Midterm Exam I ENGR 213
Applied Ordinary Differential Equations

Winter 2016 **February 18, 2016** **Version B** **Time allowed: 1h 15min**

Name **ID**

Note: This is a closed-book examination. Only calculators approved by department are allowed. Please fill your name and student ID in the space provided above, and return the question paper with your examination booklet(s).

[10 points] **Problem 1.**
Solve the following linear first-order differential equation by using the method of integrating factor

$$x\frac{dy}{dx} + (x+1)y = 3x^2e^{-x}$$

Write the solution in explicit form.

Solution.
Dividing by x we get the linear equation in standard form

$$\frac{dy}{dx} + \left(1 + \frac{1}{x}\right)y = 3xe^{-x}.$$

Multiplying the differential equation by integrating factor

$$e^{\int\left(1+\frac{1}{x}\right)dx} = e^{x+\ln x} = xe^x$$

we arrive at the following equation

$$xe^x\frac{dy}{dx} + (x+1)e^xy = 3x^2.$$

Then

$$\frac{d}{dx}\left(xe^xy\right) = 3x^2.$$

By integrating both parts with respect to x, we get

$$xe^x y(x) = \int 3x^2 dx = x^3 + C.$$

Hence,

$$y(x) = x^2 e^{-x} + \frac{C}{xe^x}.$$

[10 points] **Problem 2.**

Determine which of the following differential equations is exact and solve it

(a) $$(2 + \cos^2(xy))dx + (2y - \sin^2(xy))dy = 0$$

(b) $$3x^2(1 + \ln y)dx + (\frac{x^3}{y} - 2y)dy = 0$$

Solution.

(a) Since

$$\frac{\partial}{\partial y}(2 + \cos^2(xy)) = -2x\cos(xy)\sin(xy)$$

and

$$\frac{\partial}{\partial x}(2y - \sin^2(xy)) = -2y\sin(xy)\cos(xy),$$

i.e. they are not equal, this differential equation is not exact, and the differential expression in the left had side is not an exact differential.

(b) Since

$$\frac{\partial}{\partial y}(3x^2(1 + \ln y)) = \frac{3x^2}{y} \text{ and } \frac{\partial}{\partial x}(\frac{x^3}{y} - 2y) = \frac{3x^2}{y}$$

then it is exact differential equation, and the differential expression in the left hand side is an exact differential of the solution $F(x, y) = C$. Hence,

$$\frac{\partial F}{\partial x} = 3x^2(1 + \ln y) \text{ and } \frac{\partial F}{\partial y} = \frac{x^3}{y} - 2y.$$

Therefore,

$$F(x,y) = \int 3x^2(1+\ln y)dx = x^3(1+\ln y) + g(y).$$

From

$$\frac{\partial F}{\partial y} = \frac{x^3}{y} + g'(y) = \frac{x^3}{y} - 2y$$

we get

$$g'(y) = -2y \text{ and } g(y) = \int (-2y)dy = -y^2.$$

Thus

$$F(x,y) = x^3(1+\ln y) - y^2 = C.$$

[10 points] **Problem 3.**
Perform the proper substitution to solve the differential equation by the method of separation of variables

$$\frac{dy}{dx} = \sqrt{4x+2y-1} - 2$$

Solution.
By the substitution $z = 4x+2y-1$, and from $\frac{dz}{dx} = 4+2\frac{dy}{dx}$ we get the differential equation for $z(x)$:

$$\frac{1}{2}\frac{dz}{dx} - 2 = \sqrt{z} - 2$$

Then

$$\frac{dz}{\sqrt{z}} = 2dx.$$

Therefore,

$$2\sqrt{z} = 2x + C \quad \textit{and} \quad z(x) = (x+C)^2.$$

Hence

$$y(x) = \frac{1}{2}z - 2x + \frac{1}{2} = \frac{1}{2}(x+C)^2 - 2x + \frac{1}{2}.$$

[10 points] **Problem 4.**
Solve the initial-value problem for the following Bernoulli equation

$$2xy\frac{dy}{dx} = 2x^3 + y^2, \quad y(-1) = 4.$$

Solution.
We rewrite the equation in the form

$$\frac{dy}{dx} - \frac{1}{2x}y = x^2 y^{-1}.$$

Then by substitution $z = y^{-(-1-1)} = y^2$, and from $\frac{dz}{dx} = 2y\frac{dy}{dx}$ we get

$$y\frac{dy}{dx} - \frac{1}{2x}y^2 = x^2$$

and

$$\frac{dz}{dx} - \frac{1}{x}z = 2x^2.$$

Multiplying the last equation by integrating $e^{-\int \frac{dx}{x}} = e^{-\ln x} = \frac{1}{x}$ we get

$$\frac{1}{x}\frac{dz}{dx} - \frac{1}{x^2}z = 2x$$

or

$$\frac{d}{dx}\left(\frac{z}{x}\right) = 2x$$

and

$$\frac{z}{x} = x^2 + C.$$

Hence

$$z = Cx + x^3.$$

By inverse substitution we get

$$y^2 = Cx + x^3.$$

From the initial conditions

$$4^2 = -C + (-1)^3$$

we find

$$C = -1 - 16 = -17.$$

Therefore

$$y^2(x) = x^3 - 17x.$$

It follows from the initial conditions that

$$y(x) = \sqrt{x^3 - 17x}.$$

[5 points] **Bonus question.**
Solve the following first-order differential equation

$$ydx - (4x^2y + x)dy = 0$$

Solution
We rewrite the equation in the form $ydx - xdy - 4x^2ydy = 0$ and dividing by x^2 we get

$$\frac{ydx - xdy}{x^2} - 4ydy = 0.$$

It follows from $\frac{ydx-xdy}{x^2} = d\left(\frac{y}{x}\right)$ and $4ydy = 2d(y^2)$ that by substitution

$$z = \frac{y}{x} \quad \text{and} \quad u = y^2$$

we get the following equation

$$d\left(\frac{y}{x}\right) - 2d(y^2) = 0$$

or

$$dz - 2du = 0.$$

Then

$$z = 2u + C \quad \text{and} \quad \frac{y}{x} = 2y^2 + C$$

Thus

$$x = \frac{y}{2y^2 + C}.$$