum |a_n|$ diverges (it is convergent but not absolutely convergent).

Example to illustrate why abs. convergence matters:

$$\sum (-1)^{n+1} \cdot \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots$$

↑ converges to $\ln(2)$ ← not gonna show that.

reorder the summands:

$$\underbrace{\underbrace{1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6}}_{0.0833} + \frac{1}{3} + \frac{1}{5}}_{0.6167} \underbrace{- \frac{1}{8} - \frac{1}{10} - \frac{1}{12} - \dots - \frac{1}{22}}_{= 0.0233}$$

just above 0!!

$$\underbrace{+ \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15}}_{0.5119} - \frac{1}{24} - \frac{1}{26} - \dots$$

↑ just above $\frac{1}{2}$ ↑ just above 0!!

$+ \frac{1}{17} + \frac{1}{19} + \dots$ ↖ go until just above $\frac{1}{4}$ then subtract again as long as we're positive.

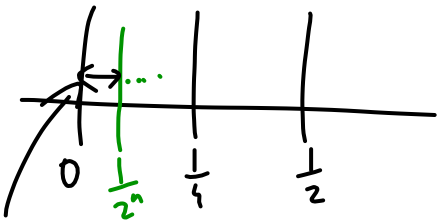
Start with 1, substract as long as > 0 ⊛

then we add until just above $\frac{1}{2}$ — $\frac{1}{2^1}$

then redo ⊛, we add again until $\frac{1}{4}$ — $\frac{1}{2^2}$

— " — ⊛, we add again until $\frac{1}{8}$ — $\frac{1}{2^3}$

$\vdots$ $\frac{1}{2^n}$



$\lim_{n \to \infty} \frac{1}{2^n} = 0$

partial sums stay here!! it means that they converge to 0!!

So the reorderd series has limit 0!!!

But I said: $\sum (-1)^{n+1} \cdot \frac{1}{n} = \ln(2)$

Order changes value!! So $3 + 5 = 5 + 3$ is not always true for infinitely many coefficients and neg. numbers.

That's the difference between absolute convergence and conditional convergence.

If a series is abs. convergent, then any rearrangement has the same limit.

We've seen in the example before, that $\sum (-1)^{n+1} \cdot \frac{1}{n}$ has at least 2 different limits depending on order (can actually create any limit).

Also If a series is abs. convergent, then it converges.

Now: 2 tests for abs. convergence (can use them to show regular convergence because of)

not abs. convergent can still mean conditionally (regularly) convergent or divergent!!

Tests for abs. convergence or divergence

RATIO TEST: (p 734)

(R1) if $\lim \frac{|a_{n+1}|}{|a_n|} = L < 1$ then $\sum a_n$ converges absolutely.

(R2) if — " — $= L > 1$ then $\sum a_n$ is divergent!

if $\lim \frac{|a_{n+1}|}{|a_n|} = 1$ then we can't say anything.
→ have to use another test.

Ex $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+5} \cdot (-1)^n$

ratio test: need $|a_{n+1}|$ and $|a_n|$

$|a_{n+1}| = \frac{\sqrt{n+1}}{(n+1)^2+5}$, $|a_n| = \frac{\sqrt{n}}{n^2+5}$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{\sqrt{n+1}}{(n+1)^2+5}}{\frac{\sqrt{n}}{n^2+5}} = \lim_{n\to\infty} \frac{\sqrt{n+1}\cdot(n^2+5)}{\sqrt{n}\cdot((n+1)^2+5)}$$

multiply out, drag in

$$= \lim_{n\to\infty} \frac{n^2\cdot\sqrt{n+1} + 5\cdot\sqrt{n+1}}{\sqrt{n}\cdot n^2 + 2n\cdot\sqrt{n} + 6\cdot\sqrt{n}} = \lim \frac{\sqrt{n^4(n+1)} + 5\cdot\sqrt{n+1}}{\sqrt{n^4\cdot n} + 2\cdot\sqrt{n^3} + 6\cdot\sqrt{n}}$$

practice

$$= \lim_{n\to\infty} \frac{\sqrt{n^5+n^4} + 5\cdot\sqrt{n+1}}{\sqrt{n^5} + 2\cdot\sqrt{n^3} + 6\cdot\sqrt{n}} =$$

highest power is $n^{5/2} = \sqrt{n^5}$

divide by $n^{5/2}$

$$= \lim_{n\to\infty} \frac{\sqrt{n^{-5}}\cdot\sqrt{n^5+n^4} + \sqrt{n^{-5}}\cdot 5\cdot\sqrt{n+1}}{\sqrt{n^{-5}}\cdot(\sqrt{n^5} + 2\cdot\sqrt{n^3} + 6\cdot\sqrt{n})}$$

$$= \lim_{n\to\infty} \frac{\sqrt{\frac{n^5}{n^5} + \frac{n^4}{n^5}} + 5\sqrt{\frac{n}{n^5} + \frac{1}{n^5}}}{\sqrt{\frac{n^5}{n^5}} + 2\cdot\sqrt{\frac{n^3}{n^5}} + 6\cdot\sqrt{\frac{n}{n^5}}} = \lim_{n\to\infty} \frac{\sqrt{1+\underbrace{\tfrac{1}{n}}_{\to 0}} + 5\sqrt{\underbrace{\tfrac{1}{n^4}}_{\to 0} + \underbrace{\tfrac{1}{n^5}}_{\to 0}}}{1 + 2\cdot\underbrace{\sqrt{\tfrac{1}{n^2}}}_{\to 0} + 6\cdot\underbrace{\sqrt{\tfrac{1}{n^4}}}_{\to 0}}$$

$= \underline{\underline{1}}$ no conclusion here....

What else?

ROOT TEST (abs. convergence or divergence)

(O1) if $\lim \sqrt[n]{|a_n|} = L < 1$: abs. convergent

(O2) if $\lim \sqrt[n]{|a_n|} = L > 1$: divergent

if $\lim \sqrt[n]{|a_n|} = 1$: no conclusion.

Useful for power series!!

Ex1 $\sum_{n=1}^{\infty} \frac{1}{n^n} \cdot (-1)^n$ ← convergent (abs. convergent) intuition!! (no proof)

n-th power: root test!!

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{n^n}} = \lim_{n\to\infty} \frac{1}{n} = \underline{\underline{0}} < 1$$

abs. convergent by root test.

Ex2 $\sum_{n=1}^{\infty} \left(-\frac{2n+3}{3n+2}\right)^n$ ← root test

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \cancel{\sqrt[n]{\left(\frac{2n+3}{3n+2}\right)^{n}}} = \lim_{n\to\infty} \frac{2n+3}{3n+2}$$ ← (divide by n)

$$= \lim_{n\to\infty} \frac{\frac{1}{n}(2n+3)}{\frac{1}{n}(3n+2)} = \lim_{n\to\infty} \frac{2+\underbrace{\frac{3}{n}}_{\to 0}}{3+\underbrace{\frac{2}{n}}_{\to 0}} = \frac{2}{3} < 1$$

↑ abs. convergent by root test!!

How do we approach any series?

(Ex10 p737) $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{n}{\sqrt{n^3+2}}$

(1) alternating? or not?

- alt. series test
- ratio test for abs. conv.
- root test for abs convergence.

- comparison test
- integral test
- limit comparison test

But: checking for abs. convergence means dropping sign.

i.e. we make an alternating series into a normal series with non-neg. coefficients.

So: can use ratio & root test to check series $\sum a_n$, $a_n \geq 0$, for convergence.

(2) do we want to check for absolute convergence or just conditional convergence?

2 tests + all convergence test for non-neg. series.

back to examples - ratio test inconclusive

- alt. series test says it converges
- comparison test for $\sum |a_n|$ with $\sum \frac{1}{n^{\frac{1}{2}}}$

$\rightarrow$ abs. convergent.