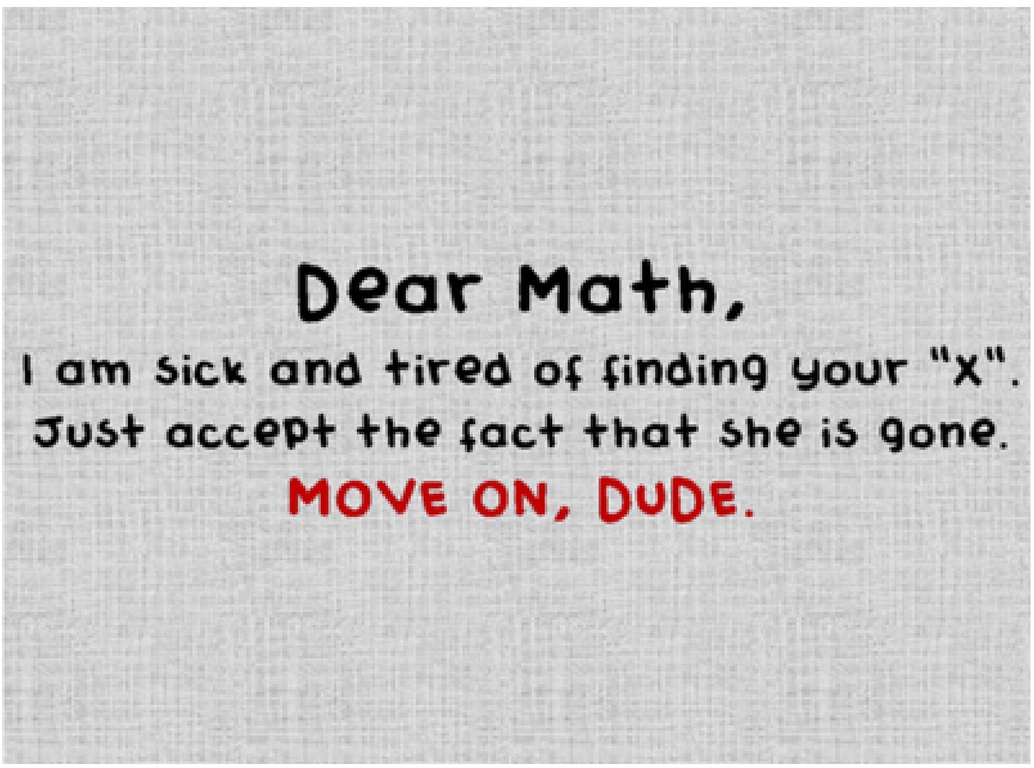
The Derivative

MAT 1300 C

Winter 2017



1 Definition of the Derivative

Let $f(x)$ be a function. Then we define **the derivative of** f **at** x, denoted $f'(x)$, as

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

if that limit exists, and if it does we say that f is **differentiable** at x.

Notation: We will denote this a few different ways. If $y = f(x)$, then all of the following denote the derivative:

$$f'(x),\ \frac{df}{dx},\ \frac{\mathrm{d}}{dx}(f(x)),\ y',\ \frac{dy}{dx}.$$

Examples:

1. Find the derivative of $f(x) = \frac{1}{2}x + 3$ at $x = 2$.

   We calculate:

$$\begin{aligned} f'(2) &= \lim_{\Delta x \to 0} \frac{f(2+\Delta x) - f(2)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\frac{1}{2}(2+\Delta x) + 3 - (\frac{1}{2}2 + 3)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\frac{1}{2}\Delta x}{\Delta x} = \frac{1}{2} \end{aligned}$$

   So the slope of the tangent line is $m = \frac{1}{2}$. This makes sense because $y = \frac{1}{2}x + 3$ is a line, and the tangent line is the line itself.

2. Find the slope for the tangent line for $g(x) = 1 - x^2$ at x (unspecified).

   We have

$$\begin{aligned} g'(x) &= \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{1 - (x+\Delta x)^2 - (1 - x^2)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{1 - (x^2 + 2x\Delta x + \Delta x^2) - (1 - x^2)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{-2x\Delta x - \Delta x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} (-2x - \Delta x) \\ &= -2x \end{aligned}$$

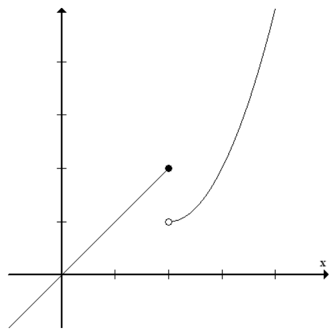
   So the slope of the tangent line at x is $-2x$.

3. Find the **equation** of the tangent line to $f(x) = \frac{1}{x}$ at the point $(1, 1)$.

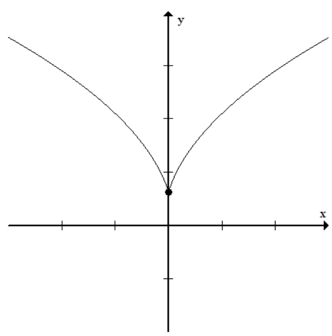
4. Find $f'(x)$, where $f(x) = \sqrt{x+1}$.

Notes:

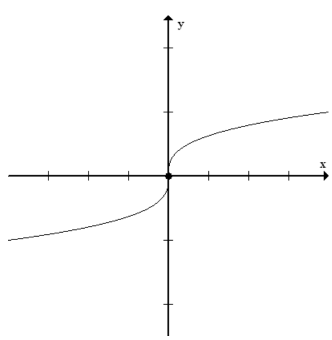
- of discontinuity,



- with corners,



- or with vertical tangents.



2 Basic Rules for Differentiation

Constant Rule: Let c be a constant function (a function whose value is c regardless of x). Then

$$\frac{d}{dx}[c] = 0.$$

Ex: Given $f(x) = -3.9$, find $f'(x)$.

Power Rule: Let n be any real number. Then

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

Ex: Given $h(z) = z^{0.6}$, find $\frac{dh}{dz}$.

Constant Multiple Rule: Let c be any constant and $f(x)$ any function. Then

$$\frac{d}{dx}[cf(x)] = cf'(x).$$

Ex: Given $g(y) = 7y^{-3}$, find $g'(y)$.

Sum Rule: Let $f(x)$ and $g(x)$ be any functions. Then

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x).$$

Examples:

1. Find the derivative of $f(t) = t^{2/3} + 3t - 3$.

   solution: We use a combination of the sum rule, constant multiple rule, and power rule:

$$\begin{aligned} \frac{\mathrm{d}}{dt}(t^{2/3} + 3t - 3) &= \frac{2}{3}t^{2/3-1} + 3t^{1-1} + 0 \\ &= \frac{2}{3}t^{-1/3} + 3 \end{aligned}$$

2. Find the derivative of $f(x) = \frac{4}{x^3}$.

3. Given $f(x) = 2x^{4/5} + 7$, find $f'(x)$.

3 The Product Rule

The Product Rule: Let $f(x)$ and $g(x)$ be differentiable functions. Then,

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

1. Differentiate $f(x) = (x^2 + 1)(2x + 5)$ (differentiate means "find the derivative of").

2. Given $h(x) = -3x^{5/2}(1 - 2x^4)$, use the product rule to find $f'(x)$.

4 The Quotient Rule

The Quotient Rule: Let $f(x)$ and $g(x)$ be differentiable functions where $g(x) \neq 0$. Then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

1. Find the derivative of $g(x) = \left(\frac{1}{x} + 3\right)\left(\frac{1}{x^2} - 4\right)$.

2. Find the derivative of $h(x) = \frac{\sqrt{x}}{x+1}$.

3. Let $f(x) = \frac{x^2-x-3}{x^2+1}$
   (a) Find $f'(x)$.
   (b) Find the equation of the tangent line to f at $(1, -\frac{3}{2})$.

5 Higher-Order Derivatives

Notice, for any function, $f(x)$, that $f'(x)$ is also a function of x. Hence, we can take the derivative of $f'(x)$ with respect to x. The **second order derivative**, or just **second derivative** of $f(x)$, if it exists, is

$$f''(x) = \frac{\mathrm{d}}{dx} f'(x)$$

The **third derivative** is

$$f'''(x) = \frac{\mathrm{d}}{dx} f''(x)$$

and so on. For higher order derivatives we use superscripts: $f^{(4)}$ = fourth derivative etc.

Example: Find $f''(x)$ given that

$$f(x) = (x^3 + 1)(2x^9 + 5).$$