Series §11.2

Review: sequences eg. $\{a_n\} = \{2^n\} = \{2, 4, 8, 16, \ldots\}$

properties of sequences:

- bounded from above: all $a_n \leq M$, $M \in \mathbb{R}$.
  from below: all $a_n \geq D$, $D \in \mathbb{R}$
- unbounded: not bounded
- convergent sequence:
  $\lim_{n \to \infty} a_n = L$, $L \in \mathbb{R}$

  if no limit exists, $\to$ divergent.

  (Can diverge, even if $\lim \neq \pm\infty$)
  eg: $\lim_{x \to \infty} \sin(x) \leftarrow$ limit does not exist.

review:

Series: special sequences, infinite sums

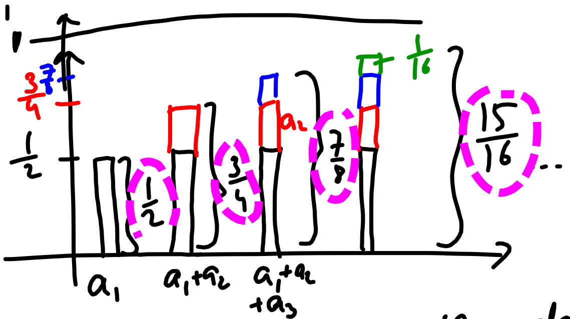
$$\sum_{n=1}^{\infty} a_n \leftarrow \text{"coefficients"}$$

Same properties as for sequences above.

eg: $\sum_{n=1}^{\infty} \frac{1}{2^n}$, here $a_n = \frac{1}{2^n}$, $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{4}$

$a_3 = \frac{1}{2^3} = \frac{1}{8}$

draw:



$a_4 = \frac{1}{2^4} = \frac{1}{16}$

observe: numerator is one less than denominator

have the form: $\sum_{n=1}^{k} \frac{1}{2^n}$ has form $\frac{a-1}{a}$ for $a = 2^k$

take limit: $\frac{2^k - 1}{2^k}$ as $k \to \infty$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{t \to \infty} \sum_{n=1}^{t} \frac{1}{2^n} = \lim_{t \to \infty} \frac{2^t - 1}{2^t} = 1$$

$\uparrow$ exercise

This was an example of a geometric series.

Geometric series: $\sum_{n=1}^{\infty} q^n$ (above we had $q=\frac{1}{2}$) $q \in \mathbb{R}$

can show: $\sum_{n=1}^{\infty} q^n = \frac{1}{1-q}$, for $q \neq 1$

turns out: converges for $|q|<1$, diverges for $|q| \geq 1$.

Partial sums: $S_n = \sum_{i=1}^{n} a_i$

n'th partial sum

i: running index variable can be any letter (better not a, or s)

If the sequence $\{S_n\}$ is convergent, then the series $\sum_{i=1}^{n} a_n$ is convergent. If $\{S_n\}$ diverges, then $\sum_{i=1}^{n} a_i$ diverges too.

More general for geometric series: $\boxed{\sum_{i=1}^{n} a q^i = \frac{a(1-q^{n+1})}{1-q}}$

now: $S_n = \frac{a(1-q^{n+1})}{1-q}$ if $S_n = \sum_{i=1}^{n} q^i$

a is any constant

limit as before:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-q^{n+1})}{1-q} =$$

limit rule for constant factors

$$= a \cdot \lim_{n\to\infty} \frac{1-q^{n+1}}{1-q} = a \cdot \frac{1-0}{1-q}$$

$q^{n+1} \to 0$ as $n \to \infty$

Case I: assume $|q|<1$

aside: $\lim_{n\to\infty} q^n = 0$ ($|q|<1$)

$\left(\frac{1}{2}\right)^5 = \frac{1}{32} < \frac{1}{4}$

"small things (<1) raised to high powers become tiny"

$$= \frac{a}{1-q}$$

Case II: $|q| \geq 1$, then $\lim_{n\to\infty} q^n = \begin{cases} \infty & \text{if } q>0 \\ \text{does not exist} & \text{if } q<0. \end{cases}$ ($|q| \geq 1$)

Here: $\lim_{n\to\infty} S_n$ is either ∞ or does not exist, so then $\sum_{i=1}^{\infty} a_i$ diverges.

Theorem: $\sum_{n=1}^{\infty} q^n$ converges if and only if $|q|<1$.
Diverges if $|q| \geq 1$.

Example: Does $\sum_{n=1}^{\infty} \frac{5}{3^n}$ converge?

note: $\sum_{n=1}^{\infty} \frac{5}{3^n} = 5 \cdot \sum_{n=1}^{\infty} \frac{1}{3^n}$

5 is indep of n

geometric series with $q = \frac{1}{3}$

use Thm: $\left|\frac{1}{3}\right| = \frac{1}{3} < 1$.

So Thm says, $\sum_{n=1}^{\infty} \frac{1}{3^n}$ converges.

So $\sum_{n=1}^{\infty} \frac{1}{3^n} = K$, $K \in \mathbb{R}$

formula for geom. series

(here: $K = \frac{1}{1-q} = \frac{1}{1-\frac{1}{3}} = \frac{1}{\frac{3-1}{3}} = \frac{3 \cdot 1}{3-1} = \frac{3}{2}$)

In particular,

$5 \cdot \sum_{n=1}^{\infty} \frac{1}{3^n} = 5 \cdot \frac{3}{2} = \frac{15}{2}$ is a finite number, so the series converges.

Often we just care if a series converges or diverges, not about the value.

$\boxed{\text{Thm: } \sum_{n=1}^{\infty} a_n \text{ can only converge, if } \lim_{n\to\infty} a_n = 0.}$

$n \cdot 1$ as $n \to \infty$

$\lim_{n\to\infty} n = \infty$

This is a necessary condition, but it alone is not enough to guarentee convergence!

eg: Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

but $\lim_{n\to\infty} \frac{1}{n} = 0$.

Thm If $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ both converge, so do

(i) $\sum_{n=1}^{\infty} c \cdot a_n = c \cdot \sum_{n=1}^{\infty} a_n$ ($c \in \mathbb{R}$)

(ii) $\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{i=1}^{\infty} a_i + \sum_{i=1}^{\infty} b_i$

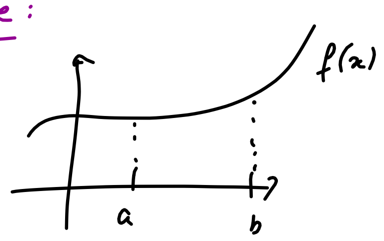
(iii) $\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

(compare to textbook)

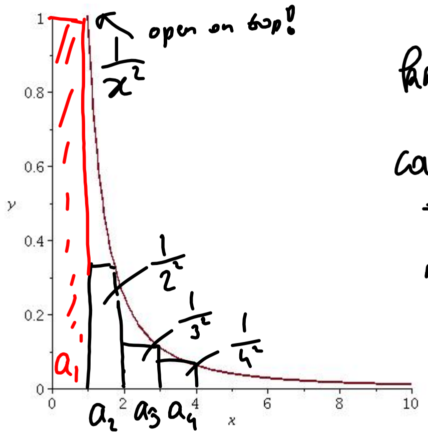
Integral test for convergence:

$$\int_a^b f(x)\,dx = $$

↑ area under the graph



Use this for series: ex: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ use $f(x) = \frac{1}{x^2}$



Know that $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges

can deduce that

$\sum_{n=2}^{\infty} a_n$ converges too

↑ 2 not 1

because sum of bars less than area under $f(x)$ from 1 to ∞.

$$\sum_{n=1}^{\infty} a_n = \underset{\uparrow\, a_1}{1} + \sum_{n=2}^{\infty} a_n = 1 + \underbrace{\sum_{n=2}^{\infty} \frac{1}{n^2}}_{< \int_1^{\infty} \frac{1}{x^2}dx \text{ converges}}$$

So $1 + \sum_{n=2}^{\infty} \frac{1}{n^2}$ is finite as well.

Thm (Integral test) (p 716) f is continuous, positive and decreasing on $[1,\infty)$ and $a_n = f(n)$

(i) If $\int_1^{\infty} f(x)\,dx$ is converging, then $\sum_{n=1}^{\infty} a_n$ converges

(ii) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ diverges!

Ex: $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$, $f(x) = \frac{1}{x^2+1}$

So: check → $\int_1^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{1+x^2}\,dx =$

$$\lim_{t\to\infty} \left[\arctan x\right]_1^t = \lim_{t\to\infty} (\arctan t - \underbrace{\arctan 1}_{\text{cole: } \frac{\pi}{4}})$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

Careful: $\frac{\pi}{4}$ is the value of the integral, not the value of the series!

But: Series converges!