
DEPARTMENT OF COMPUTER SCIENCE & SOFTWARE ENGINEERING
 COMP232 MATHEMATICS FOR COMPUTER SCIENCE
 FALL 2018

Assignment 1. Solutions

1. For each of the following statements use a truth table to determine whether it is a tautology, a contradiction, or a contingency.

$$(a) \underbrace{(p \vee r)}_a \wedge \underbrace{(q \vee r)}_b \leftrightarrow \underbrace{((p \wedge q) \vee r)}_c$$

Solution: Tautology.

p	q	r	$\underbrace{p \vee r}_a$	$\underbrace{q \vee r}_b$	$a \wedge b$	$p \wedge q$	$\underbrace{(p \wedge q) \vee r}_c$	$(a \wedge b) \leftrightarrow c$
T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T	T
T	F	T	T	T	T	F	T	T
F	T	T	T	T	T	F	T	T
T	F	F	T	F	F	F	F	T
F	T	F	F	T	F	F	F	T
F	F	T	T	T	T	F	T	T
F	F	F	F	F	F	F	F	T

(b) $(p \oplus q) \wedge (p \oplus \neg q)$

Solution: Contradiction.

p	q	$\neg q$	$\underbrace{p \oplus q}_a$	$\underbrace{p \oplus \neg q}_b$	$a \wedge b$
T	T	F	F	T	F
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	F	T	F

$$(c) \quad (p \rightarrow (q \rightarrow r)) \leftrightarrow (p \rightarrow (q \wedge r))$$

Solution: Contingency.

p	q	r	a $q \rightarrow r$	b $q \wedge r$	c $p \rightarrow \mathbf{a}$	d $p \rightarrow \mathbf{b}$	$\mathbf{c} \leftrightarrow \mathbf{d}$
T	T	T	T	T	T	T	T
T	T	F	F	F	F	F	T
T	F	T	T	F	T	F	F
F	T	T	T	T	T	T	T
T	F	F	T	F	T	F	F
F	T	F	F	F	T	T	T
F	F	T	T	F	T	T	T
F	F	F	T	F	T	T	T

$$(d) \quad \underbrace{\underbrace{(p \wedge (\underbrace{-q \rightarrow \neg p}_{\mathbf{a}}))}_{\mathbf{b}}}_{\mathbf{b}} \rightarrow q$$

Solution: Tautology.

p	q	$\neg p$	$\neg q$	a $\neg q \rightarrow \neg p$	b $p \wedge \mathbf{a}$	$\mathbf{b} \rightarrow q$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

2. For each of the following logical equivalences state whether it is valid or invalid. If invalid then give a counterexample (*e.g.*, based on a truth table). If valid then give an algebraic proof using logical equivalences from Tables 6, 7, and 8 from Section 1.3 of textbook.

(a) $(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Solution: Invalid.

If $p = T$, $q = F$, and $r = F$ then the LHS is False, while the RHS is True.

(b) $(p \rightarrow q) \vee (p \rightarrow r) \equiv (p \vee q) \rightarrow r$

Solution: Invalid.

If $p = T$, $q = T$, and $r = F$ then the LHS is True, while the RHS is False.

(c) $\left(((p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)) \rightarrow r \right) \equiv T$

Solution: Valid.

$$\begin{aligned}
 & \left((p \vee q) \wedge ((p \rightarrow r) \wedge (q \rightarrow r)) \right) \rightarrow r && \text{Assumption} \\
 \equiv & \neg \left((p \vee q) \wedge ((p \rightarrow r) \wedge (q \rightarrow r)) \right) \vee r && \text{Law for conditional} \\
 \equiv & \left(\neg(p \vee q) \vee \neg((p \rightarrow r) \wedge (q \rightarrow r)) \right) \vee r && \text{De Morgan} \\
 \equiv & \left(\neg(p \vee q) \vee (\neg(p \rightarrow r) \vee \neg(q \rightarrow r)) \right) \vee r && \text{De Morgan} \\
 \equiv & \left(\neg(p \vee q) \vee ((p \wedge \neg r) \vee (q \wedge \neg r)) \right) \vee r && \text{Table 7, law 5 twice} \\
 \equiv & \left(\neg(p \vee q) \vee ((p \vee q) \wedge \neg r) \right) \vee r && \text{Distributivity} \\
 \equiv & \left((\neg(p \vee q) \vee (p \vee q)) \wedge (\neg(p \vee q) \vee \neg r) \right) \vee r && \text{Distributivity} \\
 \equiv & \left((T) \wedge (\neg(p \vee q) \vee \neg r) \right) \vee r && \text{Excluded middle} \\
 \equiv & \left(\neg(p \vee q) \vee \neg r \right) \vee r && \text{Identity} \\
 \equiv & \neg(p \vee q) \vee (\neg r \vee r) && \text{Associativity} \\
 \equiv & \neg(p \vee q) \vee T && \text{Excluded middle} \\
 \equiv & T && \text{Domination}
 \end{aligned}$$

$$(d) \left(((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r) \right) \equiv T$$

Solution: Valid.

We shall instead prove $\neg\left(((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r) \right) \equiv F$.

The claim then follows since $\neg A \equiv F$ if and only if $A \equiv T$.

$$\begin{aligned}
& \neg\left(((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r) \right) \\
\equiv & \quad \left((p \rightarrow q) \wedge (q \rightarrow r) \right) \wedge \neg(p \rightarrow r) && \text{Table 7, law 5} \\
\equiv & \quad \left((p \rightarrow q) \wedge (q \rightarrow r) \right) \wedge (p \wedge \neg r) && \text{Table 7, law 5} \\
\equiv & \quad \left((p \rightarrow q) \wedge (q \rightarrow r) \right) \wedge (\neg r \wedge p) && \text{commutativity} \\
\equiv & \quad \left(((p \rightarrow q) \wedge (q \rightarrow r)) \wedge \neg r \right) \wedge p && \text{associativity} \\
\equiv & \quad \left((p \rightarrow q) \wedge ((q \rightarrow r) \wedge \neg r) \right) \wedge p && \text{associativity} \\
\equiv & \quad \left((p \rightarrow q) \wedge ((\neg q \vee r) \wedge \neg r) \right) \wedge p && \text{law for conditional} \\
\equiv & \quad \left((p \rightarrow q) \wedge ((\neg q \wedge \neg r) \vee (r \wedge \neg r)) \right) \wedge p && \text{distributivity} \\
\equiv & \quad \left((p \rightarrow q) \wedge ((\neg q \wedge \neg r) \vee F) \right) \wedge p && \text{excluded middle} \\
\equiv & \quad \left((p \rightarrow q) \wedge (\neg q \wedge \neg r) \right) \wedge p && \text{identity} \\
\equiv & \quad \left((\neg p \vee q) \wedge (\neg q \wedge \neg r) \right) \wedge p && \text{law for conditional} \\
\equiv & \quad \left(((\neg p \vee q) \wedge \neg q) \wedge \neg r \right) \wedge p && \text{associativity} \\
\equiv & \quad \left(((\neg p \wedge \neg q) \vee (q \wedge \neg q)) \wedge \neg r \right) \wedge p && \text{distributivity} \\
\equiv & \quad \left(((\neg p \wedge \neg q) \vee F) \wedge \neg r \right) \wedge p && \text{excluded middle} \\
\equiv & \quad \left((\neg p \wedge \neg q) \wedge \neg r \right) \wedge p && \text{identity} \\
\equiv & \quad p \wedge \left((\neg p \wedge \neg q) \wedge \neg r \right) && \text{commutativity} \\
\equiv & \quad p \wedge \left(\neg p \wedge (\neg q \wedge \neg r) \right) && \text{associativity} \\
\equiv & \quad \left(p \wedge \neg p \right) \wedge (\neg q \wedge \neg r) && \text{associativity} \\
\equiv & \quad F \wedge (\neg q \wedge \neg r) && \text{excluded middle} \\
\equiv & \quad F && \text{domination}
\end{aligned}$$

3. Which of the following conditions is *necessary* for the natural number n to be divisible by 6. The natural numbers are $\mathbb{N} = \{0, 1, 2, \dots\}$.
- (a) n is divisible by 3.
 - (b) n is divisible by 9.
 - (c) n is divisible by 12.
 - (d) $n = 24$
 - (e) n^2 is divisible by 3.
 - (f) n is even and divisible by 3.

Solution:

Necessary means *If n is divisible by 6, then condition*. Conditions (a), (e), and (f) are necessary. We have

- (a) If n is divisible by 6, then n is divisible by 3.
- (e) If n is divisible by 6, then n^2 is divisible by 3.
- (f) If n is divisible by 6, then n is even and divisible by 3.

Sufficient means *If condition, then n is divisible by 6*. Conditions (c), (d), and (f) are sufficient. We have

- (c) If n is divisible by 12, then n is divisible by 6.
- (d) If $n = 24$, then n is divisible by 6.
- (f) If n is even and divisible by 3, then n is divisible by 6.

Note that (f) is both necessary and sufficient, meaning that *n is divisible by 6 if and only if (iff) n is even and divisible by 3*

4. A set of propositions is *consistent* if there is an assignment of truth values to each of the variables in the propositions that makes each proposition true. Is the following set of propositions consistent?
- (a) *If the file system is not locked, then new messages will be queued.*
 - (b) *If the file system is not locked, then the system is functioning normally, and conversely.*
 - (c) *If new messages are not queued, then they will be sent to the message buffer.*
 - (d) *If the file system is not locked, then new messages will be sent to the message buffer.*
 - (e) *New messages will not be sent to the message buffer.*

Solution:

Let us define the following propositions:

$FL =_{\text{def}}$ *The file system is locked.*

$NQ =_{\text{def}}$ *New messages will be queued.*

$FN =_{\text{def}}$ *The system is functioning normally.*

$NB =_{\text{def}}$ *New messages will be sent to the message buffer.*

We can now formalize propositions (a) – (e):

(a) $\neg FL \rightarrow NQ$

(b) $\neg FL \leftrightarrow FN$

(c) $\neg NQ \rightarrow NB$

(d) $\neg FL \rightarrow NB$

(e) $\neg NB$

The set (a) – (e) of propositions (the conjunction of the proposition in the set) is indeed satisfiable. A satisfying truth assignment is

$$FL = \text{True}, NQ = \text{True}, FN = \text{False}, NB = \text{False}$$

5. Suppose the domain of the propositional function $P(x, y)$ consists of pairs x and y , where $x = 1, 2$, or 3 , and $y = 1, 2$, or 3 . Write out the propositions below using disjunctions and conjunctions only.

(a) $\exists x P(x, 3)$

Solution: $P(1, 3) \vee P(2, 3) \vee P(3, 3)$

(b) $\forall y \neg P(2, y)$

Solution: $\neg P(2, 1) \wedge \neg P(2, 2) \wedge \neg P(2, 3)$

(c) $\forall x \exists y P(x, y)$

Solution:

$$\begin{aligned} & \forall x \exists y P(x, y) \\ \equiv & \left(\exists y P(1, y) \right) \wedge \left(\exists y P(2, y) \right) \wedge \left(\exists y P(3, y) \right) \\ \equiv & \left(P(1, 1) \vee P(1, 2) \vee P(1, 3) \right) \\ & \wedge \left(P(2, 1) \vee P(2, 2) \vee P(2, 3) \right) \\ & \wedge \left(P(3, 1) \vee P(3, 2) \vee P(3, 3) \right) \end{aligned}$$

(d) $\exists x \forall y \neg P(x, y)$

Solution:

$$\begin{aligned} & \exists x \forall y \neg P(x, y) \\ \equiv & \left(\forall y \neg P(1, y) \right) \vee \left(\forall y \neg P(2, y) \right) \vee \left(\forall y \neg P(3, y) \right) \\ \equiv & \left(\neg P(1, 1) \wedge \neg P(1, 2) \wedge \neg P(1, 3) \right) \\ & \vee \left(\neg P(2, 1) \wedge \neg P(2, 2) \wedge \neg P(2, 3) \right) \\ & \vee \left(\neg P(3, 1) \wedge \neg P(3, 2) \wedge \neg P(3, 3) \right) \end{aligned}$$

6. Let the domain for x and x' be the set of all students in this class and the domain for y be the set of all countries in the world. Let $P(x, y)$ denote *student x has visited country y* and $Q(x, y)$ denote *student x has a friend in country y* . Express each of the following using logical operations and quantifiers, and the propositional functions $P(x, y)$ and $Q(x, y)$.

(a) *Carlos has visited Bulgaria.*

Solution: $P(\text{Carlos}, \text{Bulgaria})$

(b) *Every student in this class has visited the United States.*

Solution: $\forall x P(x, \text{UnitedStates})$

(c) *Every student in this class has visited some country in the world.*

Solution: $\forall x \exists y P(x, y)$

(d) *There is no country that every student in this class has visited.*

Solution: $\forall y \exists x \neg P(x, y)$. Equivalently $\neg(\exists y \forall x P(x, y))$

(e) *There are two students in this class, who between them, have a friend in every country in the world.*

Solution: $\exists x \exists x' (x \neq x' \wedge \forall y [Q(x, y) \vee Q(x', y)])$

(f) *Nobody in this class has visited a country in which they did not have a friend.*

Solution: $\forall x \forall y (P(x, y) \rightarrow Q(x, y))$

Equivalent solution: $\neg[\exists x \exists y (P(x, y) \wedge \neg Q(x, y))]$

7. For each part in the previous question, form the negation of the statement so that all negation symbols occur immediately in front of predicates. For example:

$$\neg(\forall x(P(x) \wedge Q(x))) \equiv \exists x(\neg((P(x) \wedge Q(x)))) \equiv \exists x((\neg P(x)) \vee (\neg Q(x)))$$

- (a) **Solution:** $\neg P(\text{Carlos}, \text{Bulgaria})$

Carlos has not visited Bulgaria

- (b) **Solution:** $\neg(\forall x P(x, \text{UnitedStates})) \equiv \exists x(\neg P(x, \text{UnitedStates}))$

There is a student in this class who has not visited the United States

- (c) **Solution:** $\neg(\forall x [\exists y P(x, y)]) \equiv \exists x \neg[\exists y P(x, y)] \equiv \exists x \forall y [\neg P(x, y)]$

There is a student in this class who has not visited any country

- (d) **Solution:**

$$\begin{aligned} \neg(\forall y [\exists x \neg P(x, y)]) &\equiv \\ \exists y \neg[\exists x \neg P(x, y)] &\equiv \\ \exists y \forall x \neg[\neg P(x, y)] &\equiv \\ \exists y \forall x [\neg\neg P(x, y)] &\equiv \\ \exists y \forall x P(x, y) & \end{aligned}$$

There is a country that every student in this class has visited

(e) **Solution:**

$$\begin{aligned} & \neg \left[\exists x \left(\exists y \left(x \neq y \wedge \forall z \left[Q(x, z) \vee Q(y, z) \right] \right) \right) \right] \equiv \\ & \forall x \left[\neg \left(\exists y \left(x \neq y \wedge \forall z \left[Q(x, z) \vee Q(y, z) \right] \right) \right) \right] \equiv \\ & \forall x \left[\forall y \neg \left(x \neq y \wedge \forall z \left[Q(x, z) \vee Q(y, z) \right] \right) \right] \equiv \\ & \forall x \left[\forall y \left(x = y \vee \neg \left(\forall z \left[Q(x, z) \vee Q(y, z) \right] \right) \right) \right] \equiv \\ & \forall x \left[\forall y \left(x = y \vee \exists z \left(\neg \left[Q(x, z) \vee Q(y, z) \right] \right) \right) \right] \equiv \\ & \forall x \left[\forall y \left(x = y \vee \exists z \left(\neg Q(x, z) \wedge \neg Q(y, z) \right) \right) \right] \end{aligned}$$

There are no two students in this class, who between them, have a friend in every country in the world.

(f) **Solution:**

$$\begin{aligned} & \neg \left[\forall x \forall y \left(P(x, y) \rightarrow Q(x, y) \right) \right] \equiv \\ & \exists x \left[\neg \left(\forall y \left(P(x, y) \rightarrow Q(x, y) \right) \right) \right] \equiv \\ & \exists x \left[\exists y \left(\neg \left(P(x, y) \rightarrow Q(x, y) \right) \right) \right] \equiv \\ & \exists x \left[\exists y \left(\neg \left(\neg P(x, y) \vee Q(x, y) \right) \right) \right] \equiv \\ & \exists x \left[\exists y \left(\neg \neg P(x, y) \wedge \neg Q(x, y) \right) \right] \equiv \\ & \exists x \left[\exists y \left(P(x, y) \wedge \neg Q(x, y) \right) \right] \end{aligned}$$

Somebody in this class has visited a country in which he/she doesn't have a friend.

8. Negate the following statements and transform the negation so that negation symbols immediately precede predicates. (See example in Question 7.)

$$(a) \exists x \exists y (P(x, y)) \vee \forall x \forall y (Q(x, y))$$

Solution:

$$\begin{aligned} & \neg [\exists x \exists y (P(x, y)) \vee \forall x \forall y (Q(x, y))] \\ \equiv & \neg [\exists x \exists y (P(x, y))] \wedge \neg [\forall x \forall y (Q(x, y))] \\ \equiv & \forall x \forall y (\neg P(x, y)) \wedge \exists x \exists y (\neg Q(x, y)) \end{aligned}$$

$$(b) \forall x \forall y (Q(x, y) \leftrightarrow Q(y, x))$$

Solution:

$$\begin{aligned} & \neg [\forall x \forall y (Q(x, y) \leftrightarrow Q(y, x))] \\ \equiv & \exists x \exists y [\neg (Q(x, y) \leftrightarrow Q(y, x))] \\ \equiv & \exists x \exists y [\neg [(Q(x, y) \rightarrow Q(y, x)) \wedge (Q(y, x) \rightarrow Q(x, y))]] \\ \equiv & \exists x \exists y [\neg (Q(x, y) \rightarrow Q(y, x)) \vee \neg (Q(y, x) \rightarrow Q(x, y))] \\ \equiv & \exists x \exists y [\neg (\neg Q(x, y) \vee Q(y, x)) \vee \neg (\neg Q(y, x) \vee Q(x, y))] \\ \equiv & \exists x \exists y [(\neg \neg Q(x, y) \wedge \neg Q(y, x)) \vee (\neg \neg Q(y, x) \wedge \neg Q(x, y))] \\ \equiv & \exists x \exists y [(Q(x, y) \wedge \neg Q(y, x)) \vee (Q(y, x) \wedge \neg Q(x, y))] \\ \equiv & \exists x \exists y [Q(x, y) \oplus Q(y, x)] \end{aligned}$$

$$(c) \forall y \exists x \exists z (T(x, y, z) \wedge Q(x, y))$$

Solution:

$$\begin{aligned} & \neg [\forall y \exists x \exists z (T(x, y, z) \wedge Q(x, y))] \\ \equiv & \exists y \forall x \forall z \neg (T(x, y, z) \wedge Q(x, y)) \\ \equiv & \exists y \forall x \forall z (\neg T(x, y, z) \vee \neg Q(x, y)) \end{aligned}$$