MCG 4102/5108 Finite Element Analysis

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Contents

Introduction

The following notes cover the theoretical material presented in the course MCG 4102 / 5108 Finite Element Analysis, which is only introductory in nature. Although the finite element method is a mathematical approach to solve many different types of partial differential and integral equations, the presentation is deeply rooted in mechanical engineering, as it is the background of most students who take this course. The document was prepared with the help of several textbooks, in particular:

N-H Kim and BV Sankar, Introduction to Finite Element Analysis and Design, Wiley, 2009

J Fish and T Belytschko, A first course in finite elements, Wiley, 2007.

FL Stasa, Applied finite element analysis for engineers, Saunders College Publishing, 1985.

IH Shames and CL Dym, Energy and finite element methods in structural mechanics, Taylor & Francis, 1985.

DL Logan, A first course in the finite element method, 4th edition, Thomson, 2007.

JN Reddy, An introduction to the finite element method, 3rd edition, McGraw Hill, 2004.

MA Bhatti, Fundamental finite element analysis and applications with Mathematica and Matlab computations, Wiley, 2005.

Questions and comments from the students over the years have also tremendously contributed to shape the document into its current form. Nicolas Sagot prepared most of the figures. All cited and anonymous contributors are gratefully acknowledged, and future comments are welcome.

Finally, the reader should keep in mind that the present document is limited to linear elasticity. Nonlinear behaviours (related to non-infinitesimally small displacements and/or deformations, material nonlinearities, such as plasticity, or changing boundary conditions/contact issues) are not covered.

Chapter 1

Introduction to FEA: Trusses

Learning Outcomes:
By the end of reading and attending the lecture of this chapter, and solving the listed practice problems, the student should be able, for a simple truss problem*, to:*
-Discretize a physical system
-Write or use the element stiffness matrix in the global coordinate system
-Write the assemblage stiffness matrix and associated displacement and load vectors
-Explain and apply boundary conditions
-Solve system of equations for displacements or forces
-Determine internal forces as well as stress and strains associated with displacements and forces
Evidence of Learning:
A student has achieved the learning outcomes when he or she is able to correctly analyze a simple problem, following all the previous steps.
Evaluation Strategies:
Test problems in at-home assignments
Teaching and Learning Activities:
-At-home reading before class
-Lecturing
-In-class practice problems
-At-home reading/reviewing after class
-At-home practice problems

Trusses: structures composed of straight members connected at joints by pins (Fig. 1.1). Most or all members in a truss do not experience bending or torsional moments.Given: external forces, geometry, material properties.

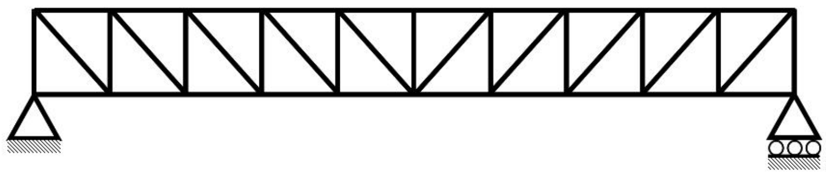

Figure 1.1: Truss bridge example.

Unknown: displacements at each joint; axial elongation, strain, stress, force for each member.

1.1 Discretization

Let us consider each member of a truss as an element (Fig. 1.2):

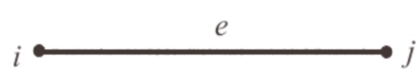

Figure 1.2: Bar element.

Element e in Fig. 1.2 has two nodes, i and j, at either end. Forces are transmitted from one element to the next at nodes. *Bar elements* can only take axial forces (*beam elements* also allow bending moments, as will be seen later). i and j (lowercase) are the local node numbers within the element. I and J (uppercase) denote global node numbers in the whole structure. Elements are also numbered (Fig. 1.3).

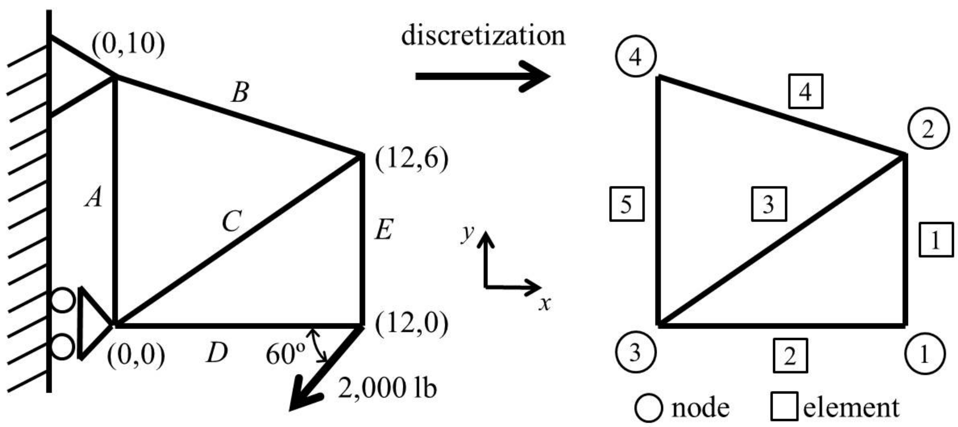

Figure 1.3: Example of discretization of a given truss.

Nodal coordinates:

Node number	x coordinate (in)	y coordinate (in)
1	12	0
2	12	6
3	0	0
4	0	10

Element data (connectivity table):

Element	Node i	Node j	Material flag
1	1	2	1 : 0.5-in (diameter) steel
2	3	1	2 : 0.4-in aluminium
3	3	2	1
4	4	2	2
5	3	4	1

1.2 Element stiffness relationship in local coordinates

Let (x', y') be the local coordinate system for Element e, with x' along the length of element from i to j, and y' perpendicular to x'.

The nodal displacements are noted as u'_i and v'_i in x' and y', respectively, at Node i (Fig. 1.4). The corresponding forces are noted F'_{xi} and F'_{yi}. Simi-

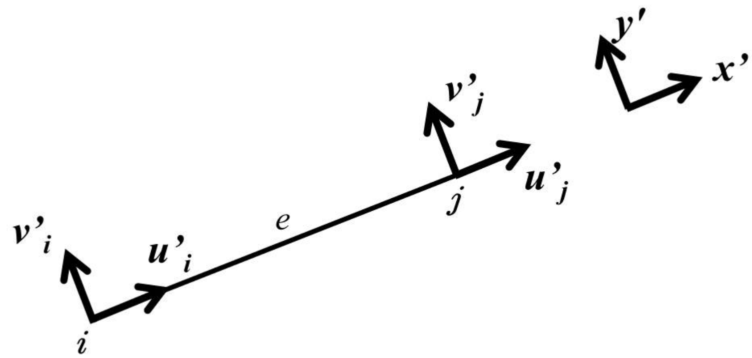

Figure 1.4: Bar element with degrees of freedom in local coordinate system.

larly, the nodal displacements are noted as u'_j and v'_j in x' and y', respectively, at Node j,with corresponding forces F'_{xj} and F'_{yj}. From elementary strength of materials, $\delta = \frac{PL}{AE}$, where δ is the axial elongation, L the member length, P the axial force, A the cross-sectional area and E the elastic modulus. It is assumed that the elastic range is not exceeded and that A is constant. In other words, writing $\delta = u'_j - u'_i$,

$$\begin{array}{l} F'_{xi} = \frac{AE}{L}(u'_i - u'_j) \\ F'_{xj} = \frac{AE}{L}(u'_j - u'_i) \end{array}, \text{ where } F'_{xi} = -F'_{xj} \text{ for equilibrium.}$$

Because bar elements do not withstand transverse forces, $F'_{yi} = F'_{yj} = 0$.

In matrix form,

$$\frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u'_i \\ v'_i \\ u'_j \\ v'_j \end{Bmatrix} = \begin{Bmatrix} F'_{xi} \\ F'_{yi} \\ F'_{xj} \\ F'_{yj} \end{Bmatrix}$$

With global nodal coordinates (x_i, y_i) and (x_j, y_j) for Nodes i and j, respectively, the element length can be computed as $L = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}$. The other two properties A and E can be specified for each element. Concisely,

$\left[K^{e'}\right]\left\{U^{e'}\right\} = \left\{F^{e'}\right\}$, where $\left[K^{e'}\right]$ represents the local stiffness matrix, $\left\{U^{e'}\right\}$ the local nodal displacement vector, and $\left\{F^{e'}\right\}$ the local nodal force vector for the element. "e" denotes element, " $'$ " denotes local coordinate system.

Example 1 *for Element 3:*

$$\left[K^{(3)'}\right] = \frac{A^{(3)}E^{(3)}}{L^{(3)}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Material flag set to 1: 0.50-in steel: $A^{(3)} = \frac{\pi}{4}(0.5)^2 = 0.196\,\text{in}^2$.
$E^{(3)} = 30 \times 10^6$ psi
$L^{(3)} = \sqrt{(0-12)^2 + (0-6)^2} = 13.42\,\text{in}$

Then, $\left[K^{(3)'}\right] = 10^3 \begin{bmatrix} 438 & 0 & -438 & 0 \\ 0 & 0 & 0 & 0 \\ -438 & 0 & 438 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ lbf/in

1.3 Transformation from local to global coordinates

In 2-D, the global (x, y) coordinate system shown in Fig. 1.5, position vector $\vec{r}$ to an arbitrary point P can be written as $\vec{r} = r_x \vec{i} + r_y \vec{j}$. In the rotated coordinate system attached to the bar, (x', y'), $\vec{r} = r_{x'} \vec{i'} + r_{y'} \vec{j'}$.

Reminder: Scalar product "." of any two vectors $\vec{u}$ and $\vec{v}$:
$\vec{u}.\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos(\vec{u},\vec{v})$. Therefore, in 2-D,

$$\begin{aligned} \vec{r}.\vec{i} &= r_x \vec{i}.\vec{i} + r_y \vec{j}.\vec{i} &&= r_{x'} \vec{i'}.\vec{i} + r_{y'} \vec{j'}.\vec{i} \\ &= r_x + 0 &&= r_{x'} \cos(x', x) + r_{y'} \cos(y', x) \\ \vec{r}.\vec{j} &= r_x \vec{i}.\vec{j} + r_y \vec{j}.\vec{j} &&= r_{x'} \vec{i'}.\vec{j} + r_{y'} \vec{j'}.\vec{j} \\ &= 0 + r_y &&= r_{x'} \cos(x', y) + r_{y'} \cos(y', y) \end{aligned}$$

In matrix form,

$$\left\{ \begin{matrix} r_x \\ r_y \end{matrix} \right\} = \begin{bmatrix} \cos(x', x) & \cos(y', x) \\ \cos(x', y) & \cos(y', y) \end{bmatrix} \left\{ \begin{matrix} r_{x'} \\ r_{y'} \end{matrix} \right\}$$

or $\{r_{2D}\} = [T_{2D}]\{r'_{2D}\}$, where

$[T_{2D}] = \begin{bmatrix} \cos(x', x) & \cos(y', x) \\ \cos(x', y) & \cos(y', y) \end{bmatrix}$. In other words,

$[T_{2D}] = \begin{bmatrix} \cos\theta & \cos(\theta + \frac{\pi}{2}) = -\sin\theta \\ \cos(\theta - \frac{\pi}{2}) = \sin\theta & \cos\theta \end{bmatrix}$ with θ as in Fig. 1.5.

More practically,

$[T_{2D}] = \begin{bmatrix} l & -m \\ m & l \end{bmatrix}$, with $l = \frac{x_j - x_i}{L}$ and $m = \frac{y_j - y_i}{L}$.

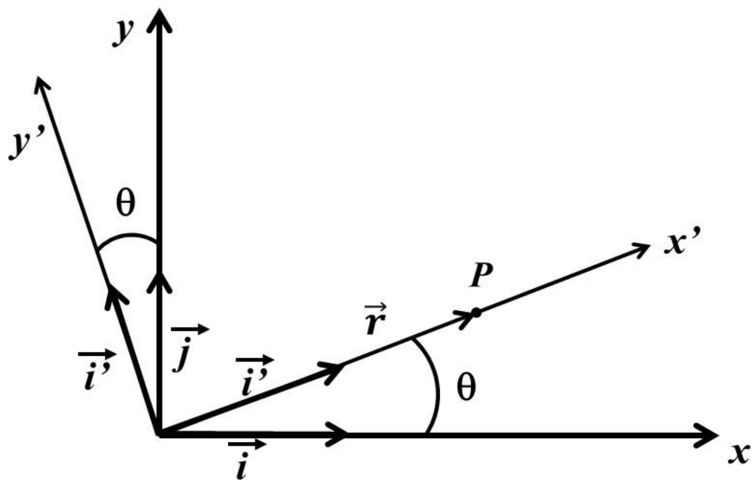

Figure 1.5: Local and global coordinate systems.

Note that computing $l = \frac{x_j - x_i}{L}$ and $m = \frac{y_j - y_i}{L}$ is a foolproof way of obtaining the direction cosines $\cos\theta_x$ and $\cos\theta_y$, respectively (Fig. 1.6), whereas calculating rotation angles first and taking their cosines and sines afterwards is associated with frequent errors in practice.

It can be shown that $[T_{2D}]$ is orthogonal, i.e. $[T_{2D}]^{-1} = [T_{2D}]^T$ .Therefore, $\{r'_{2D}\} = [T_{2D}]^T \{r_{2D}\}$.

Note that similarly in 3-D (if needed), position vector $\overrightarrow{r}$ to an arbitrary point P can be written as $\overrightarrow{r} = r_x \overrightarrow{i} + r_y \overrightarrow{j} + r_z \overrightarrow{k}$. In the rotated coordinate system, (x', y', z'), $\overrightarrow{r} = r_{x'} \overrightarrow{i'} + r_{y'} \overrightarrow{j'} + r_{z'} \overrightarrow{k'}$. From the definition of the scalar product,

$$\begin{aligned}
\overrightarrow{r}.\overrightarrow{i} &= r_x \overrightarrow{i}.\overrightarrow{i} + r_y \overrightarrow{j}.\overrightarrow{i} + r_z \overrightarrow{k}.\overrightarrow{i} & &= r_{x'} \overrightarrow{i'}.\overrightarrow{i} + r_{y'} \overrightarrow{j'}.\overrightarrow{i} + r_{z'} \overrightarrow{k'}.\overrightarrow{i} \\
&= r_x + 0 + 0 & &= r_{x'} \cos(x', x) + r_{y'} \cos(y', x) + r_{z'} \cos(z', x) \\
\overrightarrow{r}.\overrightarrow{j} &= r_x \overrightarrow{i}.\overrightarrow{j} + r_y \overrightarrow{j}.\overrightarrow{j} + r_z \overrightarrow{k}.\overrightarrow{j} & &= r_{x'} \overrightarrow{i'}.\overrightarrow{j} + r_{y'} \overrightarrow{j'}.\overrightarrow{j} + r_{z'} \overrightarrow{k'}.\overrightarrow{j} \\
&= 0 + r_y + 0 & &= r_{x'} \cos(x', y) + r_{y'} \cos(y', y) + r_{z'} \cos(z', y) \\
\overrightarrow{r}.\overrightarrow{j} &= r_x \overrightarrow{i}.\overrightarrow{k} + r_y \overrightarrow{j}.\overrightarrow{k} + r_z \overrightarrow{k}.\overrightarrow{k} & &= r_{x'} \overrightarrow{i'}.\overrightarrow{k} + r_{y'} \overrightarrow{j'}.\overrightarrow{k} + r_{z'} \overrightarrow{k'}.\overrightarrow{k} \\
&= 0 + 0 + r_z & &= r_{x'} \cos(x', z) + r_{y'} \cos(y', z) + r_{z'} \cos(z', z)
\end{aligned}$$

In matrix form,

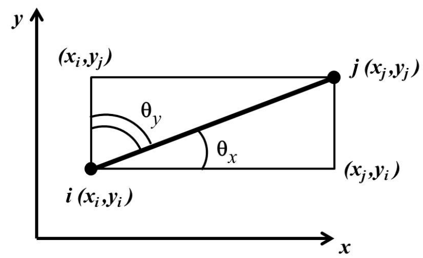

Figure 1.6: Direction cosines in 2D.

$$\left\{\begin{array}{c} r_x \\ r_y \\ r_z \end{array}\right\} = \left[\begin{array}{ccc} \cos(x',x) & \cos(y',x) & \cos(z',x) \\ \cos(x',y) & \cos(y',y) & \cos(z',y) \\ \cos(x',z) & \cos(y',z) & \cos(z',z) \end{array}\right] \left\{\begin{array}{c} r_{x'} \\ r_{y'} \\ r_{z'} \end{array}\right\}$$

or $\{r_{3D}\} = [T_{3D}]\{r'_{3D}\}$, where

$$[T_{3D}] = \left[\begin{array}{ccc} \cos(x',x) & \cos(y',x) & \cos(z',x) \\ \cos(x',y) & \cos(y',y) & \cos(z',y) \\ \cos(x',z) & \cos(y',z) & \cos(z',z) \end{array}\right].$$

It can be shown that $[T_{3D}]$ is orthogonal, i.e. $[T_{3D}]^{-1} = [T_{3D}]^T$ .Therefore, $\{r'_{3D}\} = [T_{3D}]^T \{r_{3D}\}$.

1.4 Global element stiffness relationship

The 2-D transformations of coordinates shown in Section 1.3 for the position vector also hold for displacement vectors $\left\{\begin{array}{c} u'_i \\ v'_i \end{array}\right\}$ at Node i, and $\left\{\begin{array}{c} u'_j \\ v'_j \end{array}\right\}$ at Node j.

In the following, the notation $[T_{2D}] = \left[\begin{array}{cc} l & -m \\ m & l \end{array}\right]$ is used with $l = \frac{x_j - x_i}{L}$ and $m = \frac{y_j - y_i}{L}$. We get

$$\left\{\begin{array}{c} u'_i \\ v'_i \\ u'_j \\ v'_j \end{array}\right\} = \left[\begin{array}{cccc} l & m & 0 & 0 \\ -m & l & 0 & 0 \\ 0 & 0 & l & m \\ 0 & 0 & -m & l \end{array}\right] \left\{\begin{array}{c} u_i \\ v_i \\ u_j \\ v_j \end{array}\right\}$$ or more concisely,

$$\left\{U^{e'}\right\} = [R]\left\{U^e\right\}, \text{where } [R] = \begin{bmatrix} T_{2D}^T & 0 \\ 0 & T_{2D}^T \end{bmatrix} \text{ and } \left\{U^e\right\} = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix}.$$

Similarly, $\left\{F^{e'}\right\} = [R]\left\{F^e\right\}$, with $\left\{F^e\right\} = \begin{Bmatrix} F_{xi} \\ F_{yi} \\ F_{xj} \\ F_{yj} \end{Bmatrix}$.

$\left[K^{e'}\right]\left\{U^{e'}\right\} = \left\{F^{e'}\right\}$ becomes $\left[K^{e'}\right][R]\left\{U^e\right\} = [R]\left\{F^e\right\}$, or
$[R]^T\left[K^{e'}\right][R]\left\{U^e\right\} = \left\{F^e\right\}$, since $[R]^T[R] = [R]^{-1}[R] = [I]$.

More simply, $[K^e]\left\{U^e\right\} = \left\{F^e\right\}$, where $[K^e] = [R]^T\left[K^{e'}\right][R]$ is the global element stiffness matrix

$$[K^e] = \frac{AE}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

where $L = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}$, and with $l = \frac{x_j - x_i}{L}$ and $m = \frac{y_j - y_i}{L}$.

Example 2 *for Element 3:*

$$l = \frac{x_2 - x_3}{L^{(3)}} = \frac{12-0}{13.42} = 0.8944$$
$$m = \frac{y_2 - y_3}{L^{(3)}} = \frac{6-0}{13.42} = 0.4472$$
$$\left[K^{(3)}\right] = 10^3\begin{bmatrix} 351 & 176 & -351 & -176 \\ 176 & 88 & -176 & -88 \\ -351 & -176 & 351 & 176 \\ -176 & -88 & 176 & 88 \end{bmatrix} \text{lbf/in}$$

1.5 Assemblage

The original structure is put back together from individual elements. This is done based on the compatibility of nodal displacements: the x- and y-displacements at one node must be identical (in the intended design) to those of the other nodes from other elements to be merged. Note that in the present case, each node has 2 degrees of freedom.

Example 3 *for the truss in Fig. 1.7, determine the assemblage stiffness matrix in terms of the 2×2 global stiffness submatrices* $\left[K_{I,J}^e\right]$.

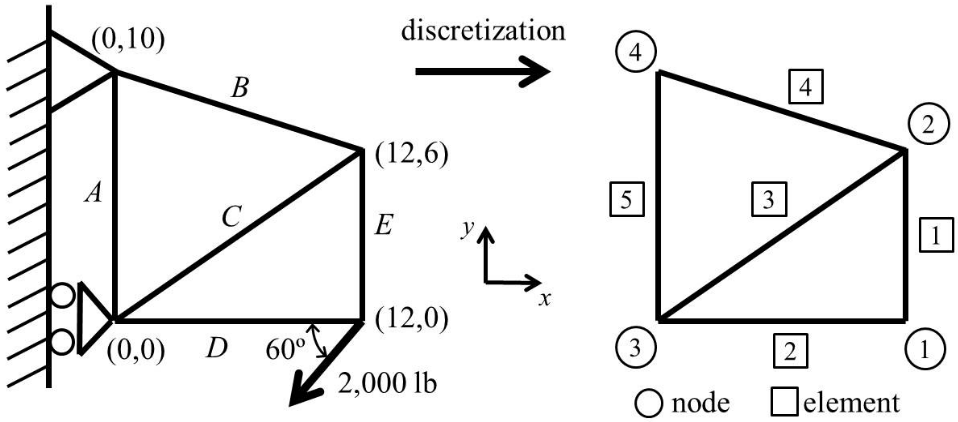

Figure 1.7: Figure 1.3 repeated here for convenience.

Procedure: let us first create a null assemblage stiffness matrix $[K^a]$ involving global node numbers 1 to 4.

$$[K^a] = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{cccc} 0_{2\times2} & 0_{2\times2} & 0_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} & 0_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} & 0_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} & 0_{2\times2} & 0_{2\times2} \end{array} \right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

From the connectivity table, Element 1 has global node numbers 1 and 2, so

$$\left[K^{(1)}\right] = \left[\begin{array}{cc} K_{1,1}^{(1)} & K_{1,2}^{(1)} \\ K_{2,1}^{(1)} & K_{2,2}^{(1)} \end{array} \right].$$

Element 2 has global node numbers 3 and 1, so

$$\left[K^{(2)}\right] = \left[\begin{array}{cc} K_{3,3}^{(2)} & K_{3,1}^{(2)} \\ K_{1,3}^{(2)} & K_{1,1}^{(2)} \end{array} \right].$$

Element 3 has global node numbers 3 and 2, so

$$\left[K^{(3)}\right] = \left[\begin{array}{cc} K_{3,3}^{(3)} & K_{3,2}^{(3)} \\ K_{2,3}^{(3)} & K_{2,2}^{(3)} \end{array} \right].$$

Element 4 has global node numbers 4 and 2, so

$$\left[K^{(4)}\right] = \left[\begin{array}{cc} K_{4,4}^{(4)} & K_{4,2}^{(4)} \\ K_{2,4}^{(4)} & K_{2,2}^{(4)} \end{array} \right].$$

Element 5 has global node numbers 3 and 4, so

$$\left[K^{(5)}\right] = \begin{bmatrix} K_{3,3}^{(5)} & K_{3,4}^{(5)} \\ K_{4,3}^{(5)} & K_{4,4}^{(5)} \end{bmatrix}.$$

Finally,

$$[K^a] = \begin{array}{c} \\ \end{array}\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}$$

$$[K^a] = \begin{bmatrix} K_{1,1}^{(1)} + K_{1,1}^{(2)} & K_{1,2}^{(1)} & K_{1,3}^{(2)} & 0_{2x2} \\ K_{2,1}^{(1)} & K_{2,2}^{(1)} + K_{2,2}^{(3)} + K_{2,2}^{(4)} & K_{2,3}^{(3)} & K_{2,4}^{(4)} \\ K_{3,1}^{(2)} & K_{3,2}^{(3)} & K_{3,3}^{(2)} + K_{3,3}^{(3)} + K_{3,3}^{(5)} & K_{3,4}^{(5)} \\ 0_{2x2} & K_{4,2}^{(4)} & K_{4,3}^{(5)} & K_{4,4}^{(4)} + K_{4,4}^{(5)} \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}$$

Note that each individual 2x2 element stiffness submatrix is symmetric, and $[K^a]$ is symmetric. Also note that, depending on the connectivity table, some submatrices may have to be transposed to fit in the assemblage matrix.

1.6 Application of loads

By design, the (known) external loads in a truss can only occur at the joints (nodes).

Example 4 *for the truss in Fig. 1.7, the x- and y- components of the applied load are* $F_x = -2,000 \cos 60^o = -1,000\,\text{lbf}$ *and* $F_y = -2,000 \sin 60^o = -1,732\,\text{lbf}$. *The load is applied to Node 1. There is an unknown reaction force in the x-direction at Node 3 (roller) and two unknown reaction forces in the x- and y-directions at Node 4 (pin).*

$$\{F^a\} = \left\{ \begin{array}{c} -1,000 \\ -1,732 \\ 0 \\ 0 \\ R_{x3} \\ 0 \\ R_{x4} \\ R_{y4} \end{array} \right\} \begin{array}{l} F_x \\ F_y \\ F_x \\ F_y \\ F_x \\ F_y \\ F_x \\ F_y \end{array} \begin{array}{l} \text{Node 1} \\ \\ \text{Node 2} \\ \\ \text{Node 3} \\ \\ \text{Node 4} \\ \\ \end{array}$$

1.7 Application of restraints on nodal displacements and solution

After assemblage, the system equation is of the form $[K^a]\{U^a\} = \{F^a\}$, where $\{U^a\}$ is the vector of nodal displacements.

$\{U^a\}^T = \{u_1 \quad v_1 \quad u_2 \quad v_2 \quad ... \quad u_N \quad v_N\}^T$, where N is the maximum number of nodes and u_I, v_I are the x- and y- displacements at global node number I. For now, no restraint on the nodal displacements have been considered, and the whole structure can fly into space!!! This is a rigid body motion, and $\{U^a\}$ cannot be determined because $[K^a]$ cannot be inverted (singular matrix, i.e. its determinant is zero). Restraints MUST be applied.

After the restraints corresponding to the boundary conditions are enforced ($u_3 = 0$, and $u_4 = v_4 = 0$ for the truss in Fig. 1.7), the system equation becomes $[K]\{U\} = \{F\}$, where $[K]$ is non-singular, therefore $[K]^{-1}$ exists, which implies a unique solution for $\{U\} = [K]^{-1}\{F\}$.

There are different methods for enforcing displacement restraints:

- **Method 1** (to be used when known displacements are not zero)

Consider the system
$$\begin{aligned} k_{11}u_1 + k_{12}u_2 + k_{13}u_3 &= f_1 \ , \\ k_{21}u_1 + k_{22}u_2 + k_{23}u_3 &= f_2 \\ k_{31}u_1 + k_{32}u_2 + k_{33}u_3 &= f_3 \end{aligned}$$
with $k_{ij} = k_{ji}$ and $i, j = 1...3$.

Let us impose $u_2 = u_{\text{known}}$, therefore the set of equations above is equivalent to e.g.
$$\begin{aligned} k_{11}u_1 + k_{12}u_2 + k_{13}u_3 &= f_1 \\ u_2 &= u_{\text{known}} \\ k_{31}u_1 + k_{32}u_2 + k_{33}u_3 &= f_3 \end{aligned}$$
The symmetry has been destroyed but can be restored by transposing terms involving u_2 to the right-hand side, and with u_{known} replacing u_2 :
$$\begin{bmatrix} k_{11} & 0 & k_{13} \\ 0 & 1 & 0 \\ k_{31} & 0 & k_{33} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1 - k_{12}u_{\text{known}} \\ u_{\text{known}} \\ f_3 - k_{32}u_{\text{known}} \end{Bmatrix}$$
. Note that f_2 disappears: a known displacement leads to an unknown reaction force.

- **Method 2** (interesting for programming of the finite element method)

Consider the same initial system as above. Let us select a large number β (6 to 12 orders of magnitude larger than the largest coefficient k_{ij}). β is added to k_{ii} if u_i is prescribed, and the right-hand side of the $i-$th equation is changed to β times the prescribed value. In matrix form, the system becomes

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & (k_{22}+\beta) & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ \beta \times u_{\text{known}} \\ f_3 \end{Bmatrix}.$$

Symmetry is preserved, and this method is very easy to implement in a computer program. For practical purposes, $u_2 = u_{\text{known}}$.

- **Method 3** (to be used when known displacements are zero)

When the boundary conditions are enforced, the initial system equation $[K^a]\{U^a\} = \{F^a\}$ is rewritten into $[K]\{U\} = \{F\}$ by simply removing the rows corresponding to the unknown reactions, and the columns corresponding to the known (zero) displacements.

1.8 Processing of results

Assemblage displacement vector

Determine $\{U^a\}$ from $\{U\}$ and the boundary conditions.

Reaction forces

Determine the unknown reaction forces (if needed) from $\{F^a\} = [K^a]\{U^a\}$.

Element resultants

Element resultants include axial elongation, strains, stresses and forces in each element, all of which need to be checked against failure criteria i.e. maximum allowable displacement, maximum allowable strain, yield stress and buckling (if compression is present). The element resultants are computed from the nodal displacements. For Element e, going back to the definition of the axial elongation, $\delta = u'_j - u'_i$, where $u'_j = lu_j + mv_j$ and $u'_i = lu_i + mv_i$.

At this stage, u_i, v_i, u_j, v_j and l, m are known therefore δ can be computed, and so can the axial strain $\varepsilon = \frac{\delta}{L}$. The material is assumed to be elastic, therefore, the axial (also called normal) stress $\sigma = E\varepsilon$ can be determined, and so can the axial force $F = \sigma A$. Note that for $\delta, \varepsilon, \sigma$ and F, positive values denote tension, while negative values denote compression. Make sure to check that these signs make physical sense.

Internal forces

Internal forces can be determined as described above, i.e. from the element strains. Another, sometimes more direct, method consists in writing the equilibrium of one element of interest and using the known displacements at its nodes. Of course, all methods must yield the same results!

Example 5 *determine the internal forces in Element 3 of the truss in Fig. 1.7, from*

$$\left[K^{(3)}\right]\left\{\begin{array}{c} u_3 \\ v_3 \\ u_2 \\ v_2 \end{array}\right\} = \left\{\begin{array}{c} F_{x3}^{(3)} \\ F_{y3}^{(3)} \\ F_{x2}^{(3)} \\ F_{y2}^{(3)} \end{array}\right\} \text{(in the global coordinate system)}$$

or from

$$\left[K^{(3)\prime}\right]\left\{\begin{array}{c} u_3' \\ v_3' \\ u_2' \\ v_2' \end{array}\right\} = \left\{\begin{array}{c} F_{x3}^{(3)\prime} \\ F_{y3}^{(3)\prime} \\ F_{x2}^{(3)\prime} \\ F_{y2}^{(3)\prime} \end{array}\right\} \text{(in the local coordinate system).}$$

1.9 Examples

Assignment 1.

For additional parctice problems, use the results in the statement of Assignment 1, and find yourself $\left[K^{(2)}\right]$, $\left[K^{(4)}\right]$ and $\left[K^{(5)}\right]$.

Chapter 2

Linear elasticity - Principle of virtual work

Learning Outcomes:
By the end of reading and attending the lecture of this chapter, the student should be able to:
-Explain and apply the concepts of stress and strain in linear elasticity
-Write equations of static equilibrium for an infinitesimally small component
-Explain and apply strain-displacement relations
–Explain and apply linear elastic constitutive relationships
-Explain and apply the principle of minimum potential energy
-Explain and apply the principle of virtual work
Evidence of Learning:
A student has achieved the learning outcomes when he or she is able to correctly analyze a simple problem combining several or all of the previous items.
Evaluation Strategies:
Test problems in at-home assignments
Teaching and Learning Activities:
-At-home reading before class
-Lecturing
-In-class practice problems
-At-home reading/reviewing after class
-At-home practice problems

This is a short review of linear elasticity in statics and equilibrium principles, including the principle of virtual work for use in finite element formulation.

2.1 Stress at a point

Consider a deformable body in static equilibrium, loaded as shown in Fig. 2.1:

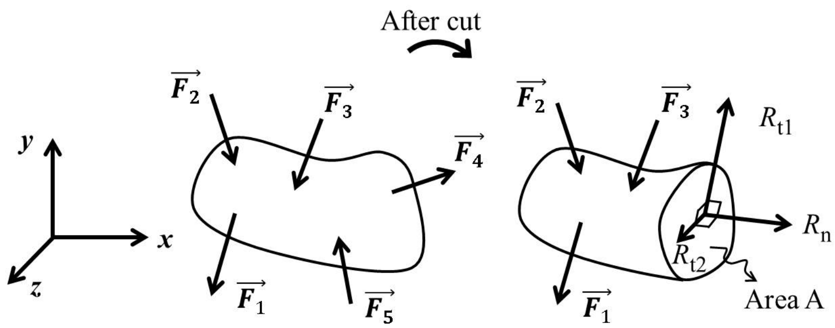

Figure 2.1: Static equilibrium of a body.

The external forces $\vec{F_1}$, $\vec{F_2}$, ... are transmitted through the deformable body in a complex manner. When the body is cut along some plane, a force $\vec{R}$ is required to maintain static equilibrium. $\vec{R}$ has a normal component R_n and two tangential orthogonal components R_{t1} and R_{t2}. Consider small area ΔA instead of A. Then, ΔR_n, ΔR_{t1}, ΔR_{t2} act on ΔA. The normal stress σ_n is defined as $\sigma_n = \lim\limits_{\Delta A \to 0} \frac{\Delta R_n}{\Delta A}$ and the two shear (tangential) stresses as $\sigma_{t1} = \lim\limits_{\Delta A \to 0} \frac{\Delta R_{t1}}{\Delta A}$ and $\sigma_{t2} = \lim\limits_{\Delta A \to 0} \frac{\Delta R_{t2}}{\Delta A}$. For an infinitesimal volume element of a body positioned at a point in a global (x, y, z) coordinate system (Fig. 2.2):

(1^{st} subscript: facet normal; 2^{nd} subscript: direction)

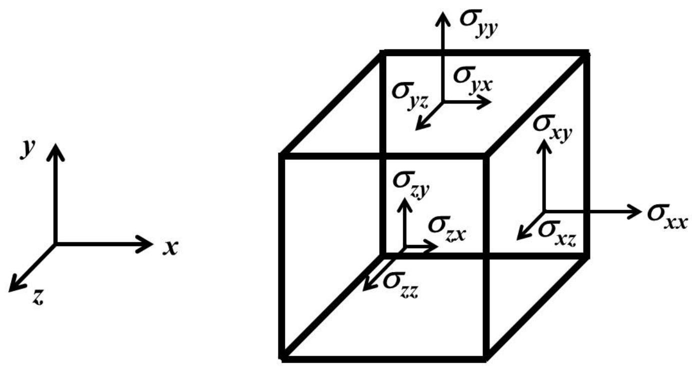

Figure 2.2: Some of the stresses acting on an infinitesimal volume element.

2.2 Equations of static equilibrium

Consider an infinitesimal 2-D volume element of thickness t (Fig. 2.3). It is assumed that the normal and shear stresses vary from point to point in the body in some continuous manner. Therefore, a first order Taylor series expansion is used.

b_x, b_y : components of body force per unit volume.

Equilibrium in x-direction:

$(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}dx)dy - \sigma_{xx}dy + (\sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y}dy)dx - \sigma_{yx}dx + b_x dxdy = 0$

Equilibrium in y-direction:

$(\sigma_{yy} + \frac{\partial \sigma_{yy}}{\partial y}dy)dx - \sigma_{yy}dx + (\sigma_{xy} + \frac{\partial \sigma_{xy}}{\partial x}dx)dy - \sigma_{xy}dy + b_y dxdy = 0$

Finally, in 2-D, $\begin{matrix} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + b_x = 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + b_y = 0 \end{matrix}$ and $\sigma_{xy} = \sigma_{yx}$ for moment equilibrium.

Similarly in 3-D, $\begin{matrix} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + b_x = 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + b_y = 0 \\ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z = 0 \end{matrix}$ and $\begin{matrix} \sigma_{xy} = \sigma_{yx} \\ \sigma_{xz} = \sigma_{zx} \\ \sigma_{zy} = \sigma_{yz} \end{matrix}$ for moment equilibrium.

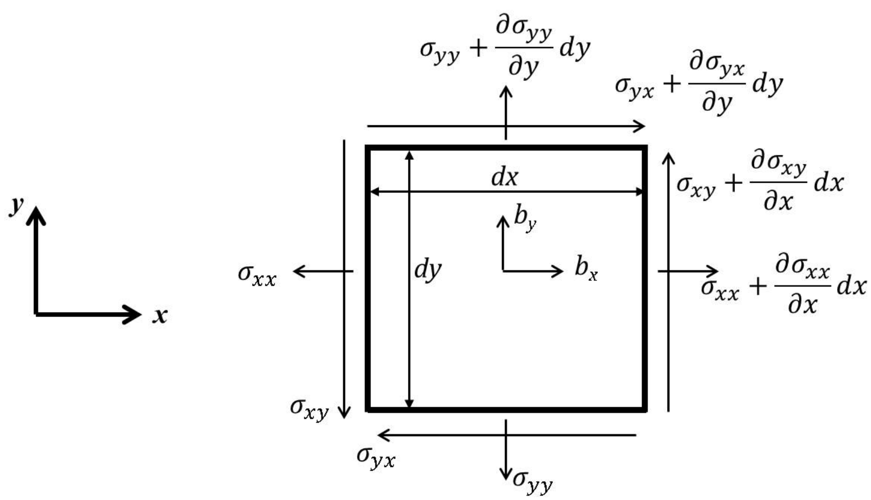

Figure 2.3: Stresses acting on a 2-D cross-section.

2.3 Strain at a point

Consider a square element whose sides are of unit length (Fig. 2.4). Under the action of the external loading, the element will deform such that the sides of the square are no longer perpendicular.

By definition, $2\varepsilon_{xy} = \gamma_1 + \gamma_2 > 0$ when angle $B'OA'$ becomes smaller than $\frac{\pi}{2}$. Strain is dimensionless.

2.4 Strain-displacement relations

For small deformations strains and displacements are related as follows:

In 2-D elements, $\varepsilon_{xx} = \frac{\partial u}{\partial x}$ $\varepsilon_{yy} = \frac{\partial v}{\partial y}$ $2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$

where u, v are the displacements of a point in the x and y directions.

In 3-D elements, $\varepsilon_{xx} = \frac{\partial u}{\partial x}$ $\varepsilon_{yy} = \frac{\partial v}{\partial y}$ $\varepsilon_{zz} = \frac{\partial w}{\partial z}$

$2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$ $2\varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$ $2\varepsilon_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$

where u, v, w are the displacements of a point in the x, y, z directions.

In matrix form, $\{\varepsilon\} = [L]\{U\}$, where,

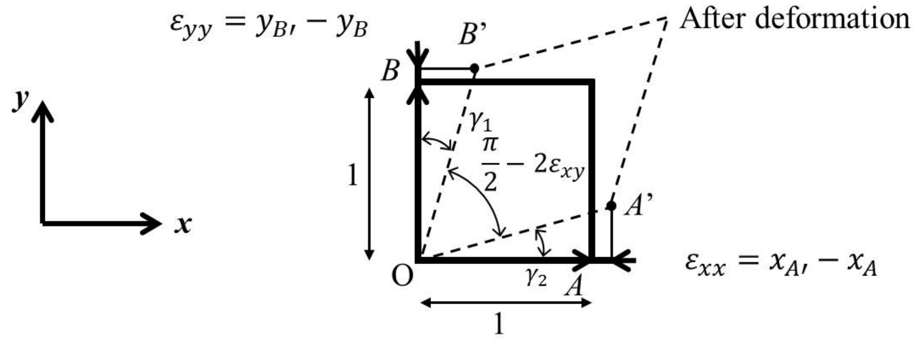

Figure 2.4: Strains in 2-D.

- in 2-D elements, $\{\varepsilon\}^T = [\varepsilon_{xx}, \varepsilon_{yy}, 2\varepsilon_{xy}]$, $[L] = \begin{bmatrix} \frac{\partial \cdot}{\partial x} & 0 \\ 0 & \frac{\partial \cdot}{\partial y} \\ \frac{\partial \cdot}{\partial y} & \frac{\partial \cdot}{\partial x} \end{bmatrix}$

  and $\{U\}^T = [u, v]$

- in 3-D elements, $\{\varepsilon\}^T = [\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, 2\varepsilon_{xy}, 2\varepsilon_{yz}, 2\varepsilon_{zx}]$, $[L] = \begin{bmatrix} \frac{\partial \cdot}{\partial x} & 0 & 0 \\ 0 & \frac{\partial \cdot}{\partial y} & 0 \\ 0 & 0 & \frac{\partial \cdot}{\partial z} \\ \frac{\partial \cdot}{\partial y} & \frac{\partial \cdot}{\partial x} & 0 \\ 0 & \frac{\partial \cdot}{\partial z} & \frac{\partial \cdot}{\partial y} \\ \frac{\partial \cdot}{\partial z} & 0 & \frac{\partial \cdot}{\partial x} \end{bmatrix}$

  and $\{U\}^T = [u, v, w]$

2.5 Compatibility equations

In 2-D, since $2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$, $\varepsilon_{xx} = \frac{\partial u}{\partial x}$ and $\varepsilon_{yy} = \frac{\partial v}{\partial y}$,
$2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = \frac{\partial^2 \partial u}{\partial x \partial y^2} + \frac{\partial^2 \partial v}{\partial x^2 \partial y} = \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2}$, hence an extra relationship, called compatibility equation, between three strains and two displacements.

Similar compatibility equations are obtained in 3-D. Compatibility equations are automatically satisfied in the stiffness approach used in the displacement-based finite element method described in this course.

2.6 A constitutive relationship - Hooke's law

The uniaxial Hooke's law $\sigma = E\varepsilon$, where E is the elastic modulus, can be generalized, and is expressed in 3-D elements as $\{\sigma\} = [D]\left(\{\varepsilon\} - \{\varepsilon_0\}\right) + \{\sigma_0\}$, where $[D]$ is the material property matrix, $\{\sigma\}$ the stress vector, $\{\varepsilon\}$ the strain vector, $\{\varepsilon_0\}$ the initial strain vector, and $\{\sigma_0\}$ the initial (or residual) stress vector.

$\{\sigma\}^T = [\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx}]$

$\{\varepsilon\}^T = [\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, 2\varepsilon_{xy}, 2\varepsilon_{yz}, 2\varepsilon_{zx}]$

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

E : elastic modulus, ν : Poisson's ratio

Other specific $[D]$ matrices will be found for plane stress, plane strain, or axisymmetric analyses (see Chapter 5). $[D]$ is symmetric for both isotropic and anisotropic materials. In the isotropic case, only E and ν are needed. $\{\sigma_0\}$ represents stresses that are known to exist in a material before it is loaded. They must be specified by the analyst. $\{\varepsilon_0\}$ may be the result of crystal growth, shrinkage, or temperature changes.

Note that in different textbooks, shear stresses are often noted with τ (e.g. σ_{xy} is noted τ_{xy}), and shear strains are often noted with γ (e.g. $2\varepsilon_{xy}$ is noted γ_{xy}).

2.7 Principle of minimum potential energy

How to find the mechanical equilibrium of a structure other than by writing the equilibrium equations for each component of the structure, and hoping to be able to solve for all of them (Note: this only works for statically determinate systems)? The principle of minimum potential energy (presented in this section, and which you've encountered many times in physics) and the principle of virtual displacements (presented in next section) are powerful tools to look for the equilibrium of a whole structure at once (including statically indeterminate systems).

The principle of minimum potential energy (PMPE) states that: "out of all the possible displacements fields that satisfy the geometric boundary conditions (i.e. prescribed displacements), the one that also satisfies the equations of static equilibrium results in the minimum of total potential energy for the structure (or body)."

The total potential energy Π is defined as the sum of the strain energy (internal potential energy U_i) and the external potential energy U_e from the external forces. $\Pi = U_i + U_e$. For conservative systems, the loss of external potential energy during the loading process must be equal to the work W_e done on the system by the external forces, or $-U_e = W_e$, and therefore $\Pi = U_i - W_e$. Π is a function of functions (strains and displacements) and is called a functional. Minimizing Π is called a variational problem. The first variation of the total potential energy $\delta\Pi = \delta U_i - \delta W_e$ must be zero, i.e. $\delta U_i = \delta W_e$.

In a global Cartesian (x, y, z) coordinate system,

$\delta U_i = \int_V (\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + \sigma_{zz}\delta\varepsilon_{zz} + 2\sigma_{xy}\delta\varepsilon_{xy} + 2\sigma_{yz}\delta\varepsilon_{yz} + 2\sigma_{zx}\delta\varepsilon_{zx})\, dV.$

Using matrix notation, $\delta U_i = \int_V \{\delta\varepsilon\}^T \{\sigma\}\, dV.$

For the work of external forces, considering a body force $\{b\}$ (per unit volume), a surface traction $\{s\}$ (per unit area) and N point loads $\{f_p\}$, then

$\delta W_e = \int_V (b_x\delta u + b_y\delta v + b_z\delta w)\, dV + \int_A (s_x\delta u + s_y\delta v + s_z\delta w)\, dA$
$+ \sum_{p=1}^{p=N} (f_{px}\delta u + f_{py}\delta v + f_{pz}\delta w).$

Using matrix notation, $\delta W_e = \int_V \{\delta U\}^T \{b\}\, dV + \int_A \{\delta U\}^T \{s\}\, dA$
$+ \sum_{p=1}^{p=N} \{\delta U\}^T \{f_p\}$, where

$\{b\}^T = [b_x, b_y, b_z]$: body force vector

$\{s\}^T = [s_x, s_y, s_z]$: surface traction vector

$\{f_p\}^T = [f_{px}, f_{py}, f_{pz}]$: point load vector

$\{\delta U\}^T = [\delta u, \delta v, \delta w]$: first variation of displacement vector

Eliminating $\{\sigma\}$ using the linear elastic stress-strain relationship, the PMPE becomes

$\int_V \{\delta\varepsilon\}^T [D] \{\varepsilon\}\, dV =$
$\int_V \{\delta\varepsilon\}^T [D] \{\varepsilon_0\}\, dV - \int_V \{\delta\varepsilon\}^T \{\sigma_0\}\, dV$
$+ \int_V \{\delta U\}^T \{b\}\, dV + \int_A \{\delta U\}^T \{s\}\, dA + \sum_{p=1}^{p=N} \{\delta U\}^T \{f_p\}.$

Going back to the total potential energy, it is given by

$\Pi = \frac{1}{2} \int_V \{\varepsilon\}^T [D] \{\varepsilon\}\, dV - \int_V \{\varepsilon\}^T [D] \{\varepsilon_0\}\, dV + \int_V \{\varepsilon\}^T \{\sigma_0\}\, dV$
$- \int_V \{U\}^T \{b\}\, dV - \int_A \{U\}^T \{s\}\, dA - \sum_{p=1}^{p=N} \{U\}^T \{f_p\}$

Example 6 *consider the elongation Δ and an axial force P in a uniaxial* *stress member (bar) of uniform cross-sectional area A, length L and modulus of elasticity E (Fig. 2.5). One end of the bar is fixed.*

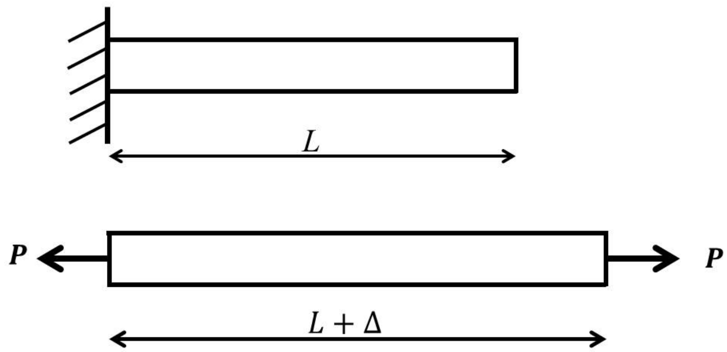

Figure 2.5: Extension of a bar.

$\varepsilon_0 = 0$, $\sigma_0 = 0$, $b = 0$ and $s = 0$.
$\Pi = \frac{1}{2}E\varepsilon^2 AL - P\Delta$ with $\varepsilon = \frac{\Delta}{L}$, therefore $\Pi = \frac{1}{2}E(\frac{\Delta}{L})^2 AL - P\Delta$.
We want Δ for equilibrium, i.e. Δ for minimum Π.
$\frac{d\Pi}{d\Delta} = E\frac{\Delta}{L^2}AL - P = 0 \Longleftrightarrow \Delta = \frac{PL}{AE}$: this is well known!
Note that $\frac{d^2\Pi}{d\Delta^2} = \frac{EA}{L} > 0$, therefore, Δ really minimizes Π.

2.8 Principle of virtual work

This principle, also known as the principle of virtual displacements, will be very convenient and useful for finite element formulation of complex problems. Work is the product of a displacement and the component of the force in the direction of the displacement. Virtual work is imagined to occur when the forces are real and the displacements are virtual (imagined), or vice-versa, but this not used herein.

Statement of the principle of virtual work (PVW): if the work done by the external forces on the structure is equal to the increase in strain energy for any set of admissible virtual displacements (i.e. satisfying the prescribed displacements), then the system is in equilibrium.

Let us denote the virtual displacements in x, y, z directions as δu, δv, δw (not variations!). The virtual displacements will cause virtual strains $\delta\varepsilon_{xx}, \delta\varepsilon_{yy}, \delta\varepsilon_{zz}, \delta\varepsilon_{xy}, \delta\varepsilon_{yz}, \delta\varepsilon_{zx}$.

In matrix form, the PVW becomes

$\int_V \{\delta\varepsilon\}^T \{\sigma\} \, dV = \int_V \{\delta U\}^T \{b\} \, dV + \int_A \{\delta U\}^T \{s\} \, dA + \sum_{p=1}^{p=N} \{\delta U\}^T \{f_p\}$, $\forall \{\delta U\}, \{\delta\varepsilon\}$.

with notations as in Section 2.7.

This is known as a weak form of the equilibrium equations because this equation only contains first derivatives of displacements whereas the original equilibrium equations (see Section 2.2 + Hooke's law) contain second order derivatives of the displacements.

Chapter 3

Finite element for beams

Learning Outcomes:

By the end of reading and attending the lecture of this chapter, and solving the listed practice problems, the student should be able, for a planar beam structure*, to:*

-Explain the theory of beam bending in one plane

-Explain the discretization and interpolation of beam elements

-Explain the determination of the local stiffness matrix and load vector for beam elements

-Write the element stiffness matrix and associated displacement and load vectors in the global coordinate system

-Write the assemblage stiffness matrix and associated displacement and load vectors

-Explain and apply boundary conditions

-Solve system of equations for displacements or loads

-Determine internal loads as well as stress and strains associated with displacements and external loads

Evidence of Learning:

A student has achieved the learning outcomes when he or she is able to correctly analyze a problem combining several or all of the previous items.

Evaluation Strategies:

Test problems in at-home assignments

Teaching and Learning Activities:

-At-home reading before class

-Lecturing

-In-class practice problems

-At-home reading/reviewing after class
-At-home practice problems

3.1 Theory - Beam bending in one plane

The deflection of the neutral axis of a beam at any location x is represented by $v(x)$ (bending in a plane). The deflections are supposed to be small compared to the length of the beam (typically less than 3% of length – if this is not the case, a more advanced theory must be used). The material is also supposed to be linearly elastic. Finally, it is assumed that the beam cross-section has an axis of symmetry in the plane of bending, and that planar cross-sections remain planar during deformation. Let us take point P on the beam neutral axis, and point Q at distance y from the neutral axis (Fig. 3.1, left). After deformation (Fig. 3.1, right, in which the deflection $v(x)$ and rotation $\theta(x)$ of the cross-section are grossly exaggerated), relationships between different variables can be established as follows. Let us introduce $u(x)$ the longitudinal displacement of Q due to the deformation.

v : local deflection of the beam

$\frac{dv}{dx}$: local slope of the beam, physically interpreted as the rotation of the local cross-section of the beam, therefore $\frac{dv}{dx} = \theta$.

From trigonometry, $\tan\theta = \frac{-u}{y}$, and because θ is very small, $\theta \simeq \tan\theta$. Therefore, one can write

$-y\frac{dv}{dx} = u(x)$: longitudinal displacement due to $v(x)$.

Recalling that a longitudinal strain is obtained by taking the first derivative of the longitudinal displacement, one obtains

$-y\frac{d^2v}{dx^2} = \varepsilon_x$: longitudinal strain due to $v(x)$.

Because the material of the beam is assumed linearly elastic, the local constitutive equation is such that the longitudinal (i.e. normal) stress in the beam at point Q is $\sigma_x = E\varepsilon_x$, where E is the material's elastic modulus. In other words, $\sigma_x = -yE\frac{d^2v}{dx^2}$.

In beam cross-section A, the external normal force is $N(x) = \int_A \sigma_x dA$. Because σ_x varies linearly in the y-direction, $N(x) = 0$.

In beam cross-section A, the external bending moment is

$M(x) = -\int_A y\sigma_x dA = E\frac{d^2v}{dx^2}\int_A y^2 dA$. Recalling $\int_A y^2 dA = I$: second mo-

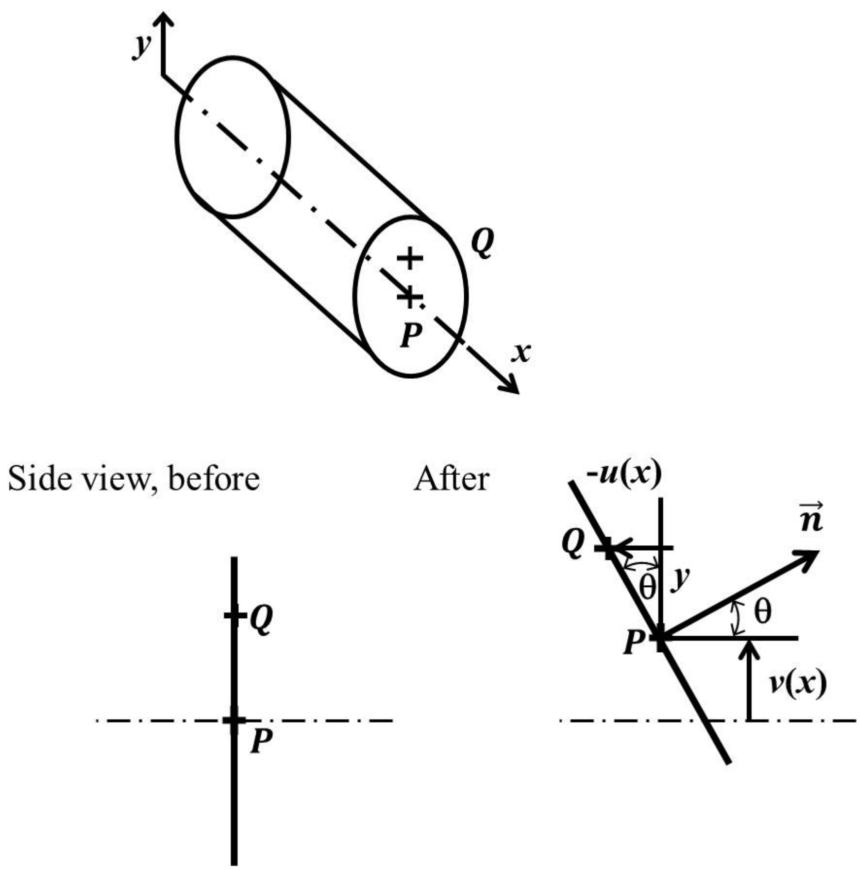

Figure 3.1: Beam deformation.

ment of area, one obtains, $M(x) = EI\frac{d^2v}{dx^2}$. This is the global constitutive equation for the beam, i.e. it expresses the relationship between the deflection and the global external loads applied to the beam.

Note that beam deflection analysis also often aims at determining the bending stresses in the beam. To do so, one simply applies

$\sigma_x = -yE\frac{d^2v}{dx^2} = -\frac{M(x)y}{I}$ for each location of interest in the beam.

Considering the free-body diagram of an elemental beam segment of length dx (Fig.3.2), one can get additional relationships as follows. Let us

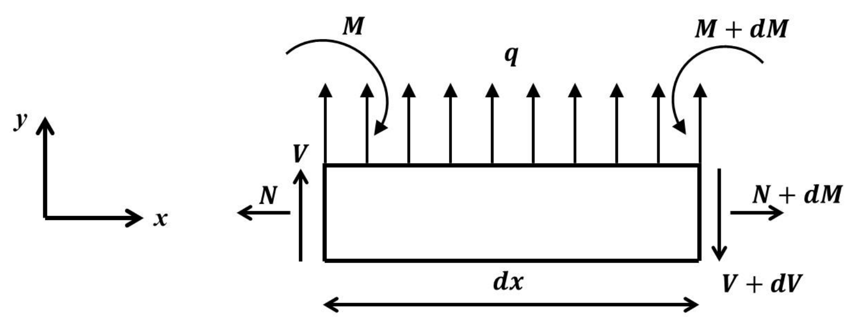

Figure 3.2: Free-body diagram of an elemental segment of beam.

introduce $q(x)$ the transverse distributed load (force per unit length – not necessarily constant).

$\sum M = 0 : -Vdx + dM = 0$, therefore $V(x) = \frac{dM(x)}{dx}$.

$\sum F = 0 : -dV + qdx = 0$, therefore $\frac{dV(x)}{dx} = q(x)$.

In the absence of distributed load, obviously, $q(x) = 0$.

Finally, the governing equations for the beam are:

$EI\frac{d^2v}{dx^2} = M$ bending moment

$EI\frac{d^3v}{dx^3} = \frac{dM}{dx} = V$ transverse shear force

$EI\frac{d^4v}{dx^4} = \frac{dV}{dx} = q$ transverse distributed load

3.2 Discretization

Consider the beam element shown in Fig. 3.3:

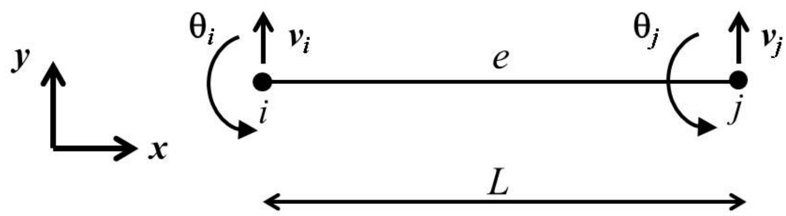

Figure 3.3: Beam element and its degrees of freedom.

In the absence of distributed loads, the equilibrium equation for the element is $\frac{d^4v}{dx^4} = 0$, for which the general solution is a third-order polynomial $v(x) = c_1 + c_2x + c_3x^2 + c_4x^3$.

The element's end conditions can be expressed in terms of the following nodal values:

Node i: $v(0) = v_i \quad = c_1$

$\frac{dv(0)}{dx} = \theta_i \quad = c_2$

Node j: $v(L) = v_j \quad = c_1 + c_2L + c_3L^2 + c_4L^3$

$\frac{dv(L)}{dx} = \theta_j \quad = c_2 + 2c_3L + 3c_4L^2$

In matrix form, the same relationships can be written as:

$\left\{ \begin{array}{c} v_i \\ \theta_i \\ v_j \\ \theta_j \end{array} \right\} = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{array} \right] \left\{ \begin{array}{c} c_1 \\ c_2 \\ c_3 \\ c_4 \end{array} \right\}$ or $\{U^e\} = [A]\{Q\}$, where $[A]$ is called coefficient matrix.

By inverting $[A]$, we can solve for $\{Q\}$ in terms of $\{U^e\}$.

$[A]^{-1} = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{L^2} & -\frac{2}{L} & \frac{3}{L^2} & -\frac{1}{L} \\ \frac{2}{L^3} & \frac{1}{L^2} & -\frac{2}{L^3} & \frac{1}{L^2} \end{array} \right]$

Finally, $v(x)$ can be written in matrix form as:

$v(x) = \left[\begin{array}{cccc} 1 & x & x^2 & x^3 \end{array} \right] \{Q\} = \left[\begin{array}{cccc} 1 & x & x^2 & x^3 \end{array} \right] [A]^{-1} \{U^e\}$,

or $v(x) = [N]\{U^e\}$ with $[N] = \left[\begin{array}{cccc} 1 & x & x^2 & x^3 \end{array} \right] [A]^{-1}$

$[N] = \left[\begin{array}{cccc} N_1(x) & N_2(x) & N_3(x) & N_4(x) \end{array} \right]$, or more explicitly,

$[N] = \left[\begin{array}{cccc} 1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} & x - \frac{2x^2}{L} + \frac{x^3}{L^2} & \frac{3x^2}{L^2} - \frac{2x^3}{L^3} & -\frac{x^2}{L} + \frac{x^3}{L^2} \end{array} \right]$.

We note that
$$\begin{aligned}
[N]\,|_{x=0} &= \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}\\
[N]\,|_{x=L} &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}\\
\left[\tfrac{dN}{dx}\right]|_{x=0} &= \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}\\
\left[\tfrac{dN}{dx}\right]|_{x=L} &= \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

3.3 Element stiffness and load vectors

Let us assume that at Node i of the beam element, nodal force F_i works in displacement v_i, nodal moment M_i works in rotation θ_i, and similarly at Node j, F_j works in displacement v_j, nodal moment M_j works in rotation θ_j.

The principle of virtual work for the element is:

$$\int_V \{\delta\varepsilon\}^T \{\sigma\}\, dV = \{\delta U^e\}^T \begin{Bmatrix} F_i \\ M_i \\ F_j \\ M_j \end{Bmatrix}, \qquad \forall \{\delta U^e\}, \forall \{\delta\varepsilon\}.$$

Here, $\{\delta\varepsilon\}^T = \delta\varepsilon_x$, and $\{\sigma\} = \sigma_x$, as the other components are zero. Then,

$$\int_V \delta\varepsilon_x \sigma_x dV = \int_0^L \left(\int_A (-y(\tfrac{d^2\delta v(x)}{dx^2})(-yE\tfrac{d^2 v(x)}{dx^2}) dA \right) dx$$

$$= \int_0^L EI(\tfrac{d^2\delta v(x)}{dx^2})(\tfrac{d^2 v(x)}{dx^2}) dx \text{ since } \int_A y^2 dA = I.$$

From $v(x) = [N]\{U^e\}$, one can derive $\frac{d^2 v(x)}{dx^2} = \frac{d^2}{dx^2}[N]\{U^e\} = [B]\{U^e\}$, and $\frac{d^2\delta v(x)}{dx^2} = [B]\{\delta U^e\}$,

where $[B] = \begin{bmatrix} -\frac{6}{L^2} + \frac{12x}{L^3} & \frac{-4}{L} + \frac{6x}{L^2} & \frac{6}{L^2} - \frac{12x}{L^3} & -\frac{2}{L} + \frac{6x}{L^2} \end{bmatrix}$.

Then,

$$\int_0^L EI(\tfrac{d^2\delta v(x)}{dx^2})(\tfrac{d^2 v(x)}{dx^2}) dx = \int_0^L EI \{\delta U^e\}^T [B]^T [B] \{U^e\}\, dx$$

$$= \{\delta U^e\}^T \int_0^L [B]^T EI\, [B]\, dx\, \{U^e\}.$$

Finally, the PVW yields:

$$\{\delta U^e\}^T \int_0^L [B]^T EI\, [B]\, dx\, \{U^e\} = \{\delta U^e\}^T \begin{Bmatrix} F_i \\ M_i \\ F_j \\ M_j \end{Bmatrix}, \qquad \forall \{\delta U^e\}.$$

In other words,

$$\int_0^L [B]^T EI\,[B]\,dx\,\{U^e\} = \left\{ \begin{array}{c} F_i \\ M_i \\ F_j \\ M_j \end{array} \right\}, \qquad \text{or } [K^e]\{U^e\} = \{F^e\},$$

with $[K^e] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$. Note that this is for an element aligned with the $x-$axis, and bending in the $xy-$plane.

So far, we only considered point forces and moments acting in the displacements and rotations, respectively. For a distributed load $w(x)$, the virtual work of external forces is:
$\int_0^L \delta v(x) w(x) dx = \{\delta U^e\}^T \int_0^L [N]^T w(x) dx = \{\delta U^e\}^T \{F_w\}$,where $\{F_w\}$ is the nodal force vector representing a distributed load on the basis of work equivalence.

For $w(x) = q$ (constant),

$$\{F_w\} = \left\{ \begin{array}{c} \int_0^L N_1(x) q dx \\ \int_0^L N_2(x) q dx \\ \int_0^L N_3(x) q dx \\ \int_0^L N_4(x) q dx \end{array} \right\} = \left\{ \begin{array}{c} \frac{qL}{2} \\ \frac{qL^2}{12} \\ \frac{qL}{2} \\ -\frac{qL^2}{12} \end{array} \right\} \text{(Fig. 3.4)}.$$

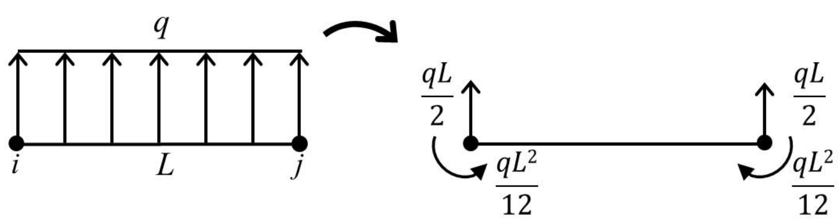

Figure 3.4: Constant distributed load and its nodal equivalents.

3.4 Assemblage and solution

Consider the example shown in Fig. 3.5:

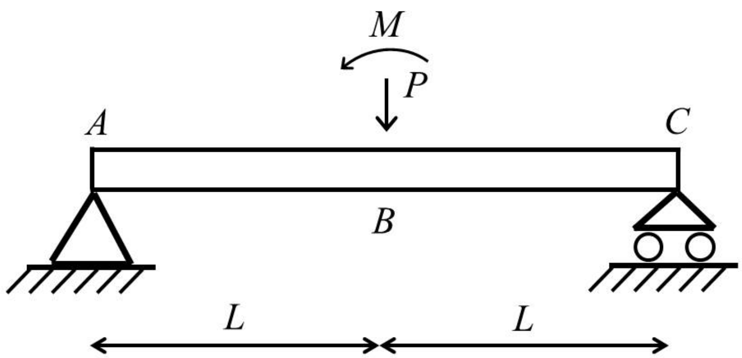

Figure 3.5: Example of beam problem.

Let us descritize the beam into two elements of same length (Fig. 3.6). Having nodes where loads and reactions are applied makes the analysis convenient.

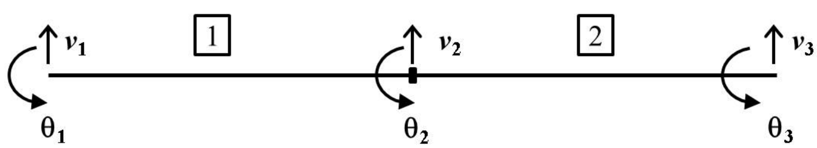

Figure 3.6: Degrees of freedom for problem in Fig. 3.5.

In this example, there are two elements, with two nodes per element, and two degrees of freedom per node.

Procedure: let us first create a null assemblage stiffness matrix $[K^a]$ involving global node numbers 1 to 3 (Fig. 3.6).

$$
\begin{array}{cc}
 & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\
[K^a] = & \left[\begin{array}{ccc} 0_{2\times2} & 0_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} & 0_{2\times2} \end{array} \right] \begin{array}{c} 1 \\ 2 \\ 3 \end{array}
\end{array}
$$

From the connectivity table, Element 1 has global node numbers 1 and 2, so

$$\left[K^{(1)}\right] = \begin{bmatrix} K^{(1)}_{1,1} & K^{(1)}_{1,2} \\ K^{(1)}_{2,1} & K^{(1)}_{2,2} \end{bmatrix}.$$

Element 2 has global node numbers 2 and 3, so

$$\left[K^{(2)}\right] = \begin{bmatrix} K^{(2)}_{2,2} & K^{(2)}_{2,3} \\ K^{(2)}_{3,2} & K^{(2)}_{3,3} \end{bmatrix}.$$

Finally,

$$[K^a] = \begin{array}{c} \begin{matrix} 1 & & 2 & & 3 \end{matrix} \\ \begin{bmatrix} K^{(1)}_{1,1} & K^{(1)}_{1,2} & 0_{2\times 2} \\ K^{(1)}_{2,1} & K^{(1)}_{2,2} + K^{(2)}_{2,2} & K^{(2)}_{2,3} \\ 0_{2\times 2} & K^{(2)}_{3,2} & K^{(2)}_{3,3} \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \end{array}$$

Elements 1 and 2 have the same stiffness matrices

$$[K] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}. \text{ Therefore,}$$

$$[K^a] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{matrix} \text{ works in } \begin{matrix} F_1 \\ M_1 \\ F_2 \\ M_2 \\ F_3 \\ M_3 \end{matrix}$$

The assemblage nodal displacement vector is
$\{U^a\}^T = \left[\begin{array}{cccccc} v_1 & \theta_1 & v_2 & \theta_2 & v_3 & \theta_3 \end{array}\right].$

Let us now determine the assemblage nodal force vector, or assemblage load vector. Can we identify some components in the generic load vector $\{F^a\}^T = \left[\begin{array}{cccccc} F_1 & M_1 & F_2 & M_2 & F_3 & M_3 \end{array}\right]$?

A free-body diagram of the system (Fig. 3.7) gives $F_1 \equiv R_A$, $F_2 = -P$, $M_2 \equiv M$, $F_3 \equiv R_C$ and $M_1 \equiv 0$ and $M_3 \equiv 0$ (obviously, $H_A \equiv 0$).

So far, we have 6 equations and 8 unknowns: R_A, R_C, and $v_1, \theta_1, v_2, \theta_2, v_3, \theta_3$. However, the boundary conditions are such that $v_1 = v_3 \equiv 0$, therefore the actual number of unknowns is 6, and the problem can be solved.

Because $v_1 = v_3 \equiv 0$, we eliminate the corresponding columns of $[K^a]$, and because R_A and R_C are unknown, we eliminate the corresponding rows of $[K^a]$ to solve for the unknowns. Then,

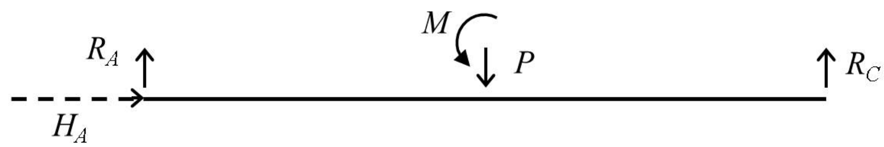

Figure 3.7: Free-body diagram of beam in Fig. 3.5.

$$\frac{EI}{L^3}\begin{bmatrix} 4L^2 & -6L & 2L^2 & 0 \\ -6L & 24 & 0 & 6L \\ 2L^2 & 0 & 8L^2 & 2L^2 \\ 0 & 6L & 2L^2 & 4L^2 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ M \\ 0 \end{Bmatrix}$$

For the case where $P = 1,000\,\text{lbf}$ and $M = 0$,
$\theta_1 = -\frac{250L^2}{EI}$, $v_2 = -\frac{167L^3}{EI}$, $\theta_2 = 0$, $\theta_3 = \frac{250L^2}{EI}$.

From these solutions, and extracting two rows of the assemblage system, one can determine the reaction forces R_A and R_C corresponding to $v_1 = v_3 \equiv 0$ as

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ 0 & 0 & -12 & -6L & 12 & -6L \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} R_A \\ R_C \end{Bmatrix}.$$

Finally, $R_A = 500\,\text{lbf}$ and $R_C = 500\,\text{lbf}$. Note that a value slightly different from 500 may be obtained due to round-off errors during calculation.

Once the displacements and rotations are known along with all the shear forces and moments, the bending stresses can be calculated at critical locations, using the classical equations of strength of materials. Do not forget that, in practice, design criteria based on maximum allowable stress or displacement must include safety factors. Review your notes from Machine Design or Strength of Materials courses as necessary.

3.5 Example

Assignment 2.

3.6 Beam element with axial loading

Outside of buckling and stress stiffening (e.g. a taut guitar string), which are nonlinear cases, we can linearly superimpose the results from bar and beam elements, because displacements, strains and rotations are assumed to be small. Note that in many textbooks and software packages, such combination is described by the term frame element (Fig. 3.8).

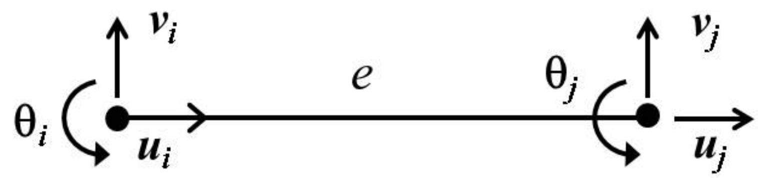

Figure 3.8: Combined bar and beam element, or frame element.

$$[K^e] = \begin{bmatrix} [k^e_{\text{Bar}}] & [0] \\ [0] & [k^e_{\text{Beam}}] \end{bmatrix} \text{ with } \{U^e\}^T = \begin{bmatrix} u_i & u_j & v_i & \theta_i & v_j & \theta_j \end{bmatrix}.$$

With more convenient $\{U^e\}^T = \begin{bmatrix} u_i & v_i & \theta_i & u_j & v_j & \theta_j \end{bmatrix}$, $[K^e]$ can be re-organized into:

$$[K^e] = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & 0 & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ 0 & \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ -\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ 0 & \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & 0 & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix}.$$

Note that this is for an element aligned with the $x-$axis, and bending in the $xy-$plane. If the element is oriented at an arbitrary angle θ from the $x-$ axis of the global reference frame, we have (Fig. 3.9), and $[K^e]$ above becomes $\left[K^{e'}\right]$.

$$\begin{aligned} u'_i &= lu_i + mv_i \\ v'_i &= -mu_i + lv_i \\ \theta'_i &= \theta_i \\ u'_j &= lu_j + mv_j \\ v'_j &= -mu_j + lv_j \\ \theta'_j &= \theta_j \end{aligned}$$

with l and m determined as in Section1.4.

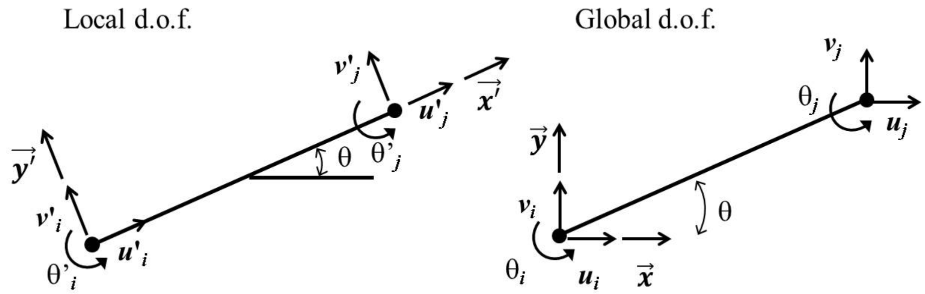

Figure 3.9: Local and global degrees of freedom for a 2-D frame element.

In matrix form,

$$\left\{\begin{array}{c} u_i' \\ v_i' \\ \theta_i' \\ u_j' \\ v_j' \\ \theta_j' \end{array}\right\} = \left[\begin{array}{cccccc} l & m & 0 & 0 & 0 & 0 \\ -m & l & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & 0 \\ 0 & 0 & 0 & -m & l & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \left\{\begin{array}{c} u_i \\ v_i \\ \theta_i \\ u_j \\ v_j \\ \theta_j \end{array}\right\},$$

or $\left\{U^{e'}\right\} = [R]\left\{U^e\right\}$. Note that $[R]$ is different from that in Section 1.4. Then, the element stiffness matrix in the global system is $[K^e] = [R]^T [K^{e\prime}] [R]$.

3.7 General 3-D beam (or frame) element

We want to include

- axial behaviour along $x-$axis (bar)
- bending behaviour in $xy-$plane (beam)
- bending behaviour in $xz-$plane (beam)
- torsional behaviour about $x-$axis

Items 1 and 2 have been taken care of in Section 3.6. Bending in $xz-$plane needs attention because it is similar to bending in $xy-$plane, but instead of $\theta_z = \frac{dv}{dx}$, we have $\theta_y = -\frac{dw}{dx}$, and instead of $M_z = EI_z\frac{d^2v}{dx^2}$, we have $M_y = -EI_y\frac{d^2w}{dx^2}$ (Fig. 3.10).

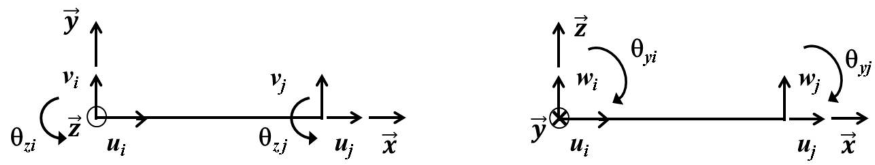

Figure 3.10: Degrees of freedom for a general 3-D beam element.

Therefore, $[k^e]_{xz} = \frac{EI_y}{L^3}\begin{bmatrix} 12 & -6L & -12 & -6L \\ -6L & 4L^2 & 6L & 2L^2 \\ -12 & 6L & 12 & 6L \\ -6L & 2L^2 & 6L & 4L^2 \end{bmatrix}$.

Torsion about longitudinal axis is represented by stiffness matrix $[k^e_{\text{Torsion}}] = \frac{JG}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, nodal displacement vector $\begin{Bmatrix} \theta_{xi} \\ \theta_{xj} \end{Bmatrix}$ and load vector $\begin{Bmatrix} M_{xi} \\ M_{xj} \end{Bmatrix}$. Finally, for the general 3-D beam element:

$$\begin{bmatrix} [k^e_{\text{Bar}}] & [0] & [0] & [0] \\ [0] & [k^e_{\text{Beam}}]_{xy} & [0] & [0] \\ [0] & [0] & [k^e_{\text{Beam}}]_{xz} & [0] \\ [0] & [0] & [0] & [k^e_{\text{Torsion}}] \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \\ v_i \\ \theta_{zi} \\ v_j \\ \theta_{zj} \\ w_i \\ \theta_{yi} \\ w_j \\ \theta_{yj} \\ \theta_{xi} \\ \theta_{xj} \end{Bmatrix} = \begin{Bmatrix} F_{xi} \\ F_{xj} \\ F_{yi} \\ M_{zi} \\ F_{yj} \\ M_{zj} \\ F_{zi} \\ M_{yi} \\ F_{zj} \\ M_{yj} \\ M_{xi} \\ M_{xj} \end{Bmatrix}.$$

With more convenient
$\{U^e\}^T = \begin{bmatrix} u_i & v_i & w_i & \theta_{xi} & \theta_{yi} & \theta_{zi} & u_j & v_j & w_j & \theta_{xj} & \theta_{yj} & \theta_{zj} \end{bmatrix}$,
$[K^e]$ can be re-organized into:
$[K^e] = \begin{bmatrix} K^e_{1...6,1...6} & K^e_{1...6,7...12} \\ K^e_{7...12,1...6} & K^e_{7...12,7...12} \end{bmatrix}$.

$$\left[K^e_{1\ldots6,1\ldots6}\right] = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\ 0 & 0 & \frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 \\ 0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\ 0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 \\ 0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} \end{bmatrix},$$

$$\left[K^e_{1\ldots6,7\ldots12}\right] = \begin{bmatrix} -\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & \frac{6EI_z}{L^2} \\ 0 & 0 & -\frac{12EI_y}{L^3} & 0 & -\frac{6EI_y}{L^2} & 0 \\ 0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 \\ 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 \\ 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} \end{bmatrix},$$

$$\left[K^e_{7\ldots12,1\ldots6}\right] = \begin{bmatrix} -\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} \\ 0 & 0 & -\frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\ 0 & 0 & 0 & -\frac{GJ}{L} & 0 & 0 \\ 0 & 0 & -\frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 \\ 0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} \end{bmatrix},$$

$$\left[K^e_{7\ldots12,7\ldots12}\right] = \begin{bmatrix} \frac{AE}{L} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & 0 & 0 & 0 & -\frac{6EI_z}{L^2} \\ 0 & 0 & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 \\ 0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\ 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & 0 \\ 0 & -\frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L} \end{bmatrix}.$$

Note that this is for an element aligned with the $x-$axis. If the element is oriented at an arbitrary angle from the $x-$ axis of the global reference frame, $[K^e]$ above becomes $\left[K^{e'}\right]$, and the components of displacements and rotations at each node of the element can be transformed from the local (rotated) coordinate system to the global coordinate system as done in Sections 1.3 and 1.4. Overall, the degrees of freedom for the element can be expressed as $\left\{U^{e'}\right\} = [R]\{U^e\}$, where $[R] = \begin{bmatrix} T_{3D}^T & 0 & 0 & 0 \\ 0 & T_{3D}^T & 0 & 0 \\ 0 & 0 & T_{3D}^T & 0 \\ 0 & 0 & 0 & T_{3D}^T \end{bmatrix}$ and $[T_{3D}] = \begin{bmatrix} \cos(x',x) & \cos(y',x) & \cos(z',x) \\ \cos(x',y) & \cos(y',y) & \cos(z',y) \\ \cos(x',z) & \cos(y',z) & \cos(z',z) \end{bmatrix}$. Then, the element stiffness matrix in the global system is $[K^e] = [R]^T [K^e] [R]$. Note that $[R]$ is different from that in Section 1.4.

There are several ways to define a coordinate system attached to a 3-D beam. We now show one that is compatible with all the notations above.

Let the beam element extend along the $x'-$axis between Nodes i and j. To properly define the 3-D beam element, a third arbitrary node (Node k) needs to be created in the $x'z'-$plane outside of line $i - j$ (Fig. 3.11). Let $\overrightarrow{V}_{ij}$ denote the vector between Nodes i and j, and $\overrightarrow{V}_{ik}$ the vector between Nodes i and k. The coordinate system attached to the beam is constructed such that:

$\overrightarrow{x'} = \overrightarrow{V}_{ij} / \left\| \overrightarrow{V}_{ij} \right\|$,

in other words, $\left\{ \overrightarrow{x'} \right\} = \left\{ \begin{array}{c} (x_j - x_i)/L \\ (y_j - y_i)/L \\ (z_j - z_i)/L \end{array} \right\}$

with $L = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2}$;

$\overrightarrow{y'} = \overrightarrow{V}_{ik} \times \overrightarrow{V}_{ij} / \left\| \overrightarrow{V}_{ik} \times \overrightarrow{V}_{ij} \right\|$;

$\overrightarrow{z'} = \overrightarrow{x'} \times \overrightarrow{y'}$.

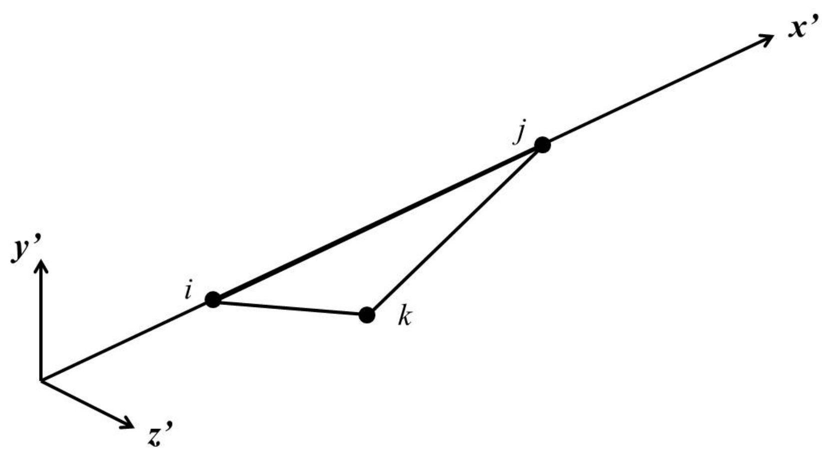

Figure 3.11: Third node for 3-D beam element orientation.

The stresses in a frame element are determined as follows. After solution of $[K]\{U\} = \{F\}$ for the whole sturcture (i.e. after the assembalge system of equations was built, and boundary conditions applied, and resulting system inverted), $\{U^e\}$ for any specific element in the structure is knwon. For a specific element, $\left\{U^{e'}\right\} = [R]\{U^e\}$ needs to be computed, from which $[K^{e'}]\left\{U^{e'}\right\} = \left\{F^{e'}\right\}$ can be obtained. $\left\{F^{e'}\right\}$ contains the axial force $F_i^{'}$, the shear force $V_i^{'}$ and the bending moment $M_i^{'}$ at Node i, and the axial force $F_j^{'}$, the shear force $V_j^{'}$ and the bending moment $M_j^{'}$ at Node j. At Node i (and similarly at Node j), the axial (or normal) stress $\sigma_{\text{axial}} = \frac{F_i^{'}}{A_i}$, the maximum bending stress $\sigma_{\text{bending}} = \pm\frac{M_i^{'}c}{I_i}$, where A_i is the cross-sectional area, I_i is the second moment of area, and c is the distance from the neutral axis to the extremal face (e.g. top or bottom) of the beam, all at Node i. The resulting maximum total stress is $\sigma_{\text{total}} = \sigma_{\text{axial}} + \sigma_{\text{bending}}$. Note that shear stresses are negligible for long members in bending; they are only relevant in direct shear.

Chapter 4

Interpolation functions - Integration formulas

Learning Outcomes:

By the end of reading and attending the lecture of this chapter, the student should be able to:

-Explain the concepts of compatibility and completeness of the interpolation functions

-Explain the need for mesh sensitivity analysis and know how to do one

-Explain how various interpolation functions can be constructed in 1-, 2- or 3-D elements

-Apply efficient integration rules over elements

Evidence of Learning:

A student has achieved the learning outcomes when he or she is able to reproduce the knowledge and abilities listed above.

Evaluation Strategies:

Test problems in at-home assignments

Teaching and Learning Activities:

-At-home reading before class

-Lecturing

-In-class practice problems

-At-home reading/reviewing after class

-At-home practice problems

From the principle of virtual work, it is evident that in stress analysis problems, the variables of interest are the displacements (vector). In thermal analysis, the variable of interest would be the temperature (scalar). In fluid flow problems, the variables of interest would be the fluid velocities (vector) and the pressure (scalar). Below are ideas and results regarding interpolation functions for these variables and integration over element length, surface or volume that can be used for yet other studies (electromagnetism, mass transfer, etc...).

4.1 Compatibility and completeness requirements

Compatibility

For C^0-continuous problems, the interpolation function must be continuous along the boundaries of the element. For C^1-continuous problems, the function and its first derivative must be continuous (Fig. 4.1).

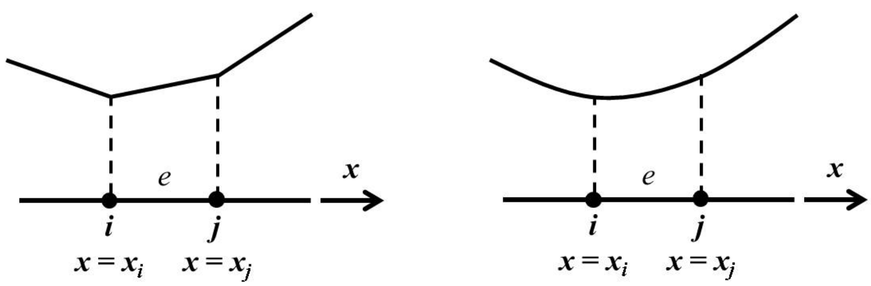

Figure 4.1: C^0-continuous function (left), and C^1-continuous function (right).

Many problems are C^0-continuous once formulated using the weak form (2-D stress, strain, axisymmetric stress, 3-D stress analyses), but some are C^1-continuous or higher. Elements that obey the compatibility requirement are said to be conforming (vs. non-conforming). Some non-conforming elements are useful, but they should be used with extra caution.

Completeness

For C^n-continuous problems, the interpolation function must be capable of representing a constant value of the variable as well as partial derivatives of up to order $n+1$ as the element size decreases to a point. Example in uniaxial stress analysis: let us assume $u = c_1+c_2x$. If $c_2 = 0$, $u = c_1 =$ constant : rigid body mode (displacement of the whole body without straining). Also, $\frac{du}{dx} = c_2 =$ constant : this allows for a constant strain in the element. Therefore, the interpolation function $u = c_1 + c_2x$ can be used for a C^0-continuous problem.

Mesh sensitivity analysis

With both compatibility and completeness requirements satisfied, convergence of the solution during mesh refinement can be achieved (Fig. 4.2). Regardless of the expectations, convergence of the solution during mesh refinement is a verification that MUST be done for every analysis (except for bar elements) when using a finite element code. Note that, typically, convergence of displacements can be obtained with a coarser mesh than convergence of stresses.

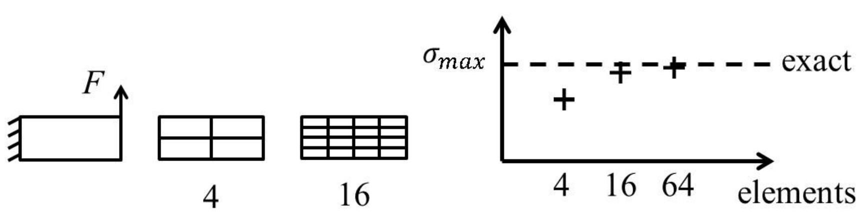

Figure 4.2: Example of convergence of solution upon mesh refinement.

4.2 One dimensional (1-D) elements

Note that a 1-D element can be used in 2-D or 3-D problems! An element is one-dimensional if, in its own coordinate system, the element has only one dimension. The same idea applies to 2-D and 3-D elements.

Interpolation functions considered herein are based on polynomials or rational functions. The functions can be developed with global, natural (or serendipity) or length coordinates, depending on the base of integration. Many different types of elements can be created, but not all will be useful.

4.2.1 2-node C^0-continuous element

Global coordinates (Fig. 4.3)

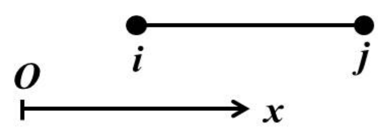

Figure 4.3: Global coordinates.

For variable ϕ, we pose $\phi(x) = c_1 + c_2 x$,
or $\phi(x) = [1 \quad x] \left\{ \begin{array}{c} c_1 \\ c_2 \end{array} \right\} = [1 \quad x] \{Q\}$.
At Nodes i, j, we require $\begin{array}{l} \phi(x_i) = \phi_i = c_1 + c_2 x_i \\ \phi(x_j) = \phi_j = c_1 + c_2 x_j \end{array}$.
In matrix form, $\left\{ \begin{array}{c} \phi_i \\ \phi_j \end{array} \right\} = \left[\begin{array}{cc} 1 & x_i \\ 1 & x_j \end{array} \right] \left\{ \begin{array}{c} c_1 \\ c_2 \end{array} \right\}$ or $\{\phi^e\} = [A] \{Q\}$.
Solving for the vector of constants yields $\left\{ \begin{array}{c} c_1 \\ c_2 \end{array} \right\} = \left[\begin{array}{cc} 1 & x_i \\ 1 & x_j \end{array} \right]^{-1} \left\{ \begin{array}{c} \phi_i \\ \phi_j \end{array} \right\}$.
Then, $\phi(x) = [1 \quad x] \left[\begin{array}{cc} 1 & x_i \\ 1 & x_j \end{array} \right]^{-1} \left\{ \begin{array}{c} \phi_i \\ \phi_j \end{array} \right\}$ or
$\phi(x) = [1 \quad x] [A]^{-1} \{\phi^e\} = [N] \{\phi^e\}$ with $[N] = [\frac{x_j - x}{x_j - x_i} \quad \frac{x - x_i}{x_j - x_i}]$.

Natural coordinates (Fig. 4.4)

Let us define a local, normalized coordinate r defined relative to the global coordinate x.
$r = 0$ at $x = \overline{x} = \frac{1}{2}(x_i + x_j)$, and $-1 \le r \le +1$ with $r = \frac{2(x - \overline{x})}{x_j - x_i}$.
With these coordinates, $[N] = [\frac{1}{2}(1 - r) \quad \frac{1}{2}(1 + r)]$.

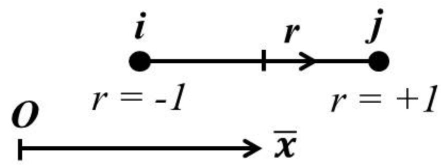

Figure 4.4: Natural coordinates.

Length coordinates (Fig.4.5)

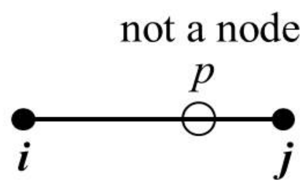

Figure 4.5: Length coordinates.

Let us consider an internal point p (not a node) in Element e, and define $L_i = \frac{\text{length } pj}{\text{length } ij}$ and $L_j = \frac{\text{length } ip}{\text{length } ij}$.

Obviously, $0 \leq L_i \leq 1, \quad 0 \leq L_j \leq 1$, and $L_i + L_j = 1$.

Then, $[N] = [L_i \quad L_j]$.

4.2.2 2-node C^1-continuous element

See development of beam element in global coordinates in Chapter 3 (Section 3.2, Discretization). Note that this element is C^2-continuous in v, but C^1-continuous in θ.

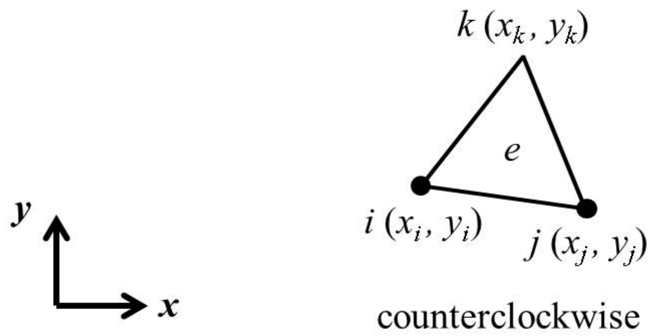

Figure 4.6: Global 2-D coordinates.

4.3 2-D elements

4.3.1 Triangular C^0-continuous element

Global coordinates (Fig. 4.6)

We pose $\phi(x,y) = c_1 + c_2 x + c_3 y$ or

$$\phi(x,y) = [1 \quad x \quad y] \left\{ \begin{array}{c} c_1 \\ c_2 \\ c_3 \end{array} \right\} = [1 \quad x \quad y] \{Q\}.$$

At Nodes i, j, k, we require $\phi(x_l, y_l) = \phi_l = c_1 + c_2 x_l + c_3 y_l$ for $l = i, j$ or k.

In matrix form, $\left\{ \begin{array}{c} \phi_i \\ \phi_j \\ \phi_k \end{array} \right\} = \left[\begin{array}{ccc} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{array} \right] \left\{ \begin{array}{c} c_1 \\ c_2 \\ c_3 \end{array} \right\}$ or $\{\phi^e\} = [A]\{Q\}$.

Solving for the vector of constants yields $\{Q\} = [A]^{-1} \{\phi^e\}$.

Then, $\phi(x,y) = [1 \quad x \quad y] [A]^{-1} \{\phi^e\} = [N] \{\phi^e\}$

with $[N] = [N_i \quad N_j \quad N_k]$

and $N_i(x,y) = m_{11} + m_{21} x + m_{31} y$

$N_j(x,y) = m_{12} + m_{22} x + m_{32} y$

$N_k(x,y) = m_{13} + m_{23} x + m_{33} y$

where $m_{11} = (x_j y_k - x_k y_j)/2A \quad m_{21} = (y_j - y_k)/2A \quad m_{31} = (x_k - x_j)/2A$

$m_{12} = (x_k y_i - x_i y_k)/2A \quad m_{22} = (y_k - y_i)/2A \quad m_{32} = (x_i - x_k)/2A$

$m_{13} = (x_i y_j - x_j y_i)/2A \quad m_{23} = (y_i - y_j)/2A \quad m_{33} = (x_j - x_i)/2A$

and $A = \frac{1}{2}\det\begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix}$ = area of triangle ijk.

Area coordinates (Fig. 4.7)

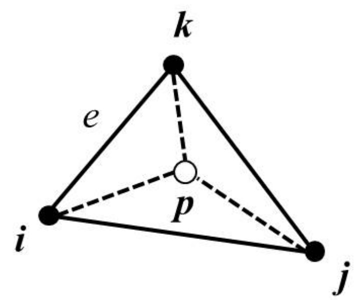

Figure 4.7: Area coordinates.

Let us consider an internal point p (not a node) in Element e, and define $L_i = \frac{\text{area } pjk}{\text{area } ijk}$, $L_j = \frac{\text{area } ipk}{\text{area } ijk}$, and $L_k = \frac{\text{area } ijp}{\text{area } ijk}$

Obviously, $0 \le L_l \le 1$ for $l = i, j$ or k, and $L_i + L_j + L_k = 1$.

Also, $L_i(x_i, y_i) = 1 \quad L_i(x_j, y_j) = 0 \quad L_i(x_k, y_k) = 0$

$L_j(x_i, y_i) = 0 \quad L_j(x_j, y_j) = 1 \quad L_j(x_k, y_k) = 0$

$L_k(x_i, y_i) = 0 \quad L_k(x_j, y_j) = 0 \quad L_k(x_k, y_k) = 1$

As can be seen in Fig. 4.8, triangles p_1ij and p_2ij have the same area (they have the same base and height), therefore L_k = constant is a line parallel to the opposite leg ij. L_k varies linearly between 0 and 1. It can be shown that $[N] = [L_i \quad L_j \quad L_k]$.

Example 7 *show that* $L_i = N_i$.

$$L_i = \frac{\text{area } pjk}{\text{area } ijk} = \frac{\frac{1}{2}\det\begin{bmatrix} 1 & x & y \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix}}{A}$$

$$L_i = \frac{(x_j y_k - x_k y_j) + (y_j - y_k)x + (x_k - x_j)y}{2A}$$

$$L_i = m_{11} + m_{21}x + m_{31}y = N_i \qquad \text{QED.}$$

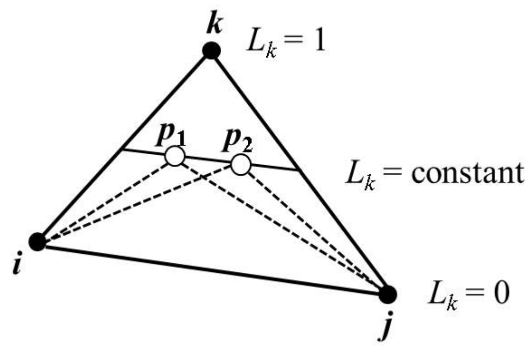

Figure 4.8: Area coordinates with L_k = constant.

4.3.2 Rectangular (quad) C^0-continuous element

Natural coordinates (Fig. 4.9)

We have $-1 \leq r \leq +1$ and $-1 \leq s \leq +1$ with $r = \frac{x-\overline{x}}{a}$ and $s = \frac{y-\overline{y}}{b}$.

Then, $\phi(x,y) = [N_i \quad N_j \quad N_k \quad N_l]\,\{\phi^e\}$

with
$$N_i = \tfrac{1}{4}(1+r)(1-s) \quad \text{and } \{\phi^e\} = \left\{ \begin{array}{c} \phi_i \\ \phi_j \\ \phi_k \\ \phi_l \end{array} \right\}.$$
$$N_j = \tfrac{1}{4}(1+r)(1+s)$$
$$N_k = \tfrac{1}{4}(1-r)(1+s)$$
$$N_l = \tfrac{1}{4}(1-r)(1-s)$$

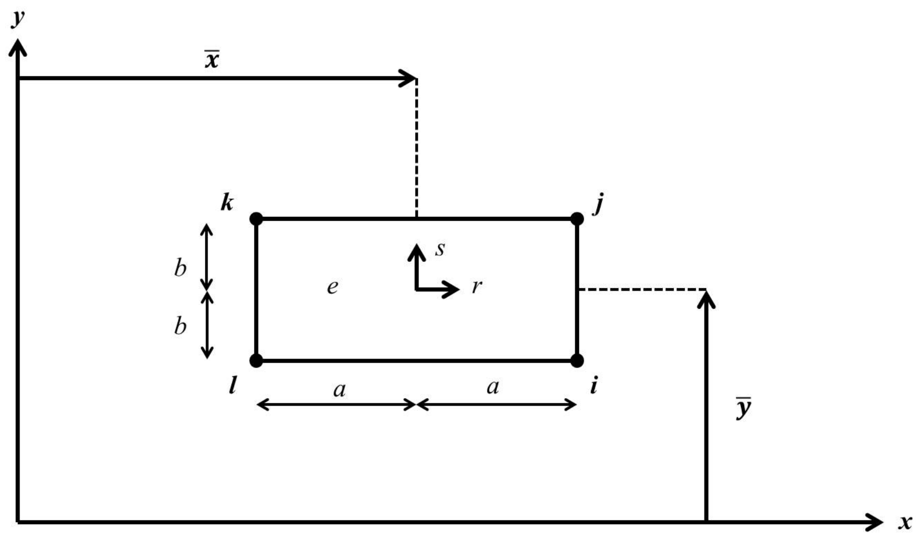

Figure 4.9: Natural 2-D coordinates.

Three rules

1. $N_m = 1$ at Node m, and 0 at other nodes, for $m = i, j, k$ or l.

2. The value of N_m varies from 1 (at Node m) to 0 (at any other node), for $m = i, j, k$ or l.

3. $N_i + N_j + N_k + N_l = 1 \quad \forall r, s$

4.3.3 Curved elements

So far, all the elements considered have had straight edges. Curved elements (with more nodes) may be needed to describe curved boundaries more accurately (or with fewer elements). A mapping can be used between the straight-edged parent element and the curved element (Fig. 4.10). In an isoparametric mapping, the same interpolation functions are used both for the variable of interest (ϕ) and the description of the geometry.

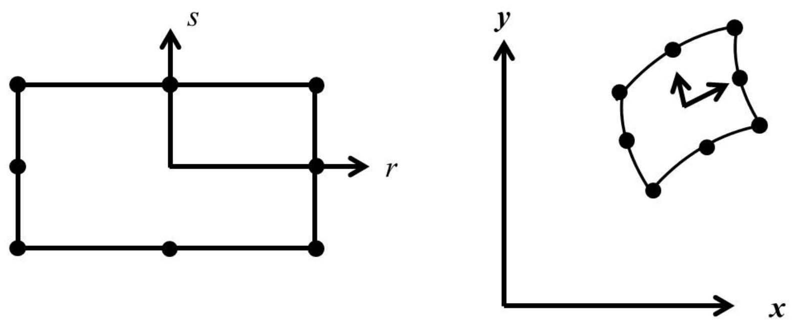

Figure 4.10: Parent element (left), and curved element (right).

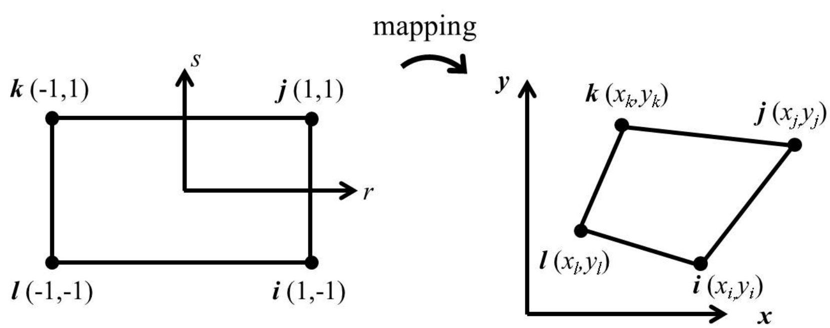

Figure 4.11: Mapping between parent and real geometries.

For simplicity of presentation, let us stick with linear elements (straight edges), as in Fig. 4.11:

To map the geometry, we want $x = [G_i \quad G_j \quad G_k \quad G_l] \begin{Bmatrix} x_i \\ x_j \\ x_k \\ x_l \end{Bmatrix}$ and

$$y = [G_i \quad G_j \quad G_k \quad G_l] \begin{Bmatrix} y_i \\ y_j \\ y_k \\ y_l \end{Bmatrix}.$$

We may want to use the interpolation functions defined previously for the variables of interest, such that $N_m(r,s) = G_m$, for $m = i, j, k$ or l.

When both the geometry and the variables of interest are interpolated with the same functions, an element is called an isoparametric element.

4.4 3-D elements

4.4.1 Four-node, C^0-continuous tetrahedral element (pyramid)

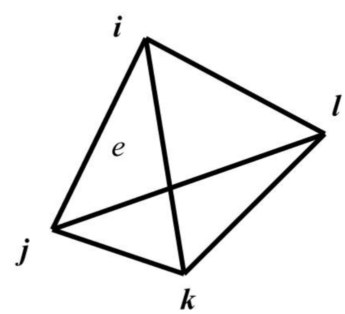

Figure 4.12: Tetrahedral element.

$\phi(x,y,z) = c_1 + c_2x + c_3y + c_4z$

4.4.2 Eight-node, C^0-continuous brick element

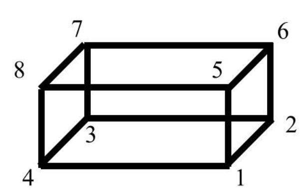

Figure 4.13: Brick element.

$$\phi(x, y, z) = c_1 + c_2x + c_3y + c_4z + c_5xy + c_6yz + c_7zx + c_8xyz$$

4.4.3 Axisymmetric elements

A problem is axisymmetric if the body of interest is a body of revolution AND if the material properties, boundary conditions and loads do not change with θ in a global cylindrical coordinate system attached to the body (Fig. 4.14).

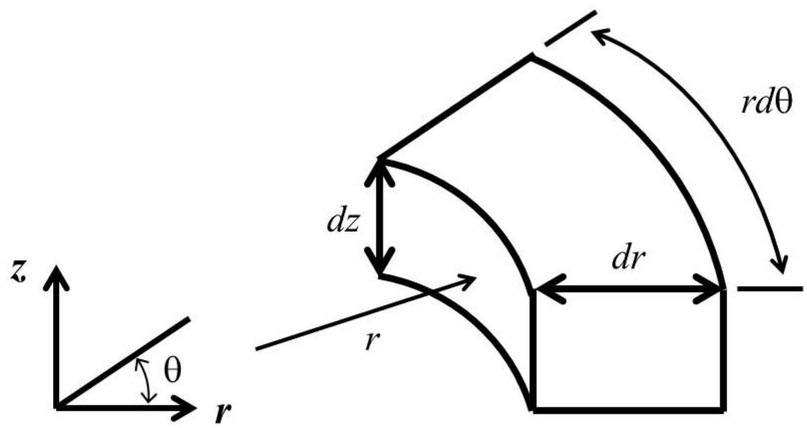

Figure 4.14: Cylindrical coordinate system for axisymmetric problem.

In an axisymmetric problem, the variable of interest is a function of r and z only, therefore the problem is actually two-dimensional. For volume integrations, $dV = rdrd\theta dz = 2\pi rdrdz = 2\pi rdA$.

4.5 Integration formulas

Building a stiffness matrix or a force vector calls for many integrations of the interpolation functions and their derivatives over the element length, surface or volume.

4.5.1 Direct integration

Length coordinates

$\int_l L_i^\alpha L_j^\beta dl = \frac{\alpha!\beta!}{(\alpha+\beta+1)!}l$ where dl is an elemental length between nodes i and j, and l is the length of line between nodes i and j. Exponents α and β must be positive integers. Recall that $n! = n \times (n-1) \times \cdots \times 1$.

Area coordinates

$\int_A L_i^\alpha L_j^\beta L_k^\gamma dA = \frac{\alpha!\beta!\gamma!}{(\alpha+\beta+\gamma+2)!}2A$ where dA is an elemental area of the element, and A is the area of the triangle formed by nodes i, j and k. Exponents α, β and γ must be positive integers.

Such formulas will be used when we build elements in the following chapters.

4.5.2 Numerical integration - Gaussian quadrature

1-D formulas

Consider $I = \int_{x_i}^{x_j} f(x)dx$. First we need to use natural coordinates, such that, when $x = x_i$, $r = -1$ and when $x = x_j$, $r = +1$.

Take $x = x_i + \frac{1}{2}(x_j - x_i)(1+r)$. Then, $dx = \frac{dx}{dr}dr = Jdr$, where J is the Jacobian of the transformation (Note: the notation is not useful in 1-D, but becomes very convenient in 2-D and 3-D). Here, $J = \frac{1}{2}(x_j - x_i)$. Finally, according to the Gauss-Legendre quadrature,

$$I = \int_{x_i}^{x_j} f(x)dx = \int_{-1}^{+1} f(x(r))Jdr = \int_{-1}^{+1} f(r)Jdr \simeq J\sum_{k=1}^{n} f(x(r_k))w_k$$

(index k has nothing to do with the node index).

where w_k are the weight factors, r_k the base points, and n the number of Gauss points (see numerical methods textbook for values of w_k and r_k). A polynomial of degree p is integrated exactly by employing $n = \frac{1}{2}(p+1)$ Gauss points or nearest larger integer. Note that the location of Gauss points has nothing to do with that of the element nodes.

2-D and 3-D formulas:

Similarly in 2-D,

$$I = \int_{-1}^{+1}\int_{-1}^{+1} f(r,s)drds \simeq \sum_{j=1}^{m}\sum_{i=1}^{n} f(r_i,s_j)w_i^r w_j^s$$

and in 3-D,

$$I = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} f(r,s,t)drdsdt \simeq \sum_{k=1}^{l}\sum_{j=1}^{m}\sum_{i=1}^{n} f(r_i,s_j,t_k)w_i^r w_j^s w_k^t.$$

Therefore, in 2-D, with $dxdy = \det[J]\, drds$,

$$[K^e] = \iint_A [B]^T [D] [B]\, dxdy = \int_{-1}^{+1}\int_{-1}^{+1} [B]^T [D] [B] \det[J]\, drds,$$

and in 3-D, with $dxdydz = \det[J]\, drdsdt$,

$$[K^e] = \iiint_V [B]^T [D] [B]\, dxdydz = \int_{-1}^{+1}\int_{-1}^{+1}\int_{-1}^{+1} [B]^T [D] [B] \det[J]\, drdsdt$$

where $[J]$ is the Jacobian matrix of the transformation.

$[J]$ shows up in many other instances, anytime coordinate changes are needed. For example, for each interpolation function

N_i, $\quad \frac{\partial N_i}{\partial r} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial N_i}{\partial z}\frac{\partial z}{\partial r}$ or, in matrix form,

$$\left\{ \begin{array}{c} \frac{\partial N_i}{\partial r} \\ \frac{\partial N_i}{\partial s} \\ \frac{\partial N_i}{\partial t} \end{array} \right\} = \left[\begin{array}{ccc} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \end{array} \right] \left\{ \begin{array}{c} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{array} \right\} = [J] \left\{ \begin{array}{c} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{array} \right\}.$$

Thus, we have

$$\left\{ \begin{array}{c} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{array} \right\} = [J]^{-1} \left\{ \begin{array}{c} \frac{\partial N_i}{\partial r} \\ \frac{\partial N_i}{\partial s} \\ \frac{\partial N_i}{\partial t} \end{array} \right\}.$$

4.6 Example

Assignment 3.

Chapter 5

Stress analysis in 2-D with linear triangular element

Learning Outcomes:

By the end of reading and attending the lecture of this chapter, and solving the listed practice problems, the student should be able, for a 2-D stress analysis with linear triangular elements, *to:*

-Distinguish between plane stress, plane strain and axisymmetric problems

-Explain the full derivation of a finite element (leading to the stiffness matrix and load vectors) from the principle of virtual work

-Write the assemblage stiffness matrix and associated displacement and load vectors

-Explain and apply boundary conditions

-Solve system of equations for displacements or loads

-Determine element resultants

Evidence of Learning:

A student has achieved the learning outcomes when he or she is able to correctly analyze a problem combining several or all of the previous items.

Evaluation Strategies:

Test problems in at-home assignments

Teaching and Learning Activities:

-At-home reading before class

-Lecturing

-In-class practice problems

-At-home reading/reviewing after class

-At-home practice problems

5.1 Plane stress

The plane stress model is appropriate for a thin plate loaded uniformly across its thickness t in a direction parallel to the mid-plate plane (Fig. 5.1). The thickness need not be constant. Plane stress state implies: $\sigma_{zz} = 0$ and $\sigma_{xz} = \sigma_{yz} = 0$.

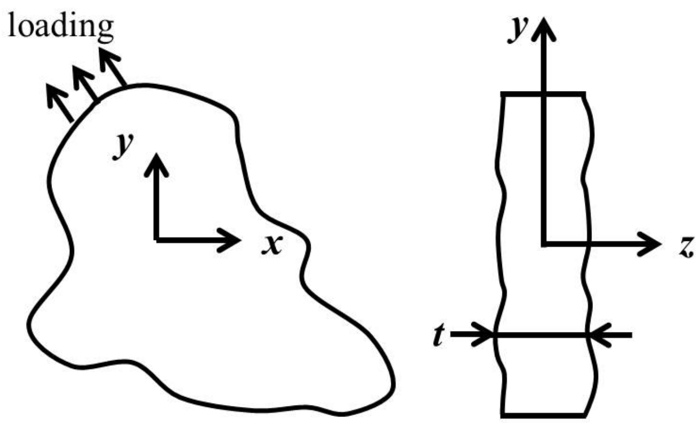

Figure 5.1: Plane stress problem (top view, left; side view, right).

5.1.1 The shape function matrix $[N]$

In plane stress, element nodes have 2 degrees of freedom: the x and y components of the displacements. For a triangular element (see Chapter 4)

$$\left\{ \begin{array}{c} u(x,y) \\ v(x,y) \end{array} \right\} = \left[\begin{array}{cccccc} N_i & 0 & N_j & 0 & N_k & 0 \\ 0 & N_i & 0 & N_j & 0 & N_k \end{array} \right] \left\{ \begin{array}{c} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \end{array} \right\}$$

or $\{U\} = [N]\{U^e\}$.

5.1.2 The strain-nodal displacement matrix $[B]$

Recalling that for 2-D small deformations,

$\varepsilon_{xx} = \frac{\partial u}{\partial x}$, $\varepsilon_{yy} = \frac{\partial v}{\partial y}$ and $2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$. In matrix form,

$$\left\{ \begin{array}{c} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{array} \right\} = \left[\begin{array}{cc} \frac{\partial \cdot}{\partial x} & 0 \\ 0 & \frac{\partial \cdot}{\partial y} \\ \frac{\partial \cdot}{\partial y} & \frac{\partial \cdot}{\partial x} \end{array} \right] \left\{ \begin{array}{c} u \\ v \end{array} \right\} \text{ or } \{\varepsilon\} = [L]\{U\}.$$

Finally, $\{\varepsilon\} = [L][N]\{U^e\} = [B]\{U^e\}$,

$$\text{with } [B] = [L][N] = \left[\begin{array}{cccccc} \frac{\partial N_i}{\partial x} & 0 & \frac{\partial N_j}{\partial x} & 0 & \frac{\partial N_k}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 & \frac{\partial N_j}{\partial y} & 0 & \frac{\partial N_k}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial y} & \frac{\partial N_j}{\partial x} & \frac{\partial N_k}{\partial y} & \frac{\partial N_k}{\partial x} \end{array} \right].$$

If, as we found in Chapter 4 for a triangular element,

$N_i(x, y) = m_{11} + m_{21}x + m_{31}y$
$N_j(x, y) = m_{12} + m_{22}x + m_{32}y$
$N_k(x, y) = m_{13} + m_{23}x + m_{33}y$

$$\text{then } [B] = \left[\begin{array}{cccccc} m_{21} & 0 & m_{22} & 0 & m_{23} & 0 \\ 0 & m_{31} & 0 & m_{32} & 0 & m_{33} \\ m_{31} & m_{21} & m_{32} & m_{22} & m_{33} & m_{23} \end{array} \right].$$

Since the m_{ij}s are known functions of the nodal coordinates, $[B]$ is known. Note that in this case, $[B]$ is composed of constant entries. In other words, in a 2-D triangular linear element, strains are constant over an element, and so are stresses. If a better resolution is needed, more elements or different elements (of different shape and/or higher order) are required.

5.1.3 Constitutive relationship

For plane stress, the constitutive relationship for a linear elastic material $\{\sigma\} = [D](\{\varepsilon\} - \{\varepsilon_0\}) + \{\sigma_0\}$ is written with $\{\sigma\} = \left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array} \right\}$,

$$\{\sigma_0\} = \left\{ \begin{array}{c} \sigma_{xx0} \\ \sigma_{yy0} \\ \sigma_{xy0} \end{array} \right\}, \{\varepsilon_0\} = \left\{ \begin{array}{c} \varepsilon_{xx0} \\ \varepsilon_{yy0} \\ 2\varepsilon_{xy0} \end{array} \right\} \text{ and } [D] = \frac{E}{1-\nu^2} \left[\begin{array}{ccc} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{array} \right].$$

If the self-strain vector $\{\varepsilon_0\}$ is a result of temperature change ΔT, then $\{\varepsilon_0\}$ is given by $\{\varepsilon_0\} = \left\{ \begin{array}{c} \alpha_t \Delta T \\ \alpha_t \Delta T \\ 0 \end{array} \right\}$ (no shear in isotropic materials) where α_t is the coefficient of thermal expansion. Note that ε_{zz} is not zero in general: $\varepsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) + \varepsilon_{zz0}$.

5.1.4 Finite element formulation (triangular element)

Recalling the principle of virtual work written for one element:

$\int_{V^e} \{\delta\varepsilon\}^T \{\sigma\} dV = \int_{V^e} \{\delta U\}^T \{b\} dV + \int_{A^e} \{\delta U\}^T \{s\} dA + \sum_{p=1}^{p=N} \{\delta U\}^T \{f_p\}$, $\forall \{\delta U\}, \{\delta\varepsilon\}$.

Using $\{\varepsilon\} = [B]\{U^e\}$, then $\{\delta\varepsilon\} = \delta([B]\{U^e\}) = [B]\{\delta U^e\}$ because $[B]$ is independent of $\{U^e\}$, and $\{\delta\varepsilon\}^T = \{\delta U^e\}^T [B]^T$.

Similarly, $\{U\} = [N]\{U^e\}$, leading to $\{\delta U\}^T = \{\delta U^e\}^T [N]^T$. Therefore, the PVW is equivalent to

$\int_{V^e} [B]^T \{\sigma\} dV - \int_{V^e} [N]^T \{b\} dV$
$- \int_{A^e} [N]^T \{s\} dA - \sum_{p=1}^{p=N} [N]^T \{f_p\} = \{0\}$,

but since $\{\sigma\} = [D](\{\varepsilon\} - \{\varepsilon_0\}) + \{\sigma_0\}$ and $\{\varepsilon\} = [B]\{U^e\}$,

$(\int_{V^e} [B]^T [D][B] dV)\{U^e\} - \int_{V^e} [B]^T [D]\{\varepsilon_0\} dV$
$+ \int_{V^e} [B]^T \{\sigma_0\} dV - \int_{V^e} [N]^T \{b\} dV$
$- \int_{A^e} [N]^T \{s\} dA - \sum_{p=1}^{p=N} [N]^T \{f_p\} = \{0\}$.

Finally, $[K^e]\{U^e\} = \{f^e\}$ where

$[K^e] = \int_{V^e} [B]^T [D][B] dV$ is the element stiffness matrix and $\{f^e\}$ the nodal force vector is made of five components:

$$\{f^e\} = \begin{array}{c} \int_{V^e} [B]^T [D]\{\varepsilon_0\} dV - \int_{V^e} [B]^T \{\sigma_0\} dV \\ + \int_{V^e} [N]^T \{b\} dV + \int_{A^e} [N]^T \{s\} dA + \sum_{p=1}^{p=N} [N]^T \{f_p\} \end{array}, \text{ or}$$

$\{f^e\} = \{f^e_{\varepsilon 0}\} - \{f^e_{\sigma 0}\} + \{f^e_b\} + \{f^e_s\} + \{f^e_p\}$.

Note that the previous derivation is general and not limited to 2-D cases. It is done here for convenience.

5.1.5 The element stiffness matrix $[K^e]$

From above, $[K^e] = \int_{V^e} [B]^T [D][B] dV$ where both $[D]$ and $[B]$ (in our case) are composed of constants, and $dV = tdxdy$.

Therefore, $[K^e] = [B]^T [D][B] \int_{A^e} tdxdy$. For reasonably small elements, t may be taken as a constant average value. Then, $[K^e] = [B]^T [D][B] tA$. Note that in our case, $[K^e]$ is a 6x6 matrix because $[B]^T$ is 6x3, $[D]$ is 3x3 and $[B]$ is 3x6. This is consistent with the fact that there are 6 nodal displacements per triangular element.

Example 8 *determine $[K^e]$ in the case shown in Fig. 5.2:*

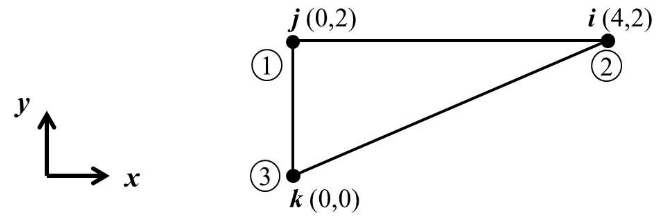

Figure 5.2: Triangular element.

Steel plate: $E = 30 \times 10^6$ psi
$\nu = 0.3$
$t = 0.25$ in (constant)

$$A = \frac{1}{2}\det\begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} = \frac{1}{2}\det\begin{bmatrix} 1 & 4 & 2 \\ 1 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix} = 4 \text{ in}^2$$

$$m_{21} = (y_j - y_k)/2A = \frac{2-0}{2\times 4} = 0.25 \text{ in}^{-1} \qquad m_{31} = (x_k - x_j)/2A = \frac{0-0}{2\times 4} = 0$$
$$m_{22} = (y_k - y_i)/2A = \frac{0-2}{2\times 4} = -0.25 \text{ in}^{-1} \qquad m_{32} = (x_i - x_k)/2A = 0.50 \text{ in}^{-1}$$
$$m_{23} = (y_i - y_j)/2A = 0 \qquad m_{33} = (x_j - x_i)/2A = -0.50 \text{ in}^{-1}$$

$$\text{Then, } [B] = \begin{bmatrix} 0.25 & 0 & -0.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.50 & 0 & -0.50 \\ 0 & 0.25 & 0.50 & -0.25 & -0.50 & 0 \end{bmatrix} \text{in}^{-1}.$$

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \begin{bmatrix} 33.0 & 9.89 & 0 \\ 9.89 & 33.0 & 0 \\ 0 & 0 & 11.5 \end{bmatrix} \times 10^6 \text{ psi (lbf/in}^2)$$

Finally,

$$[K^e] = \begin{bmatrix} 2.06 & 0 & -2.06 & 1.24 & 0 & -1.24 \\ & 0.72 & 1.44 & -0.72 & -1.44 & 0 \\ & & 4.94 & -2.68 & -2.88 & 1.24 \\ & & & 8.97 & 1.44 & -8.25 \\ & & & & 2.88 & 0 \\ \text{Sym} & & & & & 8.25 \end{bmatrix} \times 10^6 \text{ lbf/in.}$$

5.1.6 The element nodal force vector $\{f^e\}$

Self-strain

$$\{f^e_{\varepsilon 0}\} = \int_{V^e} [B]^T [D] \{\varepsilon_0\}\, dV = \int_{A^e} [B]^T [D] \{\varepsilon_0\}\, t dx dy.$$

Considering thermal strains only, with α_t and ΔT constant,
then $\{f^e_{\varepsilon 0}\} = [B]^T [D] \{\varepsilon_0\} tA$.
$\{f^e_{\varepsilon 0}\}$ is 6x1 because $[B]^T$ is 6x3, $[D]$ is 3x3 and $\{\varepsilon_0\}$ is 3x1.

Example 9 *(continued from above): determine* $\{f^e_{\varepsilon 0}\}$ *if* $\alpha_t = 6.0 \times 10^{-6}$ *in/(in.F) and the temperature increases by 150 F.*

$$\{\varepsilon_0\} = \left\{ \begin{array}{c} \alpha_t \Delta T \\ \alpha_t \Delta T \\ 0 \end{array} \right\} = \left\{ \begin{array}{c} 900 \\ 900 \\ 0 \end{array} \right\} \times 10^{-6} \text{ in/in.}$$

$$\{f^e_{\varepsilon 0}\} = \left\{ \begin{array}{c} 9,650 \\ 0 \\ -9,650 \\ 19,300 \\ 0 \\ -19,300 \end{array} \right\} \text{lbf.}$$

Prestresses

$\{f^e_{\sigma 0}\} = \int_{V^e} [B]^T \{\sigma_0\} dV = \int_{A^e} [B]^T \{\sigma_0\} tdxdy = [B]^T \{\sigma_0\} tA$.

$$\{f^e_{\sigma 0}\} \text{ is 6x1, and } \{\sigma_0\} = \left\{ \begin{array}{c} \sigma_{xx0} \\ \sigma_{yy0} \\ \sigma_{xy0} \end{array} \right\}.$$

Body forces

$$\{f^e_b\} = \int_{V^e} [N]^T \{b\} dV = \int_{A^e} \left[\begin{array}{cc} N_i & 0 \\ 0 & N_i \\ N_j & 0 \\ 0 & N_j \\ N_k & 0 \\ 0 & N_k \end{array} \right] \left\{ \begin{array}{c} b_x \\ b_y \end{array} \right\} tdxdy.$$

Recalling that for a triangular element, $N_l = L_l$ for $l = i, j$ or k, where L_l is an area coordinate,

$$\{f^e_b\} = \left\{ \begin{array}{c} \int_{A^e} L_i b_x tdxdy \\ \int_{A^e} L_i b_y tdxdy \\ \int_{A^e} L_j b_x tdxdy \\ \int_{A^e} L_j b_y tdxdy \\ \int_{A^e} L_k b_x tdxdy \\ \int_{A^e} L_k b_y tdxdy \end{array} \right\}$$

with $\int_{A^e} L_i b_x t dx dy = \frac{1!0!0!}{(1+0+0+2)!} b_x t 2A = \frac{1}{3} b_x t A$.

Finally, $\{f_b^e\} = \frac{tA}{3} \left\{ \begin{array}{c} b_x \\ b_y \\ b_x \\ b_y \\ b_x \\ b_y \end{array} \right\}$ (if b_x and b_y are constant).

Surface tractions

$$\{f_s^e\} = \int_{A^e} [N]^T \{s\} dA = \int_{A^e} \begin{bmatrix} N_i & 0 \\ 0 & N_i \\ N_j & 0 \\ 0 & N_j \\ N_k & 0 \\ 0 & N_k \end{bmatrix} \left\{ \begin{array}{c} s_x \\ s_y \end{array} \right\} dA.$$

Surface tractions are only present at the surface of the structure. Let us consider an element e with nodes i and j (but not k) on the global boundary (Fig. 5.3).

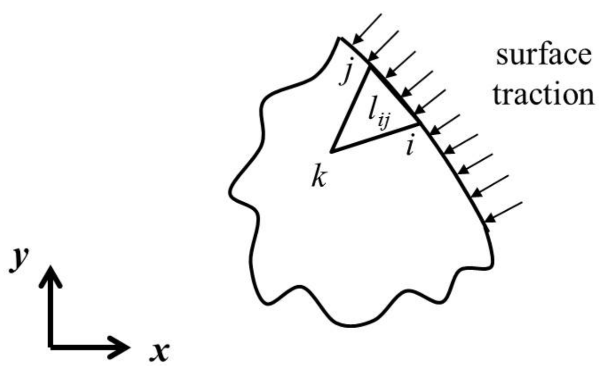

Figure 5.3: Surface traction.

A surface traction is assumed to act on leg ij. Then, $dA = tdl$, and $N_k = L_k = 0$ on leg ij.

$$\{f_s^e\} = \begin{Bmatrix} \int_{l_{ij}} L_i s_x t dl \\ \int_{l_{ij}} L_i s_y t dl \\ \int_{l_{ij}} L_j s_x t dl \\ \int_{l_{ij}} L_j s_y t dl \\ 0 \\ 0 \end{Bmatrix}.$$

If s_x and s_y are assumed constant, then $\{f_s^e\} = \frac{tl_{ij}}{2} \begin{Bmatrix} s_x \\ s_y \\ s_x \\ s_y \\ 0 \\ 0 \end{Bmatrix}$ for leg ij on the global boundary.

Example 10 *determine* $\{f_s^e\}$ *(dimensions from Fig. 5.2)*

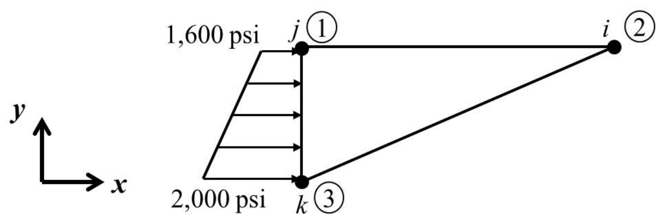

Figure 5.4: Surface traction on triangular element.

$s_y = 0$
s_x is not constant over leg jk. Effective $s_x = \frac{1{,}600+2{,}000}{2} = 1,800$ psi.

$$\{f_s^e\} = \frac{tl_{jk}}{2} \begin{Bmatrix} 0 \\ 0 \\ s_x \\ s_y \\ s_x \\ s_y \end{Bmatrix} = \frac{0.25\times 2}{2} \begin{Bmatrix} 0 \\ 0 \\ 1,800 \\ 0 \\ 1,800 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 450 \\ 0 \\ 450 \\ 0 \end{Bmatrix} \text{ lbf.}$$

(This is with respect to the local node ordering i, j, k)

Point loads

$$\left\{f_p^e\right\} = \sum_{p=1}^{p=N} [N]^T \{f_p\} = \sum_{p=1}^{p=N} \begin{bmatrix} N_i & 0 \\ 0 & N_i \\ N_j & 0 \\ 0 & N_j \\ N_k & 0 \\ 0 & N_k \end{bmatrix} \left\{ \begin{array}{c} f_{px} \\ f_{py} \end{array} \right\}$$

$$\left\{f_p^e\right\} = \sum_{p=1}^{p=N} \left\{ \begin{array}{c} N_i(x_p, y_p) f_{px} \\ N_i(x_p, y_p) f_{py} \\ N_j(x_p, y_p) f_{px} \\ N_j(x_p, y_p) f_{py} \\ N_k(x_p, y_p) f_{px} \\ N_k(x_p, y_p) f_{py} \end{array} \right\}.$$

Example 11 *determine* $\left\{f_p^e\right\}$ *for a point load acting at coordinates* $(0.6, 1.6)$ *for the element above, with* $f_{px} = 1,500$ *lbf and* $f_{py} = -2,300$ *lbf.*

Since $\begin{array}{l} N_i(x,y) = m_{11} + m_{21}x + m_{31}y \\ N_j(x,y) = m_{12} + m_{22}x + m_{32}y \\ N_k(x,y) = m_{13} + m_{23}x + m_{33}y \end{array}$, calculation of all m_{ij}s is needed.

The only ones that have not been calculated yet are:

$m_{11} = (x_j y_k - x_k y_j)/2A = 0$

$m_{12} = (x_k y_i - x_i y_k)/2A = 0$

$m_{13} = (x_i y_j - x_j y_i)/2A = 1$

Therefore, at $x = 0.6$ and $y = 1.6$,

$N_i = 0 + 0.25 \times 0.6 + 0 \times 1.6 = 0.15$

$N_j = 0 - 0.25 \times 0.6 + 0.5 \times 1.6 = 0.65$

$N_k = 1 + 0 \times 0.6 - 0.25 \times 1.6 = 0.20$

$$\left\{f_p^e\right\} = \left\{ \begin{array}{c} 225 \\ -345 \\ 975 \\ -1,495 \\ 300 \\ -460 \end{array} \right\} \text{lbf.}$$

5.1.7 Assemblage

Example 12 *see Fig. 5.5*

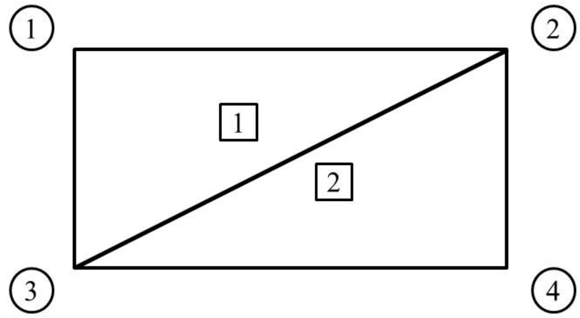

Figure 5.5: Assemblage of two triangular elements.

Connectivity table:

Element #	Node i	Node j	Node k
1	2	1	3
2	3	4	2

For Element 1, $$\left[K^{(1)}\right] = \begin{bmatrix} K_{22}^{(1)} & K_{21}^{(1)} & K_{23}^{(1)} \\ K_{12}^{(1)} & K_{11}^{(1)} & K_{13}^{(1)} \\ K_{32}^{(1)} & K_{31}^{(1)} & K_{33}^{(1)} \end{bmatrix} \quad \begin{matrix} u_2 \\ v_2 \\ u_1 \\ v_1 \\ u_3 \\ v_3 \end{matrix}$$

For Element 2, $$\left[K^{(2)}\right] = \begin{bmatrix} K_{33}^{(2)} & K_{34}^{(2)} & K_{32}^{(2)} \\ K_{43}^{(2)} & K_{44}^{(2)} & K_{42}^{(2)} \\ K_{23}^{(2)} & K_{24}^{(2)} & K_{22}^{(2)} \end{bmatrix} \quad \begin{matrix} u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_2 \\ v_2 \end{matrix}$$

By assemblage, with $\{U^a\}^T = [u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4]$.
$[K^a]$ is first zeroed out, and then entries are added.

$$[K^a] = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & K_{13}^{(1)} & 0 \\ K_{21}^{(1)} & K_{22}^{(1)} + K_{22}^{(2)} & K_{23}^{(1)} + K_{23}^{(2)} & K_{24}^{(2)} \\ K_{31}^{(1)} & K_{32}^{(1)} + K_{32}^{(2)} & K_{33}^{(1)} + K_{33}^{(2)} & K_{34}^{(2)} \\ 0 & K_{42}^{(2)} & K_{43}^{(2)} & K_{44}^{(2)} \end{bmatrix}.$$ $[K^a]$ is symmetric.

$$\{f^a\} = \left\{ \begin{array}{c} f_1^{(1)} \\ f_2^{(1)} + f_2^{(2)} \\ f_3^{(1)} + f_3^{(2)} \\ f_4^{(2)} \end{array} \right\}.$$

5.1.8 Prescribed displacements - Solution

$[K^a]$ is singular, therefore prescribed displacements must be included for solution. Finally, the system to be solved is $[K]\{U\} = \{f\}$, from which $\{U\} = [K]^{-1}\{f\}$.

5.1.9 Element resultants

From the computed displacements $\{U^e\}$, one can determine the strains and stresses in the element.

$\{\varepsilon\} = [B]\{U^e\}$. For a triangular element, the average strains across the element are:

$$\begin{array}{l} \overline{\varepsilon_{xx}} = m_{21}u_i + m_{22}u_j + m_{23}u_k \\ \overline{\varepsilon_{yy}} = m_{31}v_i + m_{32}v_j + m_{33}v_k \\ \overline{2\varepsilon_{xy}} = m_{31}u_i + m_{21}v_i + m_{32}u_j + m_{22}v_j + m_{33}u_k + m_{23}v_k \end{array}.$$

The average stresses across the element are:

$\{\overline{\sigma}\} = [D]([B]\{U^e\} - \{\varepsilon_0\}) + \{\sigma_0\}$.

5.2 Plane strain

The plane strain model is appropriate when a long prismatic member of constant cross section is held between two fixed rigid planes (Fig. 5.6). Plane strain state implies: $\varepsilon_{zz} = 0$ and $\varepsilon_{xz} = \varepsilon_{yz} = 0$.

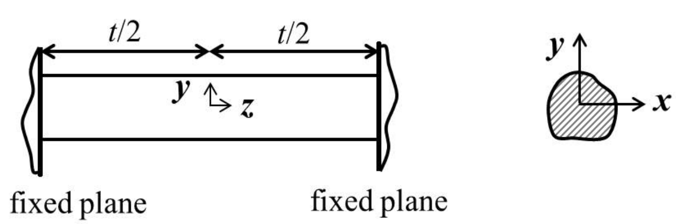

Figure 5.6: Plane strain problem (side view, left; end view, right).

All the results for plane stress apply, except that the constitutive relationship is changed: $\{\sigma\} = [D]\left(\{\varepsilon\} - \{\varepsilon_0\}\right) + \{\sigma_0\}$

with $[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$.

For a temperature change ΔT, $\{\varepsilon_0\} = \left\{ \begin{array}{c} (1+\nu)\alpha_t \Delta T \\ (1+\nu)\alpha_t \Delta T \\ 0 \end{array} \right\}$.

Note that $\sigma_{zz} \neq 0 = \nu(\sigma_{xx} + \sigma_{yy}) + \sigma_{zz0} - E\varepsilon_{zz0}$ with $\varepsilon_{zz0} = \alpha_t \Delta T$ for thermal strains.

5.3 Axisymmetric problems

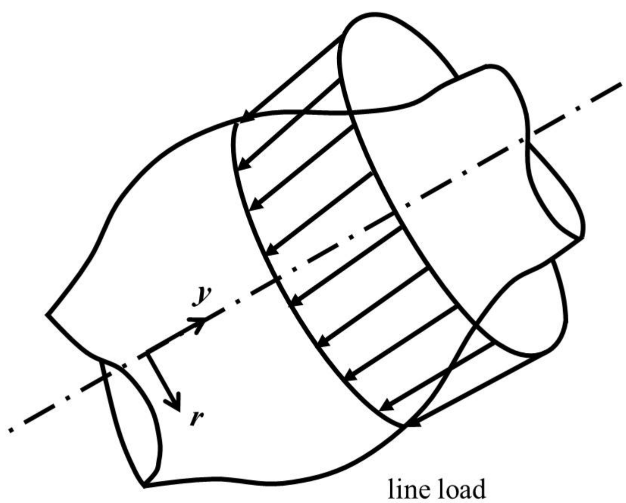

Figure 5.7: Axisymmetric problem.

Strains considered (Fig. 5.7): $\{\varepsilon\}^T = [\varepsilon_{rr} \quad \varepsilon_{\theta\theta} \quad \varepsilon_{zz} \quad 2\varepsilon_{rz}]$,
stresses considered: $\{\sigma\}^T = [\sigma_{rr} \quad \sigma_{\theta\theta} \quad \sigma_{zz} \quad \sigma_{rz}]$,
with

$$\begin{aligned} \varepsilon_{rr} &= \frac{\partial u}{\partial r}, \\ \varepsilon_{\theta\theta} &= \frac{u}{r} \\ \varepsilon_{zz} &= \frac{\partial v}{\partial z} \\ 2\varepsilon_{rz} &= \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \end{aligned}$$

where u and v are the radial and axial displacements, respectively.

In matrix form, $\{\varepsilon\} = [L]\{U\}$ with $\{U\} = \left\{ \begin{array}{c} u \\ v \end{array} \right\}$ and $[L] = \begin{bmatrix} \frac{\partial \cdot}{\partial r} & 0 \\ \frac{1}{r} & 0 \\ 0 & \frac{\partial \cdot}{\partial z} \\ \frac{\partial \cdot}{\partial z} & \frac{\partial \cdot}{\partial r} \end{bmatrix}$.

With the same shape function matrix $[N]$ and the same nodal displacements vector $\{U^e\}$ as in plane stress, $\{\varepsilon\} = [B]\{U^e\}$ with $[B] = [L][N]$.

$$[B] = \begin{bmatrix} m_{21} & 0 & m_{22} & 0 & m_{23} & 0 \\ \frac{N_i}{r} & 0 & \frac{N_j}{r} & 0 & \frac{N_k}{r} & 0 \\ 0 & m_{31} & 0 & m_{32} & 0 & m_{33} \\ m_{31} & m_{21} & m_{32} & m_{22} & m_{33} & m_{23} \end{bmatrix}.$$

Note that $\frac{N_l}{r}$ is not a constant, but a function of r and z. The constitutive relationship is $\{\sigma\} = [D](\{\varepsilon\} - \{\varepsilon_0\}) + \{\sigma_0\}$

with $[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$.

For a temperature change ΔT, $\{\varepsilon_0\} = \left\{ \begin{array}{c} \alpha_t \Delta T \\ \alpha_t \Delta T \\ \alpha_t \Delta T \\ 0 \end{array} \right\}$. Determination of the element stiffness matrix, load vectors, etc... is done using numerical integration, because the integrands are now functions of r and z.

Example 13 $[K^e] = \int_{V^e} [B]^T [D] [B] \, dV = 2\pi \int_{A^e} [B]^T [D] [B] \, r dr dz.$

5.4 Example

Assignment 4.

Chapter 6

Finite elements for plates and shells

Learning Outcomes:
By the end of reading and attending the lecture of this chapter, the student should be able to:
-Explain the theory of thin plate bending
-Explain the derivation of a plate element using the principle of virtual work
Evidence of Learning:
A student has achieved the learning outcomes when he or she is able to reproduce the knowledge listed above.
Teaching and Learning Activities:
-At-home reading before class
-Lecturing
-In-class practice problems
-At-home reading/reviewing after class
-At-home practice problems

Plates are flat; shells are curved plates.

6.1 Introduction - Bending of thin plates

6.1.1 Geometry

As shown in Fig. 6.1, the plate's surfaces are at $z = \pm t/2$ and its midsurface at $z = 0$. The assumed basic geometry of the plate is as follows: $t \ll b$ and $t \ll c$ (if $t > \simeq 0.2b$ or $0.2c$, thick plate). The deflection w due to q is assumed to be much less than the thickness: $w/t \ll 1$.

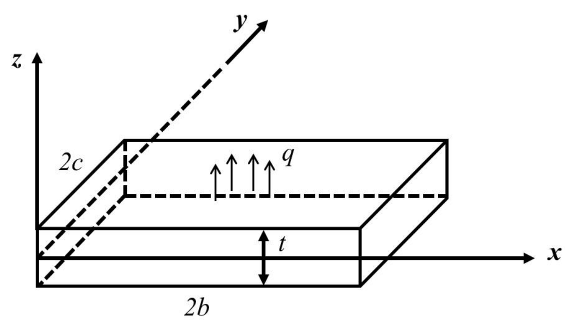

Figure 6.1: Thin plate geometry.

6.1.2 Kirchhoff assumptions

They generalize ideas taken from beam bending. Consider a differential slice of plate (Fig. 6.2). Loading q causes the plate to deform laterally in the z-direction, and the deflection w of point P is assumed to be a function of x and y only. That is, $w = w(x, y)$, and the plate does not stretch in the z-direction.

The Kirchhoff assumptions are as follows:

1. Normals remain normals. This implies that shear strains ε_{yz} and ε_{xz} are zero. However, $\varepsilon_{xy} \neq 0$: the plate may twist in its plane.

2. Thickness changes can be neglected, and normals undergo no extension, i.e. $\varepsilon_{zz} = 0$.

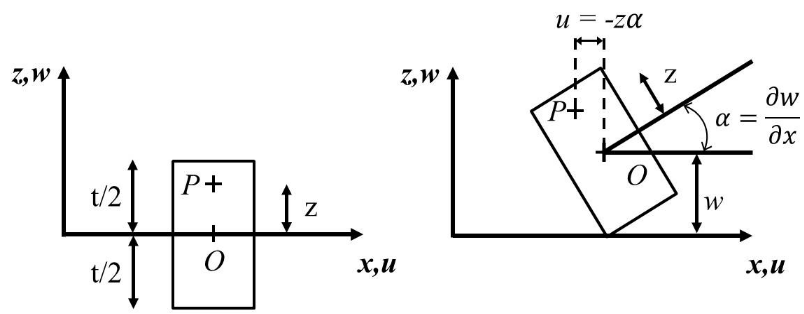

Figure 6.2: Differential slice of plate, before loading (left), and after loading (right). Similar displacements in $y-z$ plane.

3. Normal stress σ_{zz} has no effect on in-plane strains ε_{xx} and ε_{yy}, and is considered negligible.

4. Membrane or in-plane forces are neglected here (they can be superimposed at a later stage). Therefore, the in-plane deformations in the x- and y-directions at the midsurface are assumed to be zero, i.e. $u(x,y) = v(x,y) = 0$.

According to the above assumptions, any point P has displacements

$u = -z\alpha = -z(\frac{\partial w}{\partial x})$ in the x-direction

$v = -z\beta = -z(\frac{\partial w}{\partial y})$ in the y-direction

where $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ are the slopes at the midsurface.

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2}, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z} \equiv 0$$

$$2\varepsilon_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \equiv 0, \quad 2\varepsilon_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \equiv 0, \quad 2\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y}$$

Because in small deflection theory, the square of a slope may be regarded as negligible (recall that in 2-D, $\frac{1}{r_x} = \frac{\frac{d^2v}{dx^2}}{[1+(\frac{dv}{dx})^2]^{3/2}} \simeq \frac{d^2v}{dx^2}$, where r_x is the radius of curvature), the curvatures $(\frac{1}{r})$ of the plate are given simply as:

$$\kappa_{xx} = -\frac{\partial^2 w}{\partial x^2}, \quad \kappa_{yy} = -\frac{\partial^2 w}{\partial y^2} \quad \text{and} \quad \kappa_{xy} = -\frac{\partial^2 w}{\partial x \partial y}.$$

Therefore we have:

$$\varepsilon_{xx} = z\,\kappa_{xx} \quad .$$
$$\varepsilon_{yy} = z\,\kappa_{yy}$$
$$2\varepsilon_{xy} = 2z\,\kappa_{xy}$$

6.1.3 Constitutive relationships

Plane stress state can be used (Assumption 3), and then:

$$\left\{\begin{array}{c}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{array}\right\} = \frac{E}{1-\nu^2}\left[\begin{array}{ccc}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{array}\right]\left\{\begin{array}{c}z\,\kappa_{xx}\\ z\,\kappa_{yy}\\ 2z\,\kappa_{xy}\end{array}\right\}.$$

These expressions clearly show that the stresses vanish at the midsurface, and vary linearly over the plate thickness (Fig. 6.3).

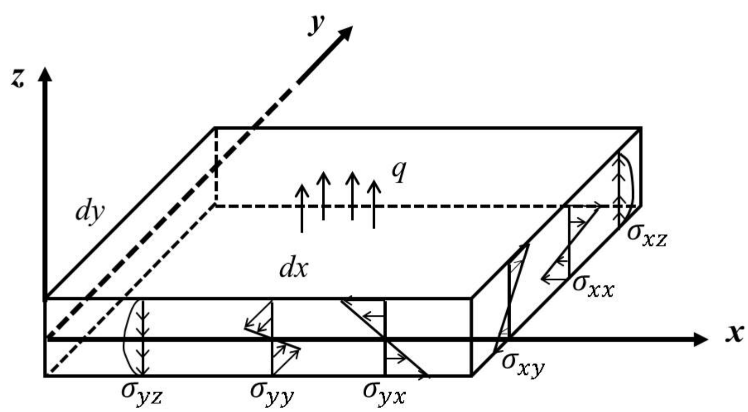

Figure 6.3: Differential plate element with stresses shown on edges.

The stresses distributed over the side surfaces of the plate, while producing no net force, result in bending and twisting moments. These moment resultants per unit length (N.m/m ≡ N) are denoted M_{xx}, M_{yy} and M_{xy} with

$$M_{xx} = \int_{-t/2}^{t/2} z\,\sigma_{xx} dz \qquad .$$

$$M_{yy} = \int_{-t/2}^{t/2} z\,\sigma_{yy} dz$$

$$M_{xy} = \int_{-t/2}^{t/2} z\,\sigma_{xy} dz$$

$M_{xy} = M_{yx}$ because $\sigma_{xy} = \sigma_{yx}$

Finally, $\left\{\begin{array}{c}M_{xx}\\ M_{yy}\\ M_{xy}\end{array}\right\} = D\left[\begin{array}{ccc}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{array}\right]\left\{\begin{array}{c}\kappa_{xx}\\ \kappa_{yy}\\ 2\kappa_{xy}\end{array}\right\}$ with $D = \frac{Et^3}{12(1-\nu^2)}$: bending rigidity of the plate.

The maximum stresses on each edge of the plate are located at the top or bottom at $z = \pm\, t/2$:

$\sigma_{xx\,\max} = \frac{6M_{xx}}{t^2}$, $\sigma_{yy\,\max} = \frac{6M_{yy}}{t^2}$, and $\sigma_{xy\,\max} = \frac{6M_{xy}}{t^2}$.

Note that these formulas are similar to the formula for beams, where $\sigma_{xx} = \frac{M_{xx}\,c}{I}$ when applied to a unit width of plate , with $c = t/2$.

Introducing the shearing forces (or transverse shear line loads, Fig. 6.4),

$$Q_x = \int_{-t/2}^{t/2} \sigma_{xz} dz$$

$$Q_y = \int_{-t/2}^{t/2} \sigma_{yz} dz$$

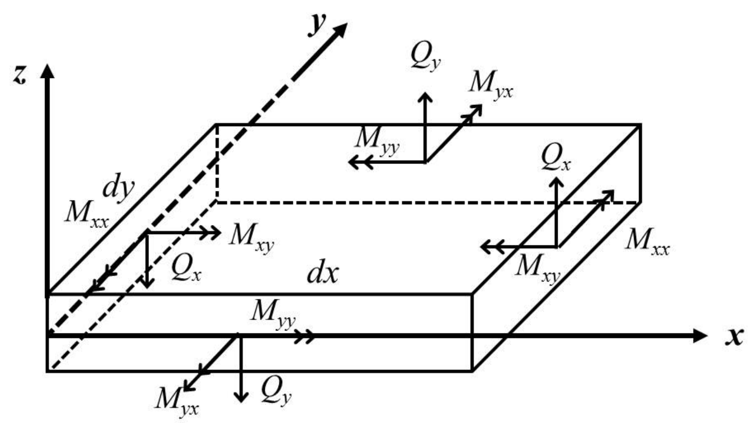

Figure 6.4: Differential plate element with differential moments and forces.

6.1.4 Equilibrium equations

From Fig. 6.5, the equilibrium in the z-direction yields:

$\frac{\partial Q_x}{\partial x} dxdy + \frac{\partial Q_y}{\partial y} dxdy + qdxdy = 0$

Equilibrium of moments about the x-axis yields:

$-(\frac{\partial M_{xy}}{\partial x} dx)dy - (\frac{\partial M_{yy}}{\partial y} dy)dx + Q_y dxdy = 0$

Equilibrium of moments about the y-axis yields:

$(\frac{\partial M_{yx}}{\partial y} dy)dx + (\frac{\partial M_{xx}}{\partial x} dx)dy - Q_x dxdy = 0$

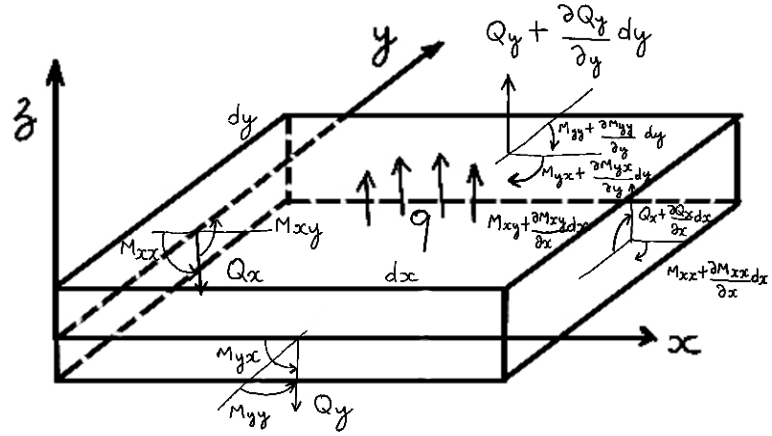

Figure 6.5: Equilibrium of the elemental plate.

Then, $\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0$, therefore $Q_x = -D\frac{\partial}{\partial x}(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2})$,

$$\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \qquad Q_y = -D\frac{\partial}{\partial y}(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2})$$

$$\frac{\partial M_{yy}}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_y = 0$$

and finally, $D(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}) = q$.

In concise form, $\nabla^4 w = q/D$. These are the governing equations of plate deflection.

Note: the theory of thin plates neglects the effects on bending of σ_{zz}, as well as $\varepsilon_{xz} = \frac{\sigma_{xz}}{G}$ and $\varepsilon_{yz} = \frac{\sigma_{yz}}{G}$. However, the shearing forces Q_x and Q_y resulting from σ_{xz} and σ_{yz} are not negligible. In fact, they are of the same order of magnitude as the lateral loads and moments.

6.2 Finite element formulation (rectangular element)

6.2.1 Element type

We will consider the 12-degree of freedom (d.o.f) flat-plate bending element shown in Fig. 6.6:

Each node has 3 d.o.f.: a transverse displacement w in the z-direction, a rotation θ_x about the x-axis and a rotation θ_y about the y-axis. The rotations

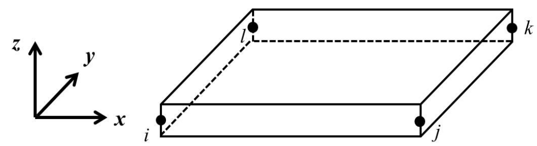

Figure 6.6: Rectangular shell element.

are related to the transverse displacement by $\theta_x = \frac{\partial w}{\partial y}$ and $\theta_y = -\frac{\partial w}{\partial x}$ (a negative w displacement is required to produce a positive rotation about the y-axis). The displacement vector for the element is:

$$\{U^e\}^T = [w_i \quad \theta_{xi} \quad \theta_{yi} \quad w_j \quad \theta_{xj} \quad \theta_{yj} \quad w_k \quad \theta_{xk} \quad \theta_{yk} \quad w_l \quad \theta_{xl} \quad \theta_{yl}] \, .$$

6.2.2 Displacement function

Because there are 12 d.o.f., we select a 12-term polynomial in x and y as follows:

$$w = c_1 + c_2 x + c_3 y + c_4 x^2 + c_5 xy + c_6 y^2 + c_7 x^3 + c_8 x^2 y + c_9 xy^2 + c_{10} y^3 + c_{11} x^3 y + c_{12} xy^3.$$

The function allows for rigid-body motion and constant strain, but continuity of slopes between elements is not ensured. Along side $i - j$ (x-axis),

$$w = c_1 + c_2 x + c_4 x^2 + c_7 x^3 \quad .$$
$$\frac{\partial w}{\partial x} = c_2 + 2c_4 x + 3c_7 x^2$$
$$\frac{\partial w}{\partial y} = c_3 + c_5 x + c_8 x^2 + c_{11} x^3$$

The displacement w is a cubic as used for the beam element, and $\frac{\partial w}{\partial x}$ is the same as in beam bending. The four constants c_1, c_2, c_4, c_7 can be defined by invoking the end point conditions of $(w_i, w_j, \theta_{yi}, \theta_{yj})$. Therefore, w and $\frac{\partial w}{\partial x}$ are completely defined along side $i - j$. On the other hand, the normal slope $\frac{\partial w}{\partial y}$ is a cubic in x, with four constants, and only two d.o.f. left $(\theta_{xi}, \theta_{xj})$. Therefore, this slope is not uniquely defined, and function w is said to be nonconforming. However, this element has proven to give acceptable results, and to converge.

Constants c_1 to c_{12} can be determined by expressing 12 simultaneous equations linking them to the values of w and its slopes at the nodes:

$$\left\{ \begin{array}{c} w \\ \frac{\partial w}{\partial y} \\ -\frac{\partial w}{\partial x} \end{array} \right\} =$$

$$\begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 & x^3y & xy^3 \\ 0 & 0 & 1 & 0 & x & 2y & 0 & x^2 & 2xy & 3y^2 & x^3 & 3xy^2 \\ 0 & -1 & 0 & -2x & -y & 0 & -3x^2 & -2xy & -y^2 & 0 & -3x^2y & -y^3 \end{bmatrix} \left\{ \begin{array}{c} c_1 \\ \vdots \\ c_{12} \end{array} \right\}$$

$$\left\{ \begin{array}{c} w \\ \frac{\partial w}{\partial y} \\ -\frac{\partial w}{\partial x} \end{array} \right\} = [P]\{Q\}.$$

At nodes i, j, k, l, we require:

$$\left\{ \begin{array}{c} w_i \\ \theta_{xi} \\ \theta_{yi} \\ \vdots \\ w_l \\ \theta_{xl} \\ \theta_{yl} \end{array} \right\} =$$

$$\begin{bmatrix} 1 & x_i & y_i & x_i^2 & x_iy_i & y_i^2 & x_i^3 & x_i^2y_i & x_iy_i^2 & y_i^3 & x_i^3y_i & x_iy_i^3 \\ 0 & 0 & 1 & 0 & x_i & 2y_i & 0 & x_i^2 & 2x_iy_i & 3y_i^2 & x_i^3 & 3x_iy_i^2 \\ 0 & -1 & 0 & -2x_i & -y_i & 0 & -3x_i^2 & -2x_iy_i & -y_i^2 & 0 & -3x_i^2y_i & -y_i^3 \\ \vdots & & & & & & \vdots & & & & & \vdots \\ 1 & x_l & y_l & x_l^2 & x_ly_l & y_l^2 & x_l^3 & x_l^2y_l & x_ly_l^2 & y_l^3 & x_l^3y_l & x_ly_l^3 \\ 0 & 0 & 1 & 0 & x_l & 2y_l & 0 & x_l^2 & 2x_ly_l & 3y_l^2 & x_l^3 & 3x_ly_l^2 \\ 0 & -1 & 0 & -2x_l & -y_l & 0 & -3x_l^2 & -2x_ly_l & -y_l^2 & 0 & -3x_l^2y_l & -y_l^3 \end{bmatrix} \left\{ \begin{array}{c} c_1 \\ \vdots \\ c_{12} \end{array} \right\}$$

In compact matrix form, $\{U^e\} = [A]\{Q\}$ and $\{Q\} = [A]^{-1}\{U^e\}$.

Finally, $\left\{ \begin{array}{c} w \\ \frac{\partial w}{\partial y} \\ -\frac{\partial w}{\partial x} \end{array} \right\} = [P][A]^{-1}\{U^e\} = [N]\{U^e\}.$

The specific form of $[N]$ can be found in the documentation of finite element packages.

6.2.3 Principle of virtual work

For an element,

$\int_{V^e} \{\delta\varepsilon\}^T \{\sigma\} \, dV = \int_{V^e} \{\delta U\}^T \{b\} \, dV + \int_{A^e} \{\delta U\}^T \{s\} \, dA + \sum_{p=1}^{p=N} \{\delta U\}^T \{f_p\},$
$\forall \{\delta U\}, \{\delta\varepsilon\}.$

Focusing on the left-hand side,

$$\int_{V^e} \{\delta\varepsilon\}^T \{\sigma\}\, dV = \int_{V^e} \left\{ \begin{array}{c} \delta\varepsilon_{xx} \\ \delta\varepsilon_{yy} \\ 2\delta\varepsilon_{xy} \end{array} \right\}^T \left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array} \right\} dV$$

$$\int_{V^e} \{\delta\varepsilon\}^T \{\sigma\}\, dV = \int_{A^e} (\int_{-t/2}^{t/2} z \left\{ \begin{array}{c} \delta\kappa_{xx} \\ \delta\kappa_{yy} \\ 2\delta\kappa_{xy} \end{array} \right\}^T \left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array} \right\} dz) dA$$

$$\int_{V^e} \{\delta\varepsilon\}^T \{\sigma\}\, dV = \int_{A^e} \left\{ \begin{array}{c} \delta\kappa_{xx} \\ \delta\kappa_{yy} \\ 2\delta\kappa_{xy} \end{array} \right\}^T (\int_{-t/2}^{t/2} z \left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array} \right\} dz) dA$$

$$\int_{V^e} \{\delta\varepsilon\}^T \{\sigma\}\, dV = \int_{A^e} \left\{ \begin{array}{c} \delta\kappa_{xx} \\ \delta\kappa_{yy} \\ 2\delta\kappa_{xy} \end{array} \right\}^T \left\{ \begin{array}{c} M_{xx} \\ M_{yy} \\ M_{xy} \end{array} \right\} dA$$

$$\int_{V^e} \{\delta\varepsilon\}^T \{\sigma\}\, dV = \int_{A^e} \left\{ \begin{array}{c} \delta\kappa_{xx} \\ \delta\kappa_{yy} \\ 2\delta\kappa_{xy} \end{array} \right\}^T [D] \left\{ \begin{array}{c} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{array} \right\} dA$$

with $[D] = D \left[\begin{array}{ccc} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{array} \right]$.

From $\{\kappa\} = \left\{ \begin{array}{c} \kappa_{xx} \\ \kappa_{yy} \\ 2\kappa_{xy} \end{array} \right\} = \left\{ \begin{array}{c} -2c_4 - 6c_7 x - 2c_8 y - 6c_{11} xy \\ -2c_6 - 2c_9 x - 6c_{10} y - 6c_{12} xy \\ -2c_5 - 4c_8 x - 4c_9 y - 6c_{11} x^2 - 6c_{12} y^2 \end{array} \right\}$

or $\{\kappa\} = [R]\{Q\}$, with $\{Q\} = [A]^{-1}\{U^e\}$, one can write $\{\kappa\} = [B]\{U^e\}$, where $[B] = [R][A]^{-1}$.

Finally,
$\int_{V^e} \{\delta\varepsilon\}^T \{\sigma\}\, dV = \{\delta U^e\}^T \int_{A^e} [B]^T [D] [B]\, dA \{U^e\}$.

6.2.4 Element stiffness matrix $[K^e]$

From above, $[K^e] = \int_{A^e} [B]^T [D] [B]\, dA$.

$[K^e]$ is a 12x12 matrix. It can be found in the documentation of finite element packages.

6.2.5 Element nodal force vector $\{f^e\}$

- Body forces:

$\{f_b^e\} = \int_{V^e} [N]^T \{b\}\, dV$

- Surface tractions:

$\{f_s^e\} = \int_{A^e} [N]^T \, q dA$

For a uniform load q acting over the surface of an element of dimensions $2b \times 2c$, the force and moments at node i are (similar expressions for j, k, l):

$$\left\{ \begin{array}{c} f_{w_i} \\ f_{\theta_{xi}} \\ f_{\theta_{yi}} \end{array} \right\} = 4qbc \left\{ \begin{array}{c} 1/4 \\ -c/12 \\ b/12 \end{array} \right\}.$$

Finally, $[K^e]\{U^e\} = \{f^e\}$. Apply boundary conditions and solve.

Chapter 7

Method of weighted residuals

Learning Outcomes:
By the end of reading and attending the lecture of this chapter, and solving the listed practice problems, the student should be able to:
-Explain and apply different methods of weighted residuals to solve a governing equation
-Explain and apply the Galerkin methods to formulate a 1-D finite element problem
Evidence of Learning:
A student has achieved the learning outcomes when he or she is able to to reproduce the knowledge and abilities listed above.
Evaluation Strategies:
Test problems in at-home assignments
Teaching and Learning Activities:
-At-home reading before class
-Lecturing
-In-class practice problems
-At-home reading/reviewing after class
-At-home practice problems

So far, we have always used the principle of virtual work (or, equivalently, the minimum of potential energy) towards finite element (FE) formulation. What happens when such principles are not available? FE formulation can in fact be generalized to virtually all problems describable by ordinary or partial differential equations (ODE or PDE). For simplicity, let us limit ourselves here to linear statics, with $[K]\{U\} = \{f\}$ being the end point of the formulation.

7.1 The method of weighted residuals

7.1.1 General concepts

Consider a governing equation involving one independent variable:

$f\ [T(x)] = 0$ in domain Ω

where T represents the function sought (e.g. temperature), which is a function of x only. In addition, let us specify the boundary conditions (BC):

$g_1\ [T(x)] = 0$ on Γ_1

$g_2\ [T(x)] = 0$ on Γ_2

where Γ_1 and Γ_2 include only those parts of Ω that are on the boundary. Let us approximate the solution with an approximate function $\widehat{T}$ given by $\widehat{T} = \widehat{T}(x; a_1, \cdots, a_n) = \sum_{i=1}^{n} a_i\ N_i(x)$ which has n unknown constant parameters a_i and satisfies the boundary conditions exactly. The $N_i(x)$ are referred to as trial functions. If the approximate solution $\widehat{T}$ is substituted in the governing equation, some residual error $R(x; a_1, \cdots, a_n)$ appears:

$f\ [\widehat{T}(x; a_1, \cdots, a_n)] = R(x; a_1, \cdots, a_n)$.

The method of weighted residuals requires that parameters a_i be determined by satisfying:

$\int_\Omega w_i(x)\ R(x; a_1, \cdots, a_n)dx = 0 \quad , 1 \leq i \leq n$

where the $w_i(x)$ are n arbitrary weighting functions. Many functions can be used; the most popular methods are described below.

7.1.2 Point collocation

$w_i(x) = \delta(x - x_i)$ (Dirac function) such that $\int_a^b \delta(x - x_i)dx = 1$ for $x = x_i$.

$$\int_a^b \delta(x - x_i)dx = 0 \text{ for } x \neq x_i$$

The x_i are the collocation points and are selected arbitrarily by the analyst.

If $\int_\Omega \delta(x - x_i)\ R(x; a_1, \cdots, a_n)dx = 0$ is evaluated at n collocation points, n algebraic equations in n unknowns (a_i) result:

$$\begin{aligned} &R(x_1; a_1, \cdots, a_n) = 0 \\ &R(x_2; a_1, \cdots, a_n) = 0 \\ &\qquad\vdots \\ &R(x_n; a_1, \cdots, a_n) = 0 \end{aligned}$$

Once the a_i are determined, the approximate solution $\widehat{T}$ is found, and the problem is solved.

Example 14 *solve the ODE:* $\frac{d^2T}{dx^2} + 1000x^2 = 0$ *for* $0 \leq x \leq 1$ *with* $T(0) = T(1) = 0$, *using* $N_1(x) = x(1 - x^2)$ *(satisfies BC)*

$\widehat{T} = a_1 N_1(x) = a_1 x(1 - x^2)$

$R(x; a_1) = \frac{d^2\widehat{T}}{dx^2} + 1000x^2 = -6a_1 x + 1000x^2$

Collocation point (chosen): $x_1 = \frac{1}{2}$

$$\begin{aligned} \int_0^1 \delta(x - \tfrac{1}{2})\ R(x; a_1)dx = 0 &\Leftrightarrow \int_0^1 \delta(x - \tfrac{1}{2})\ (-6a_1 x + 1000x^2)dx = 0 \\ &\Leftrightarrow R(x_1; a_1) = 0 \\ &\Leftrightarrow -6a_1(\tfrac{1}{2}) + 1000(\tfrac{1}{2})^2 = 0 \\ &\Leftrightarrow a_1 = \tfrac{1000}{12} \end{aligned}$$

Finally, $\widehat{T}(x) = \frac{1000}{12} x(1 - x^2)$ (dashed line in Fig. 7.2).

7.1.3 Subdomain collocation

The domain Ω is subdivided into n subdomains over which $w_i(x)$ is unity and 0 elsewhere.

$$w_1(x) = \begin{array}{l} 1 \text{ for } x \text{ in } \Omega_1 \\ 0 \text{ for } x \text{ not in } \Omega_1 \end{array}$$

$$\vdots$$

$$w_n(x) = \begin{array}{l} 1 \text{ for } x \text{ in } \Omega_n \\ 0 \text{ for } x \text{ not in } \Omega_n \end{array}$$

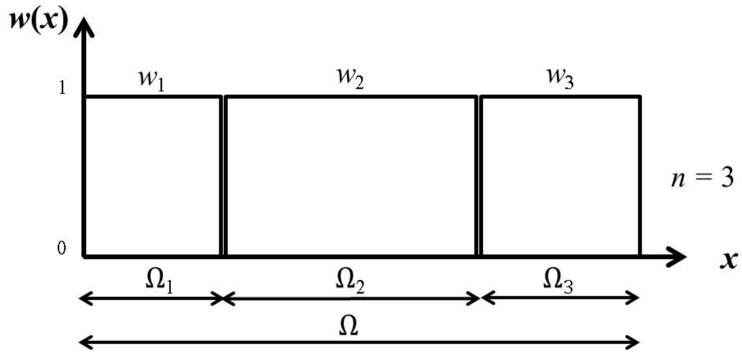

Figure 7.1: Subdomain collocation.

If $\int_{\Omega} w_i(x)\ R(x; a_1, \cdots, a_n)dx = 0$ is evaluated for $1 \leq i \leq n$, n integral equations result:

$$\int_{\Omega_1} R(x; a_1, \cdots, a_n)dx = 0$$

$$\vdots$$

$$\int_{\Omega_n} R(x; a_1, \cdots, a_n)dx = 0$$

These equations can be solved for a_i, and $\widehat{T}$ determined.

Example 15 *same problem as above: solve the ODE:* $\frac{d^2T}{dx^2} + 1000x^2 = 0$ *for* $0 \leq x \leq 1$ *with* $T(0) = T(1) = 0$, *using* $N_1(x) = x(1 - x^2)$

$\widehat{T} = a_1 N_1(x) = a_1 x(1 - x^2)$

Only one integral of the residual needs to be evaluated (because one parameter, a_1).

$R(x; a_1) = \frac{d^2\widehat{T}}{dx^2} + 1000x^2 = -6a_1x + 1000x^2$

$\int_0^1 R(x; a_1)dx = 0 \Leftrightarrow a_1 = \frac{1000}{9}$

Finally, $\widehat{T}(x) = \frac{1000}{9}x(1-x^2)$ (thin solid line in Fig. 7.2).
Note: this method is equivalent to the Ritz (variational) method.

7.1.4 Least squares

The method of least squares requires that the integral I of the square of the residual R be minimized:

$$I = \int_\Omega [R(x; a_1, \cdots, a_n)]^2 dx.$$

In other words, parameters a_i need to be determined so that I is minimized. This is done by writing:

$$\frac{\partial I}{\partial a_1} = 0 = \frac{\partial}{\partial a_1}\int_\Omega [R(x; a_1, \cdots, a_n)]^2 dx = 0$$

$$\vdots$$

$$\frac{\partial I}{\partial a_n} = 0 = \frac{\partial}{\partial a_n}\int_\Omega [R(x; a_1, \cdots, a_n)]^2 dx = 0$$

Because the limits on the integral are not functions of a_i, the order of integration and differentiation may be interchanged to give:

$$\int_\Omega \frac{\partial}{\partial a_1}[R(x; a_1, \cdots, a_n)]^2 dx = 0 \text{ etc... , or } \int_\Omega R\frac{\partial R}{\partial a_1} dx = 0 \text{ etc...}$$

Finally, we observe that in this method, $w_i(x) = \frac{\partial R}{\partial a_i}, 1 \leq i \leq n$.
The n equations obtained will be used to solve for the n parameters a_i.

Example 16 *from above:* $R(x; a_1) = \frac{d^2\widehat{T}}{dx^2} + 1000x^2 = -6a_1x + 1000x^2$

$$\frac{\partial R}{\partial a_1} = -6x$$

$$\int_0^1 R\frac{\partial R}{\partial a_1} dx = 0 \Leftrightarrow a_1 = \frac{1000}{8}$$

Finally, $\widehat{T}(x) = \frac{1000}{8}x(1-x^2)$ (dotted line in Fig. 7.2).

7.1.5 Galerkin

In the Galerkin method, the trial functions $N_i(x)$ are used as weighting functions, that is $w_i(x) = N_i(x)$.

If $\int_\Omega N_i(x)\ R(x; a_1, \cdots, a_n)dx = 0$ is evaluated for $1 \leq i \leq n$, n integral equations result and can be solved for n parameters a_i.

Example 17 $N_1(x) = x(1-x^2)$ *from above*

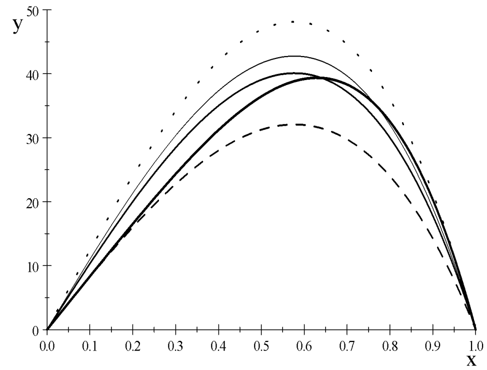

Figure 7.2: Comparison between solutions.

$\int_0^1 x(1-x^2)(-6a_1x+1000x^2)dx = 0 \Leftrightarrow a_1 = \frac{5000}{48}$
Finally, $\widehat{T}(x) = \frac{5000}{48}x(1-x^2)$ (medium solid line in Fig. 7.2).
Note: this method is equivalent to the Rayleigh-Ritz (variational) method.

7.1.6 Comparison with exact solution

The exact solution to the above problem is $T(x) = \frac{1000}{12}x(1-x^3)$ (thick solid line in Fig. 7.2), while our approximate solution was assumed to be of the form $\widehat{T}(x) = a_1x(1-x^2)$. None of the methods seems to do an outstanding job, but the trial functions have not been worked on. In what follows, we will use the Galerkin method in combination with trial functions that are no longer applied to the entire domain, but applied locally over each element. This is a very successful method.

7.2 The Galerkin finite element method

7.2.1 Formulation

Piecewise continuous trial functions (or shape functions) will be employed for each element. For a 1-D problem, as in Fig. 7.3:

The problem domain $a \leq x \leq b$ is divided into M elements. A typical element e has two nodes j and k with coordinates x_j and x_k. The field variable

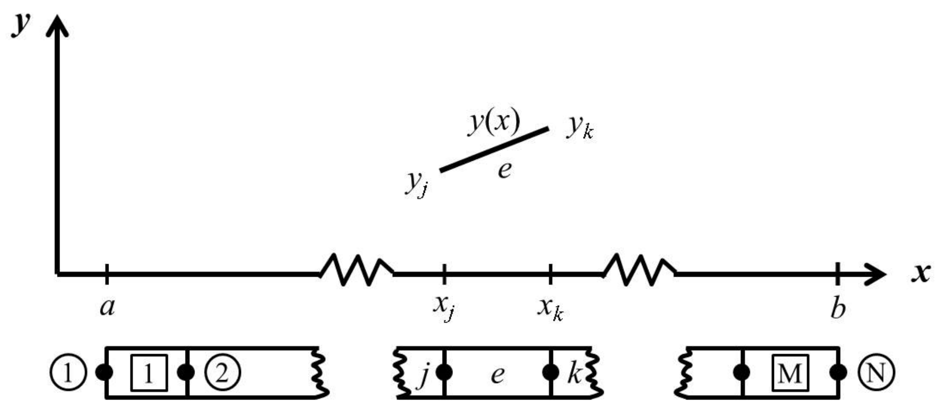

Figure 7.3: Piecewise formulation.

y at nodes j and k has values y_j and y_k. Since each element has two nodes and since a unique straight line may be drawn between two points (Fig. 7.4), the interpolation polynomial to be used must be of the form $y^e = mx + b$. In other words, we wish to express $y^e(x)$ in the form $y^e(x) = y_j N_j(x) + y_k N_k(x)$ with $N_j(x) = \frac{x_k - x}{x_k - x_j}$ and $N_k(x) = \frac{x - x_j}{x_k - x_j}$. Clearly, $y^e(x_j) = y_j$ and $y^e(x_k) = y_k$.

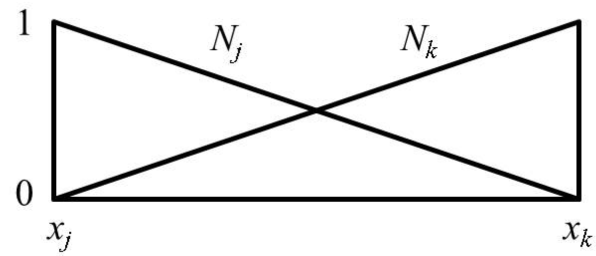

Figure 7.4: Linear interpolation over one element.

Let us now resolve the problem in Section 1 by the Galerkin FE method using six equally spaced nodes.

Over one element, $T^e = N_j(x)T_j + N_k(x)T_k$, where T_j and T_k are analogous to the previous unknown parameters a_1 and a_2. For all the elements, from 1 to M,

$\sum_{e=1}^{M} \int_{x_j}^{x_k} N_j(x) R^e(x; T_j, T_k) dx = 0$
$\sum_{e=1}^{M} \int_{x_j}^{x_k} N_k(x) R^e(x; T_j, T_k) dx = 0$

In matrix notation, with $[N]^T = \left\{ \begin{array}{c} N_j(x) \\ N_k(x) \end{array} \right\}$, the previous equations become:
$\sum_{e=1}^{M} \int_{x_j}^{x_k} [N]^T R^e(x; T_j, T_k) dx = \{0\}$.

The residual R^e for a typical element for the present problem is
$R^e = \frac{d^2T^e}{dx^2} + 1000x^2$.

Therefore, $\sum_{e=1}^{M} \int_{x_j}^{x_k} [N]^T (\frac{d^2T^e}{dx^2} + 1000x^2) dx = \{0\}$. For simplicity, let us drop the summation sign (corresponding to a routine assemblage).
$\int_{x_j}^{x_k} [N]^T (\frac{d^2T^e}{dx^2} + 1000x^2) dx = \{0\}$
$\Leftrightarrow \int_{x_j}^{x_k} [N]^T \frac{d^2T^e}{dx^2} dx + \int_{x_j}^{x_k} [N]^T 1000x^2 dx = \{0\}$

Recalling the integration by parts rule: $\int_a^b u \frac{dv}{dx} dx = [uv]_a^b - \int_a^b \frac{du}{dx} v dx$,
$\int_{x_j}^{x_k} [N]^T \frac{d^2T^e}{dx^2} dx = \left[[N]^T \frac{dT^e}{dx}\right]_{x_j}^{x_k} - \int_{x_j}^{x_k} \frac{d}{dx}[N]^T \frac{dT^e}{dx} dx$, and then,
$\left[[N]^T \frac{dT^e}{dx}\right]_{x_j}^{x_k} - \int_{x_j}^{x_k} \frac{d}{dx}[N]^T \frac{dT^e}{dx} dx + \int_{x_j}^{x_k} [N]^T 1000x^2 dx = \{0\}$.

Since we already know that $T^e = [N_j \quad N_k] \left\{ \begin{array}{c} T_j \\ T_k \end{array} \right\} = [N]\{U^e\}$,
$\left[[N]^T \frac{dT^e}{dx}\right]_{x_j}^{x_k} - \int_{x_j}^{x_k} \frac{d}{dx}[N]^T \frac{d}{dx}[N]\{U^e\} dx + \int_{x_j}^{x_k} [N]^T 1000x^2 dx = \{0\}$.

Note that not using the interpolation to calculate $\left[[N]^T \frac{dT^e}{dx}\right]_{x_j}^{x_k}$ allows for imposing gradient boundary conditions, and is convenient.

Finally, $[K^e]\{U^e\} = \{f^e\}$ where
$[K^e] = \int_{x_j}^{x_k} \frac{d}{dx}[N]^T \frac{d}{dx}[N] dx$ (element stiffness matrix) and
$\{f^e\} = \left[[N]^T \frac{dT^e}{dx}\right]_{x_j}^{x_k} + \int_{x_j}^{x_k} [N]^T 1000x^2 dx$ (nodal force vector).

After assemblage and imposition of boundary conditions, the system $[K]\{U\} = \{f\}$ can be solved. The solution matches the exact solution.

Two linear shape functions applied locally give much superior results than a cubic function applied globally (as done in Section 1 of this chapter).

7.2.2 Application: 1-D heat transfer in a pin fin

Let us derive the governing equation for the temperature $T(x)$ in a circular pin fin of varying cross sectional area A and perimeter P (Fig. 7.5):

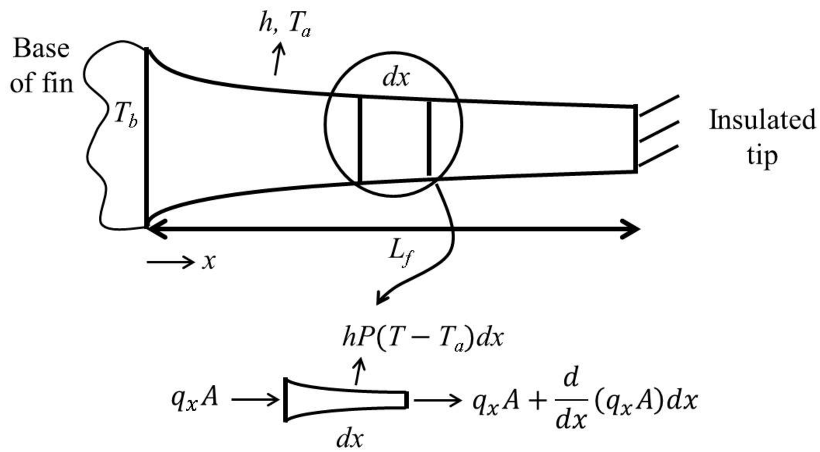

Figure 7.5: Pin fin.

Conduction (through fin) and convection (with surrounding fluid) are present.

q_xA	$-(q_xA+\frac{d}{dx}(q_xA)dx)$	$-hP(T-T_a)dx$	$=0$
conduction in	conduction out	convection out	

q_x : heat flux
h : convective heat transfer coefficient
T_a : ambient fluid temperature far removed from fin
Finally, the steady state energy balance on an infinitesimal slice yields:
$-\frac{d}{dx}(q_xA)-hP(T-T_a)=0.$

Applying Fourier's law of heat conduction (this is a constitutive relationship, like $\sigma=E\varepsilon$) whereby $q_x=-k\frac{dT}{dx}$, where k is the thermal conductivity, and the minus sign is necessary because q_x is assumed positive in the direction of decreasing temperature, the governing equation for 1-D heat transfer is:

$\frac{d}{dx}(kA\frac{dT}{dx}) - hP(T - T_a) = 0$ for $0 \leq x \leq L_f$, subject to two boundary conditions $T(0) = T_b$.
$\frac{dT}{dx}(L_f) = 0$

Let us now derive equations to obtain an approximate solution using the Galerkin FE method.

Element characteristics:

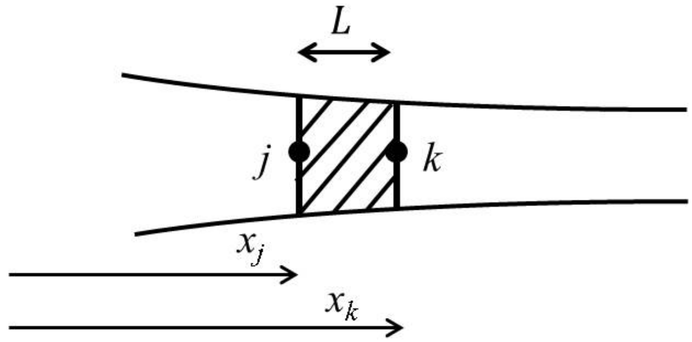

Figure 7.6: 1-D element with nodes j and k.

For a typical element (Fig. 7.6), the Galerkin formulation yields:

$$\int_{x_j}^{x_k} [N]^T \left[\frac{d}{dx}(kA\frac{dT}{dx}) - hP(T - T_a)\right] dx = \{0\}.$$

Integrating $\int_{x_j}^{x_k} [N]^T \frac{d}{dx}(kA\frac{dT}{dx})dx$ by parts,

$$\int_{x_j}^{x_k} [N]^T \frac{d}{dx}(kA\frac{dT}{dx})dx =$$

$$\left[[N]^T kA\frac{dT}{dx}\right]_{x_j}^{x_k} \qquad - \int_{x_j}^{x_k} \frac{d}{dx}[N]^T kA\frac{dT}{dx}dx.$$

These terms will cancel out upon assemblage,
and at $x = L_f$, $\frac{dT}{dx} = 0$

Therefore,

$$-\int_{x_j}^{x_k} \frac{d}{dx}[N]^T kA\frac{dT}{dx}dx - \int_{x_j}^{x_k} [N]^T hPTdx + \int_{x_j}^{x_k} [N]^T hPT_a dx = \{0\}.$$

By using $T = [N]\{U^e\}$ and noting that $[N]$ is only a function of x, we get $[K^e]\{U^e\} = \{f^e\}$ where

$[K^e] = [K_x^e] + [K_{cv}^e]$ and

$[K_x^e] = \int_{x_j}^{x_k} \frac{d}{dx}[N]^T kA \frac{d}{dx}[N]dx,$

$[K_{cv}^e] = \int_{x_j}^{x_k} [N]^T hP[N]dx$, and

$\{f^e\} = \int_{x_j}^{x_k} [N]^T hPT_a dx.$

The vector of nodal unknowns $\{U^e\}$ is $\{U^e\}^T = [T_j \quad T_k]$, and the shape function matrix $[N]$ is $[N] = [N_j \quad N_k]$.

Let us consider parameters k, h, A, P and T_a to be constant in any given element by using the values at the midpoint of the element, where $x = \overline{x}$. Then,

$[K_x^e] = \frac{\overline{kA}}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$

$[K_{cv}^e] = \frac{\overline{hP}L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, and

$\{f^e\} = \frac{\overline{hP}L\overline{T_a}}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$, where $L = x_k - x_j$.

The assemblage of these 2x2 matrices is performed as usual to give a system of N equations in the N unknown nodal temperatures, where $M = N-1$ is the number of elements. Overall, $[K^a]\{U^a\} = \{f^a\}$, and the prescribed boundary temperature is applied, leading to $[K]\{U\} = \{f\}$ which may be solved for $\{U\}$.

Note: the Galerkin formulation leads to what is called a weak formulation: because of the integration by parts, the order of the derivative in the governing equation (strong formulation) is reduced in the working equations.

Chapter 8

Steady state thermal analysis

Learning Outcomes:

By the end of reading and attending the lecture of this chapter, the student should be able to:

-Explain the governing equation in 1-D steady state heat conduction

-Apply the Green Gauss theorem

-Explain and apply the Galerkin methods to formulate 2-D and axisymmetric finite element problems in heat transfer

Evidence of Learning:

A student has achieved the learning outcomes when he or she is able to to reproduce the knowledge and abilities listed above.

Evaluation Strategies:

Test problems in at-home assignments

Teaching and Learning Activities:

-At-home reading before class

-Lecturing

-In-class practice problems

-At-home reading/reviewing after class

-At-home practice problems

In the illustrative heat conduction problem that was considered in the previous chapter, thermal radiation and heat generation were not included. In addition, we need to look into 2-D and axisymmetric problems. Let us start with a 1-D problem.

8.1 1-D heat conduction

8.1.1 Governing equation

The governing equation for temperature T is given by (Fig. 8.1):

$\frac{d}{dx}(kA\frac{dT}{dx}) - hP(T - T_a) - \varepsilon\sigma P(T^4 - T_r^4) + QA = 0$ where

k : thermal conductivity of the material (W/m$^{\text{o}}$C)

h : conductive heat transfer coefficient (W/m$^{2\text{o}}$C)

ε : emissivity of the body surface (-)

P : perimeter

A : cross-sectional area of body

T_a : ambient fluid temperature far removed from body

T_r : receiver temperature for radiation

Q : heat generation rate per unit volume (W/m^3)

σ : Stefan-Boltzmann constant (5.670×10^{-8} W/m^{2}K^4)

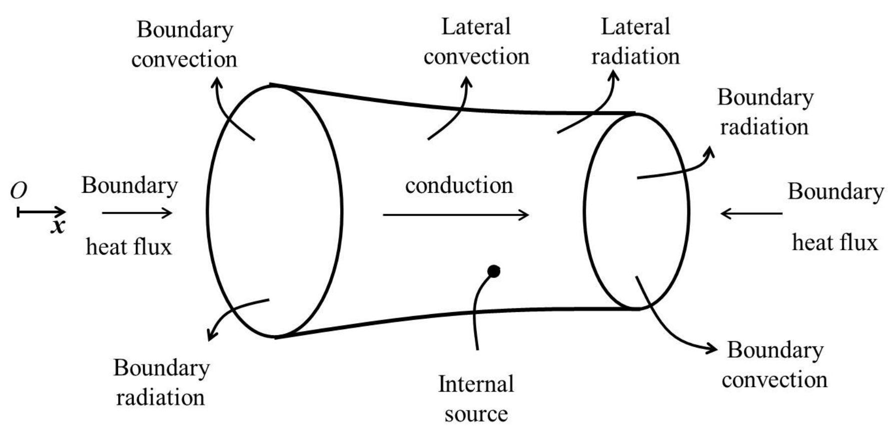

Figure 8.1: 1-D heat conduction problem.

Note that:

- k, h, ε may be temperature dependent
- absolute temperature (K) must be used if radiation is present

8.1.2 FE formulation

Using the Galerkin method for M elements such as in Fig. 8.2,

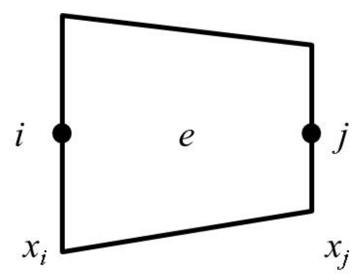

Figure 8.2: 1-D heat tranfer element.

$\sum_{e=1}^{M} \int_{l^e} [N]^T R^e dx = \{0\}$ where $[N]$ is the shape matrix function. In what follows, we drop the summation sign representing the routine assemblage process. Therefore,

$$\int_{l^e} [N]^T R^e dx =$$
$$\int_{l^e} [N]^T \left[\frac{d}{dx}(kA\frac{dT}{dx}) - hP(T - T_a) - \varepsilon\sigma P(T^4 - T_r^4) + QA \right] dx = \{0\}.$$

Integrating the first term by parts,

$$\left[[N]^T kA\frac{dT}{dx}\right]_{x_i}^{x_j} - \int_{l^e} \frac{d}{dx}[N]^T kA\frac{dT}{dx} dx$$
$$-\int_{l^e} [N]^T hPT dx + \int_{l^e} [N]^T hPT_a dx - \int_{l^e} [N]^T \varepsilon\sigma PT^4 dx$$
$$+\int_{l^e} [N]^T \varepsilon\sigma PT_r^4 dx + \int_{l^e} [N]^T QA dx = \{0\}.$$

The first term is related to the heat transfer rate per unit area in the x-direction: $q_x = -k\frac{dT}{dx}$. It will contribute to the assemblage equations for only the two elements at each end of the body. Therefore, this term needs to be evaluated for elements 1 and M only.

Boundary conditions

Heat transfer to and from the ends can be a combination of convection q_{cv}, radiation q_r, and prescribed heat fluxes q_s. (Think: into end: q_x, q_s ; out of end: q_{cv}, q_r, Fig. 8.3).

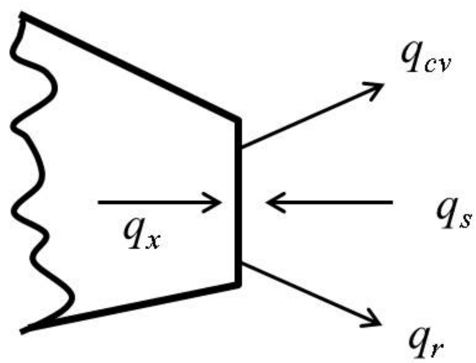

Figure 8.3: Equilibrium of one end of the body.

An energy balance on the end of the body gives:
$q_x + q_s = q_{cv} + q_r$, or $q_x = q_{cv} + q_r - q_s$.
Note that q_s is positive if the heat flux is imposed toward the surface.
At Node j, $q_{cv} = h_j(T - T_{aj})$.
$q_r = \varepsilon_j \sigma(T^4 - T_{rj}^4)$
$q_s = q_j$
Therefore, for the integrated term at $x = x_j$,

$$\begin{aligned}[N]^T kA\tfrac{dT}{dx}|_{x=x_j} &= [N]^T(-q_x A)|_{x=x_j}\\ &= [N]^T A_j(-q_{cv} - q_r + q_s)|_{x=x_j}\\ &= (-[N]^T h_j A_j T + [N]^T h_j A_j T_{aj}\\ &\quad -[N]^T \varepsilon_j A_j \sigma T^4 + [N]^T \varepsilon_j A_j \sigma T_{rj}^4 + [N]^T A_j q_j)|_{x=x_j}\end{aligned}$$

Similarly at Node i, with $q_x = -q_{cv} - q_r + q_s$ (Fig. 8.4)

$$-[N]^T kA\tfrac{dT}{dx}|_{x=x_i} = [N]^T A_i(-q_{cv} - q_r + q_s)|_{x=x_i}$$

Finally, the integrated term is given by the SUM of both right-hand sides.

Element characteristics

As usual, $T = [N]\{U^e\}$.

For convenience, we write $T^4 = T^3 T = ([N]\{U^e\})^3[N]\{U^e\}$.

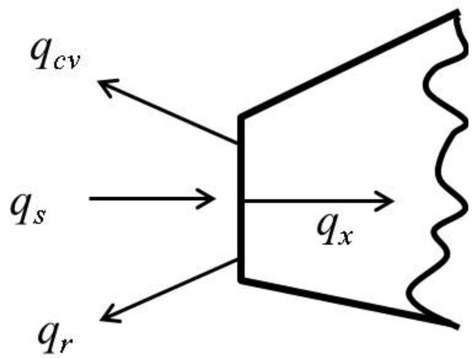

Figure 8.4: Equilibrium of other end of the body.

From the Galerkin method, we can write $[K^e]\{U^e\} = \{f^e\}$ with
$[K^e] = [K^e_x] + [K^e_{cv}] + [K^e_r] + [K^e_{cvB}] + [K^e_{rB}]$ and
$\{f^e\} = \{f^e_{cv}\} + \{f^e_r\} + \{f^e_Q\} + \{f^e_{cvB}\} + \{f^e_{rB}\} + \{f^e_{qB}\}$ where subscript B denotes those terms arising from boundary conditions. Furthermore,

$[K^e_x] = \int_{l^e} \frac{d}{dx}[N]^T kA \frac{d}{dx}[N]dx,$

$[K^e_{cv}] = \int_{l^e} [N]^T hP[N]dx,$

$[K^e_r] = \int_{l^e} [N]^T \varepsilon\sigma P([N]\{U^e\})^3[N]dx,$

$[K^e_{cvB}] = [N]^T h_i A_i [N]|_{x=x_i} + [N]^T h_j A_j [N]|_{x=x_j},$

$[K^e_{rB}] = [N]^T \varepsilon_i A_i \sigma([N]\{U^e\})^3[N]|_{x=x_i} + [N]^T \varepsilon_j A_j \sigma([N]\{U^e\})^3[N]|_{x=x_j},$

$\{f^e_{cv}\} = \int_{l^e} [N]^T hPT_a dx,$

$\{f^e_r\} = \int_{l^e} [N]^T \varepsilon\sigma PT_r^4 dx,$

$\{f^e_Q\} = \int_{l^e} [N]^T QA dx,$

$\{f^e_{cvB}\} = [N]^T h_i A_i T_{ai}|_{x=x_i} + [N]^T h_j A_j T_{aj}|_{x=x_j},$

$\{f^e_{rB}\} = [N]^T \varepsilon_i A_i \sigma T^4_{ri}|_{x=x_i} + [N]^T \varepsilon_j A_j \sigma T^4_{rj}|_{x=x_j}$, and

$\{f^e_{qB}\} = [N]^T A_i q_i|_{x=x_i} + [N]^T A_j q_j|_{x=x_j}.$

8.2 2-D heat conduction

Consider a plate of variable thickness (Fig. 8.5).

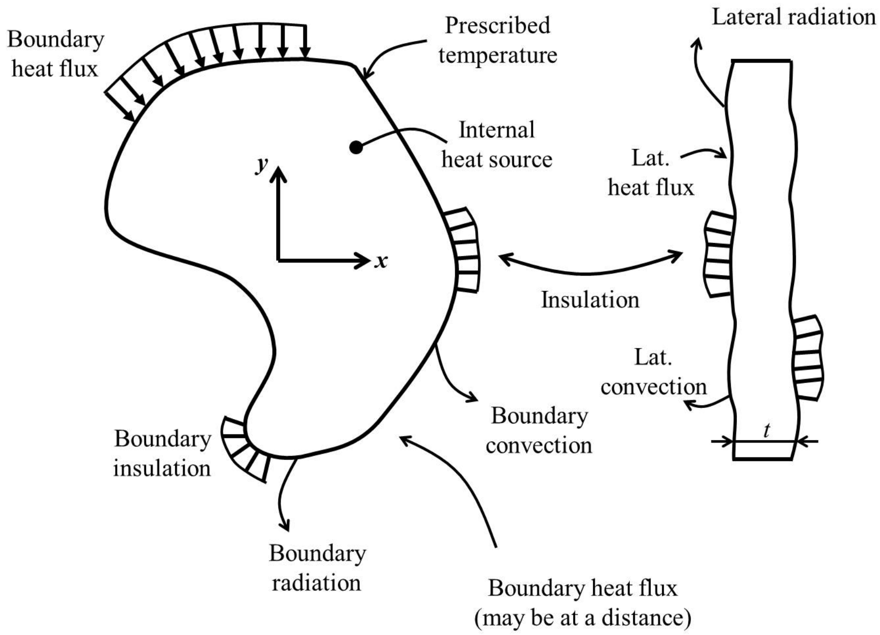

Figure 8.5: 2-D heat transfer problem (top view, left; side view, right).

8.2.1 Governing equation

Energy balance on volume $tdxdy$ yields:
$\frac{\partial}{\partial x}(kt\frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(kt\frac{\partial T}{\partial y}) - h(T - T_a) - \varepsilon\sigma(T^4 - T_r^4) + q_s + Qt = 0$, where h, ε, q_s are the sums of their contributions on two lateral faces. Note that q_s is positive if directed as shown above. Using the governing equation to derive the FE formulation will require use of the Green-Gauss theorem.

8.2.2 Green-Gauss theorem

It can be seen as a multidimensional version of integration by parts. Starting from the divergence theorem:

- in 3-D:

$\int_V \nabla \cdot \overrightarrow{q} dV = \int_S \overrightarrow{q} \cdot \overrightarrow{n} dS$, where $\nabla \cdot \overrightarrow{q} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}$,

$\overrightarrow{q}$ is any vector, dV a volume element, dS a surface element bounding V, and $\overrightarrow{n}$ a unit vector always normal to closed surface S and always directed outward (Fig. 8.6).

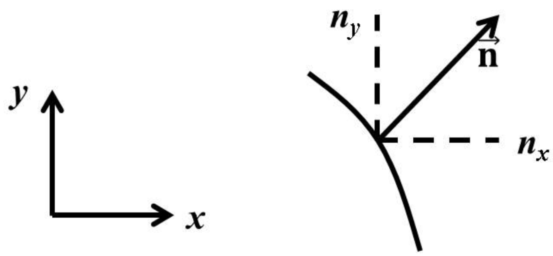

Figure 8.6: Normal to the surface.

- in 2-D:

$\int_A \nabla \cdot \overrightarrow{q} dA = \int_C \overrightarrow{q} \cdot \overrightarrow{n} dC$, where the integration path around path C must be performed counterclockwise.

Let us now derive the Green-Gauss theorem. Let $\overrightarrow{q} = \beta \overrightarrow{p}$, where β is a scalar function of x, y and z, and $\overrightarrow{p}$ a vector. Then,

$\int_V \nabla \cdot \overrightarrow{q} dV = \int_V \nabla \cdot (\beta \overrightarrow{p}) dV = \int_S \beta \overrightarrow{p} \cdot \overrightarrow{n} dS.$

With $\nabla \cdot (\beta \overrightarrow{p}) = \beta \nabla \cdot \overrightarrow{p} + \nabla \beta \cdot \overrightarrow{p}$, where $\nabla \beta$ denotes the gradient of scalar function β, i.e. $\nabla \beta = \frac{\partial \beta}{\partial x} \overrightarrow{i} + \frac{\partial \beta}{\partial y} \overrightarrow{j} + \frac{\partial \beta}{\partial z} \overrightarrow{k}$,

one obtains:

$\int_V \nabla \cdot (\beta \overrightarrow{p}) dV = \int_V \beta \nabla \cdot \overrightarrow{p} dV + \int_V \nabla \beta \cdot \overrightarrow{p} dV = \int_S \beta \overrightarrow{p} \cdot \overrightarrow{n} dS$, from which we derive

- in 3-D:

$\int_V \beta \nabla \cdot \overrightarrow{p} dV = \int_S \beta \overrightarrow{p} \cdot \overrightarrow{n} dS - \int_V \nabla \beta \cdot \overrightarrow{p} dV$

or

- in 2-D:

$$\int_A \beta\nabla\cdot\overrightarrow{p}\,dA = \int_C \beta\overrightarrow{p}\cdot\overrightarrow{n}\,dC - \int_A \nabla\beta\cdot\overrightarrow{p}\,dA.$$

Since p_x is independent of p_y and p_z and vice-versa, it follows that

- in 3-D:

$$\int_V \beta\frac{\partial p_x}{\partial x}dV = \int_S \beta p_x n_x dS - \int_V \frac{\partial\beta}{\partial x}p_x dV$$
$$\int_V \beta\frac{\partial p_y}{\partial y}dV = \int_S \beta p_y n_y dS - \int_V \frac{\partial\beta}{\partial y}p_y dV$$
$$\int_V \beta\frac{\partial p_z}{\partial z}dV = \int_S \beta p_z n_z dS - \int_V \frac{\partial\beta}{\partial z}p_z dV$$

and

- in 2-D:

$$\int_A \beta\frac{\partial p_x}{\partial x}dA = \int_C \beta p_x n_x dC - \int_A \frac{\partial\beta}{\partial x}p_x dA$$
$$\int_A \beta\frac{\partial p_y}{\partial y}dA = \int_C \beta p_y n_y dC - \int_A \frac{\partial\beta}{\partial y}p_y dA$$

8.2.3 FE formulation

Again, the Galerkin method is used, and only one element e is considered. With shape matrix function $[N]$, the Galerkin method requires

$$\int_{A^e}[N]^T\left[\begin{array}{c}\frac{\partial}{\partial x}(kt\frac{\partial T}{\partial x})+\frac{\partial}{\partial y}(kt\frac{\partial T}{\partial y})-h(T-T_a)\\ -\varepsilon\sigma(T^4-T_r^4)+q_s+Qt\end{array}\right]dxdy=\{0\}.$$

Applying the Green-Gauss theorem to the two terms containing 2^{nd} order derivatives, we get

$$\int_{C^e}[N]^T kt\frac{\partial T}{\partial x}n_x dC - \int_{A^e}\frac{\partial}{\partial x}[N]^T kt\frac{\partial T}{\partial x}dxdy +$$
$$\int_{C^e}[N]^T kt\frac{\partial T}{\partial y}n_y dC - \int_{A^e}\frac{\partial}{\partial y}[N]^T kt\frac{\partial T}{\partial y}dxdy$$
$$-\int_{A^e}[N]^T hT dxdy + \int_{A^e}[N]^T hT_a dxdy$$
$$-\int_{A^e}[N]^T\varepsilon\sigma T^4 dxdy + \int_{A^e}[N]^T\varepsilon\sigma T_r^4 dxdy$$
$$+\int_{A^e}[N]^T q_s dxdy + \int_{A^e}[N]^T Qt dxdy = \{0\}$$

One can combine $\int_{C^e}[N]^T(k\frac{\partial T}{\partial x}n_x + k\frac{\partial T}{\partial y}n_y)tdC$ into $\int_{C^e}[N]^T(-q_n)tdC$ where q_n represents the heat flux from conduction in the direction of the outward normal of the boundary (Fig. 8.7)

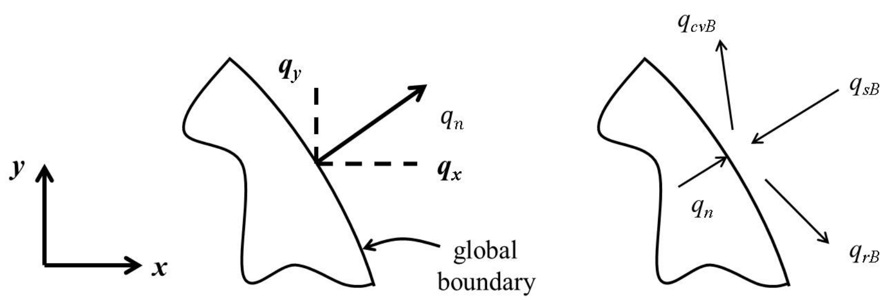

Figure 8.7: Equilibrium of boundary.

$q_n = \overrightarrow{q_n} \cdot \overrightarrow{n}$, with $\overrightarrow{q_n} = q_x\overrightarrow{i} + q_y\overrightarrow{j}$, $q_n = -k\frac{\partial T}{\partial x}n_x - k\frac{\partial T}{\partial y}n_y$ and $n_x = \overrightarrow{i} \cdot \overrightarrow{n}$, $n_y = \overrightarrow{j} \cdot \overrightarrow{n}$.

On the global boundary, an energy balance (on an area basis) gives:

$$\underset{}{q_n} + \underset{\substack{\text{imposed}\\ \text{heat flux}}}{q_{sB}} = \underset{\substack{\text{convective}\\ \text{heat flux}}}{q_{cvB}} + \underset{\substack{\text{radiative}\\ \text{heat flux}}}{q_{rB}} \quad \text{or } q_n = q_{cvB} + q_{rB} - q_{sB},$$

with $q_{cvB} = h_B(T - T_{aB})$ and $q_{rB} = \varepsilon_B\sigma(T^4 - T_{rB}^4)$.

Therefore,

$$\int_{C^e}[N]^T(-q_n)tdC =$$
$$\int_{C^e}[N]^T q_{sB}tdC - \int_{C^e}[N]^T h_B T tdC + \int_{C^e}[N]^T h_B T_{aB} tdC$$
$$-\int_{C^e}[N]^T\varepsilon_B\sigma T^4 tdC + \int_{C^e}[N]^T\varepsilon_B\sigma T_{rB}^4 tdC.$$

Recalling that $T = [N]\{U^e\}$ and $T^4 = ([N]\{U^e\})^3[N]\{U^e\}$, the Galerkin method condenses the problem into

$[K^e]\{U^e\} = \{f^e\}$ with

$[K^e] = [K^e_{xx}] + [K^e_{yy}] + [K^e_{cv}] + [K^e_r] + [K^e_{cvB}] + [K^e_{rB}]$ and

$\{f^e\} = \{f^e_{cv}\} + \{f^e_r\} + \{f^e_q\} + \{f^e_Q\} + \{f^e_{cvB}\} + \{f^e_{rB}\} + \{f^e_{qB}\}$.

The element stiffness matrices are given by:

$[K^e_{xx}] = \int_{A^e} \frac{\partial}{\partial x}[N]^T kt \frac{\partial}{\partial x}[N] dxdy, \quad [K^e_{yy}] = \int_{A^e} \frac{\partial}{\partial y}[N]^T kt \frac{\partial}{\partial y}[N] dxdy,$

$[K^e_{cv}] = \int_{A^e} [N]^T h [N] dxdy,$

$[K^e_r] = \int_{A^e} [N]^T \varepsilon\sigma([N]\{U^e\})^3 [N] dxdy,$

$[K^e_{cvB}] = \int_{C^e} [N]^T h_B t [N] dC,$

$[K^e_{rB}] = \int_{C^e} [N]^T \varepsilon_B \sigma t([N]\{U^e\})^3 [N] dC,$

and the element nodal force vectors by:

$\{f^e_{cv}\} = \int_{A^e} [N]^T h T_a dxdy,$

$\{f^e_r\} = \int_{A^e} [N]^T \varepsilon\sigma T_r^4 dxdy,$

$\{f^e_q\} = \int_{A^e} [N]^T q_s dxdy$

$\{f^e_Q\} = \int_{A^e} [N]^T Q t dxdy,$

$\{f^e_{cvB}\} = \int_{C^e} [N]^T h_B t T_{aB} dC,$

$\{f^e_{rB}\} = \int_{C^e} [N]^T \varepsilon_B \sigma t T^4_{rB} dC$, and

$\{f^e_{qB}\} = \int_{C^e} [N]^T q_{sB} t dC.$

8.3 Axisymmetric heat conduction

For simplicity, let us just consider an isotropic, heterogeneous body with a volumetric heat source. The governing equation for axisymmetric heat conduction is:

$\frac{1}{r}\frac{\partial}{\partial r}(rk\frac{\partial T}{\partial r}) + \frac{\partial}{\partial z}(k\frac{\partial T}{\partial z}) + Q = 0.$

The boundary conditions for convection and imposed heat fluxes are given later during the FE formulation, because it is more convenient then. Using the Galerkin method on an axisymmetric element basis,

$\int_{V^e} [N]^T \left[\frac{1}{r}\frac{\partial}{\partial r}(rk\frac{\partial T}{\partial r}) + \frac{\partial}{\partial z}(k\frac{\partial T}{\partial z}) + Q\right] dV = \{0\}$, where $dV = 2\pi r dr dz$.

Because r is independent of z, $\frac{\partial}{\partial z}(k\frac{\partial T}{\partial z})r = \frac{\partial}{\partial z}(rk\frac{\partial T}{\partial z})$.

Applying the Green-Gauss theorem to the two terms involving 2^{nd} order derivatives,

$$2\pi\int_{C^e}[N]^T rk\frac{\partial T}{\partial r}n_r dC - 2\pi\int_{A^e}\frac{\partial}{\partial r}[N]^T rk\frac{\partial T}{\partial r}drdz$$
$$+2\pi\int_{C^e}[N]^T rk\frac{\partial T}{\partial z}n_z dC - 2\pi\int_{A^e}\frac{\partial}{\partial z}[N]^T rk\frac{\partial T}{\partial z}drdz$$
$$+2\pi\int_{A^e}[N]^T Qrdrdz = \{0\}.$$

On the global boundary (Fig. 8.8), an energy balance gives:

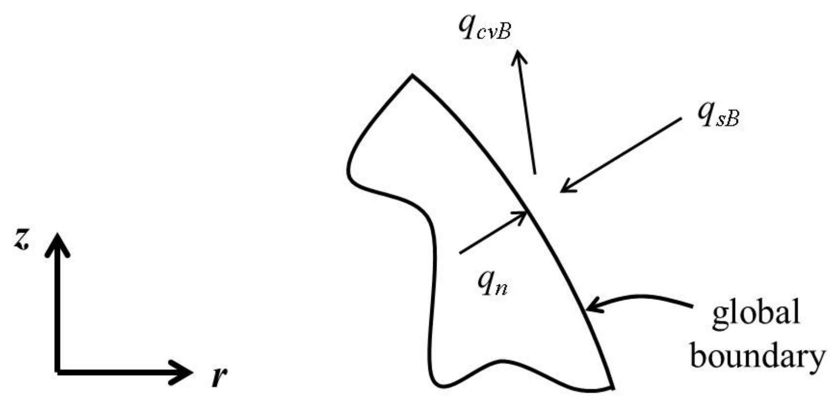

Figure 8.8: Equilibrium of boundary.

$q_n + q_{sB} = q_{cvB}$ or $q_n = q_{cvB} - q_{sB}$, with $q_n = -k\frac{\partial T}{\partial r}n_r - k\frac{\partial T}{\partial z}n_z$, and $q_{cvB} = h_B(T - T_{aB})$.

Therefore,

$$-2\pi\int_{C^e}[N]^T h_B r(T - T_{aB})dC + 2\pi\int_{C^e}[N]^T q_{sB} r dC$$
$$-2\pi\int_{A^e}\frac{\partial}{\partial r}[N]^T rk\frac{\partial T}{\partial r}drdz - 2\pi\int_{A^e}\frac{\partial}{\partial z}[N]^T rk\frac{\partial T}{\partial z}drdz$$
$$+2\pi\int_{A^e}[N]^T Qrdrdz = \{0\}.$$

Recalling that $T = [N]\{U^e\}$, we obtain $[K^e]\{U^e\} = \{f^e\}$ with
$[K^e] = [K^e_{rr}] + [K^e_{zz}] + [K^e_{cvB}]$ and
$\{f^e\} = \{f^e_Q\} + \{f^e_{qB}\} + \{f^e_{cvB}\}$.

The element stiffness matrices are given by:

$$[K^e_{rr}] = 2\pi \int_{A^e} \frac{\partial}{\partial r}[N]^T rk \frac{\partial}{\partial r}[N] dr dz,$$
$$[K^e_{zz}] = 2\pi \int_{A^e} \frac{\partial}{\partial z}[N]^T rk \frac{\partial}{\partial z}[N] dr dz,$$
$$[K^e_{cvB}] = 2\pi \int_{C^e} [N]^T h_B r [N] dC,$$

and the element nodal force vectors by:

$$\{f^e_Q\} = 2\pi \int_{A^e} [N]^T Q r dr dz,$$
$$\{f^e_{qB}\} = 2\pi \int_{C^e} [N]^T q_{sB} r dC, \text{ and}$$
$$\{f^e_{cvB}\} = 2\pi \int_{C^e} [N]^T h_B r T_{aB} dC.$$

Chapter 9

Fluid flow analysis

Learning Outcomes:

By the end of reading and attending the lecture of this chapter, the student should be able to:

-Explain the finite element formulation of a 2-D potential flow using the Galerkin method

-Explain the finite element formulation of a 2-D incompressible flow using the Galerkin method

Evidence of Learning:

A student has achieved the learning outcomes when he or she is able to to reproduce the knowledge listed above.

Evaluation Strategies:

Test problems in at-home assignments

Teaching and Learning Activities:

-At-home reading before class

-Lecturing

-In-class practice problems

-At-home reading/reviewing after class

-At-home practice problems

Note: please refer to a fluid mechanics course for details on the derivation of the governing equations used in this chapter.

9.1 2-D potential flow

An ideal fluid is incompressible and frictionless. The flow of an ideal fluid is called potential flow. No real fluid is frictionless, but outside the boundary layer, frictional effects may be neglected. The flow of an ideal fluid is also irrotational. Applications of potential flow include flows around airfoils, groundwater flow, flows through reservoirs, around corners.

The continuity equation in 2-D is given by $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, and the irrotational flow condition by $\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$, where u and v represent the x and y components of the fluid velocity. The following developments are restricted to flow around symmetric bodies without lift (Fig. 9.1).

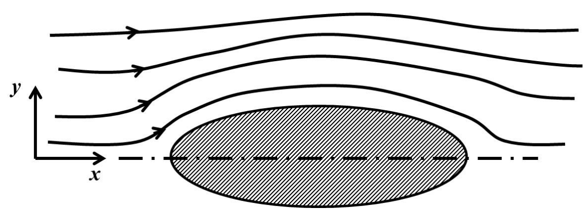

Figure 9.1: 2-D potential flow.

9.1.1 Velocity potential formulation

The irrotational flow condition is satisfied exactly if the velocity potential ϕ is defined by $u = -\frac{\partial \phi}{\partial x}$ and $v = -\frac{\partial \phi}{\partial y}$. Subbing these equations into the continuity equation yields Laplace's equation: $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$.

On an element basis, the Galerkin method gives

$\int_{A^e} [N]^T \left[\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right] dxdy = \{0\}$, where the shape function matrix $[N]$ depends on the element type used. From the Green-Gauss theorem, we get

$$\int_{C^e} [N]^T \frac{\partial \phi}{\partial x} n_x dC - \int_{A^e} \frac{\partial}{\partial x}[N]^T \frac{\partial \phi}{\partial x} dxdy$$
$$+ \int_{C^e} [N]^T \frac{\partial \phi}{\partial y} n_y dC - \int_{A^e} \frac{\partial}{\partial y}[N]^T \frac{\partial \phi}{\partial y} dxdy = \{0\}.$$

In other words,

$$-\int_{C^e}[N]^T u n_x dC - \int_{C^e}[N]^T v n_y dC$$
$$-\int_{A^e}\frac{\partial}{\partial x}[N]^T\frac{\partial\phi}{\partial x}dxdy - \int_{A^e}\frac{\partial}{\partial y}[N]^T\frac{\partial\phi}{\partial y}dxdy = \{0\}.$$

Representing the velocity potential ϕ as $\phi = [N]\{U^e\}$, where $\{U^e\}$ contains the values of the potential at the nodes of element e, the above equation can be combined into $[K^e]\{U^e\} = \{f^e\}$ with $[K^e] = [K^e_{xx}] + [K^e_{yy}]$ and $\{f^e\} = \{f^e_{uB}\} + \{f^e_{vB}\}$.

The element stiffness matrices are defined by

$$[K^e_{xx}] = \int_{A^e}\frac{\partial}{\partial x}[N]^T\frac{\partial}{\partial x}[N]dxdy,$$
$$[K^e_{yy}] = \int_{A^e}\frac{\partial}{\partial y}[N]^T\frac{\partial}{\partial y}[N]dxdy$$

and the element force vectors by

$$\{f^e_{uB}\} = -\int_{C^e}[N]^T u n_x dC \text{ and } \{f^e_{vB}\} = -\int_{C^e}[N]^T v n_y dC.$$

If a 3-node triangular element is used (see Chapter 4), then

$$[K^e_{xx}] = A\begin{bmatrix} m_{21}^2 & m_{21}m_{22} & m_{21}m_{23} \\ m_{22}m_{21} & m_{22}^2 & m_{22}m_{23} \\ m_{23}m_{21} & m_{23}m_{22} & m_{23}^2 \end{bmatrix}$$
$$[K^e_{yy}] = A\begin{bmatrix} m_{31}^2 & m_{31}m_{32} & m_{31}m_{33} \\ m_{32}m_{31} & m_{32}^2 & m_{32}m_{33} \\ m_{33}m_{31} & m_{33}m_{32} & m_{33}^2 \end{bmatrix}$$

The integrals for $\{f^e\}$ need to be evaluated on the element boundaries only. However, each direction cosine takes on opposite signs on legs shared by two elements. Therefore, the contributions from internal legs cancel out during assemblage, and $\{f^e\}$ needs to be evaluated only for those elements with legs on the global boundaries.

Example 18 *evaluate $\{f^e_{uB}\}$ in the case below (Fig.9.2):*

On leg ij, we have $N_k = L_k = 0$, thus $u = L_i u_i + L_j u_j$ (linear variation as required)

$$\{f^e_{uB}\} = -\int_{l_{ij}}\left\{\begin{array}{c} L_i \\ L_j \\ 0 \end{array}\right\}(L_i u_i + L_j u_j)n_x dl, \text{ where } l_{ij} \text{ is the length of leg } ij.$$
$$\{f^e_{uB}\} = -\frac{l_{ij}n_x}{6}\left\{\begin{array}{c} 2u_i + u_j \\ u_i + 2u_j \\ 0 \end{array}\right\}.$$

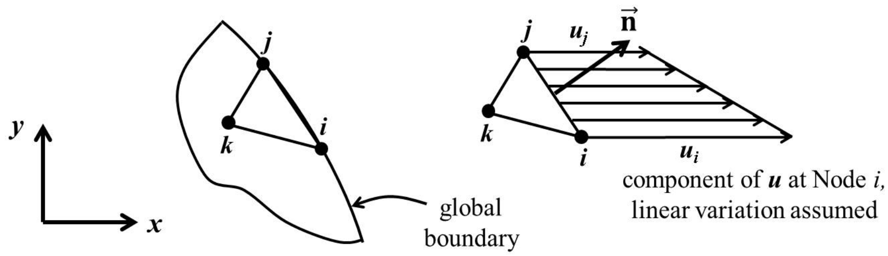

Figure 9.2: Boundary loading.

Similarly, $\{f_{vB}^e\} = -\frac{l_{ij}n_y}{6}\left\{\begin{array}{c} 2v_i + v_j \\ v_i + 2v_j \\ 0 \end{array}\right\}$.

One method for determining n_x and n_y for any leg of the element is now presented. Let us define vector $\overrightarrow{r_{ij}}$ running from node i to node j along leg ij (Fig. 9.3).

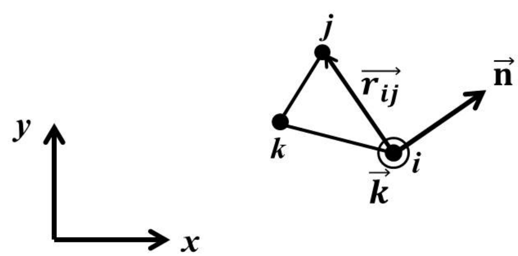

Figure 9.3: Normal components determination.

$\overrightarrow{r_{ij}} = (x_j - x_i)\overrightarrow{i} + (y_j - y_i)\overrightarrow{j}$
$\overrightarrow{n} = n_x\overrightarrow{i} + n_y\overrightarrow{j}$
$\overrightarrow{n} = \frac{\overrightarrow{r_{ij}} \times \overrightarrow{k}}{\left|\overrightarrow{r_{ij}} \times \overrightarrow{k}\right|}$, where cross product $\overrightarrow{a} \times \overrightarrow{b} = \det\begin{bmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix}$.
Finally, $n_x = \frac{y_j - y_i}{l_{ij}}$, $\quad n_y = -\frac{x_j - x_i}{l_{ij}}$, and $l_{ij} = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}$.
Similar results are obtained for leg jk and leg ki.
One may also write $V_n = un_x + vn_y$, and then

$\{f_{VB}^e\} = \{f_{uB}^e\} + \{f_{vB}^e\} = -\int_{C^e} [N]^T V_n dC$, with $V_n = -\frac{\partial \phi}{\partial n}$, where n represents the coordinate in the direction of the outward normal.

The average velocities $\overline{u}$ and $\overline{v}$ for a typical triangular element e can be determined from

$\overline{u} = -\frac{\partial \phi}{\partial x} = -\frac{\partial}{\partial x}[N]\{U^e\} = -m_{21}\phi_i - m_{22}\phi_j - m_{23}\phi_k$

$\overline{v} = -\frac{\partial \phi}{\partial y} = -\frac{\partial}{\partial y}[N]\{U^e\} = -m_{31}\phi_i - m_{32}\phi_j - m_{33}\phi_k$

where the values of ϕ_i, ϕ_j, ϕ_k are known from the solution of $[K]\{U\} = \{f\}$.

9.1.2 Stream function formulation

The continuity equation is satisfied exactly if the stream function ψ is defined by $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$. Subbing these expressions into the irrotational flow condition yields $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$.

Lines of constant ψ (streamlines) are perpendicular to the lines of constant ϕ. The FE formulation is very similar to that with the velocity potential and is not derived here.

9.2 2-D incompressible flow

Let us consider the laminar flow of a viscous fluid for the case of moderate Reynolds numbers (Fig. 9.4). For example, let us determine the 2-D velocity field for a fluid moving through a duct with an obstruction.

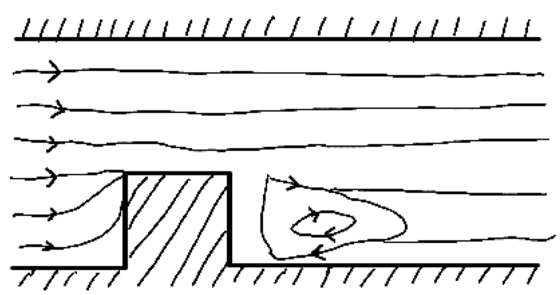

Figure 9.4: 2-D incompressible flow.

The continuity equation for an incompressible fluid is given by $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$.

The Navier-Stokes equations in the x- and y-directions are
$\rho(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}) = -\frac{\partial p}{\partial x} + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + b_x$
$\rho(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}) = -\frac{\partial p}{\partial y} + \mu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) + b_y$
where ρ is the fluid density, p is the pressure, μ the absolute viscosity, b_x and b_y body forces per unit volume in the x- and y-directions, respectively.

Let us note at once that the left-hand side of the above equations makes the problem nonlinear. Applying the Galerkin method to the continuity equation, we get
$\int_{A^e} [N]^T(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})tdxdy = \{0\}$, where t is the thickness (can be taken as unity) introduced so that the terms in the integral are volumetric flow rates.

Assuming $u = [N]\{U_u^e\}$ and $v = [N]\{U_v^e\}$ with
$\{U_u^e\}^T = [u_i \quad u_j \quad u_k]$ and $\{U_v^e\}^T = [v_i \quad v_j \quad v_k]$ for triangular elements, the previous equation can be recast in matrix form as
$[K_u^e]\{U_u^e\} + [K_v^e]\{U_v^e\} = \{0\}$, where
$[K_u^e] = \int_{A^e} [N]^T \frac{\partial}{\partial x}[N]tdxdy$ and $[K_v^e] = \int_{A^e} [N]^T \frac{\partial}{\partial y}[N]tdxdy.$

Applying now the Galerkin method to the first Navier-Stokes equation yields
$\int_{A^e} [N]^T(\rho(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}) + \frac{\partial p}{\partial x} - \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) - b_x)tdxdy = \{0\}$
where t is introduced so that the terms in the integral have units of force. Applying the Green-Gauss theorem to the terms involving 2^{nd} order derivatives, and representing pressure p over an element by $p = [N]\{U_p^e\}$, one obtains $\left([K_{u\,u}^e] + [K_{v\,u}^e] + [K_{u\,x}^e] + [K_{u\,y}^e]\right)\{U_u^e\} + [K_{px}^e]\{U_p^e\} = \{f_x^e\}$, where
$[K_{u\,u}^e] = \int_{A^e} [N]^T\rho[N]\{U_u^e\}\frac{\partial}{\partial x}[N]tdxdy,$
$[K_{v\,u}^e] = \int_{A^e} [N]^T\rho[N]\{U_v^e\}\frac{\partial}{\partial y}[N]tdxdy,$
$[K_{u\,x}^e] = \int_{A^e} \frac{\partial}{\partial x}[N]^T\mu\frac{\partial}{\partial x}[N]tdxdy,$
$[K_{u\,y}^e] = \int_{A^e} \frac{\partial}{\partial y}[N]^T\mu\frac{\partial}{\partial y}[N]tdxdy,$
$[K_{px}^e] = \int_{A^e} [N]^T\frac{\partial}{\partial x}[N]tdxdy$, and
$\{f_x^e\} = \int_{A^e} [N]^T b_x tdxdy.$

The integrals around the element bounding C^e need not be considered, because the velocity components are usually prescribed on the global boundary. In a similar fashion, if the Galerkin method is applied to the second Navier-Stokes equation, $\left([K^e_{u\,v}] + [K^e_{v\,v}] + [K^e_{v\,x}] + [K^e_{v\,y}]\right)\{U^e_v\} + [K^e_{py}]\{U^e_p\} = \{f^e_y\}$, where

$[K^e_{u\,v}] = [K^e_{u\,u}]$,
$[K^e_{v\,v}] = [K^e_{v\,u}]$,
$[K^e_{v\,x}] = [K^e_{u\,x}]$,
$[K^e_{v\,y}] = [K^e_{u\,y}]$,
$[K^e_{py}] = \int_{A^e} [N]^T \frac{\partial}{\partial y}[N] t dx dy$, and
$\{f^e_y\} = \int_{A^e} [N]^T b_y t dx dy$.

Combining all the equations relative to one element, one gets

$$\begin{bmatrix} [K^e_u] & [K^e_v] & [0] \\ [K^e_{u\,u}] + [K^e_{v\,u}] + [K^e_{u\,x}] + [K^e_{u\,y}] & [0] & [K^e_{px}] \\ [0] & [K^e_{u\,v}] + [K^e_{v\,v}] + [K^e_{v\,x}] + [K^e_{v\,y}] & [K^e_{py}] \end{bmatrix} \begin{Bmatrix} \{U^e_u\} \\ \{U^e_v\} \\ \{U^e_p\} \end{Bmatrix}$$
$$= \begin{Bmatrix} \{0\} \\ \{f^e_x\} \\ \{f^e_y\} \end{Bmatrix}.$$

A possible assemblage would produce a vector of nodal unknowns as $\{U\}^T = [u_1 \quad \cdots \quad u_n \quad v_1 \quad \cdots \quad v_n \quad p_1 \quad \cdots \quad p]$, but this would require increased bandwidth of the assemblage system equations. Instead, Let us denote the above submatrices and subvectors as follows

$$\begin{bmatrix} [K^{11}] & [K^{12} & [K^{13}] \\ [K^{21}] & [K^{22}] & [K^{23}] \\ [K^{31}] & [K^{32}] & [K^{33}] \end{bmatrix} \begin{Bmatrix} \{U^e_u\} \\ \{U^e_v\} \\ \{U^e_p\} \end{Bmatrix} = \begin{Bmatrix} \{f^1\} \\ \{f^2\} \\ \{f^3\} \end{Bmatrix}$$

and let us now use subscripts to indicate the entries within each submatrix, e.g.

$$[K^e_u] = [K^{11}] = \begin{bmatrix} [K^{11}_{11}] & [K^{11}_{12}] & [K^{11}_{13}] \\ [K^{11}_{21}] & [K^{11}_{22}] & [K^{11}_{23}] \\ [K^{11}_{31}] & [K^{11}_{32}] & [K^{11}_{33}] \end{bmatrix} \text{ and } \{f^e_x\} = \{f^2\} = \begin{Bmatrix} f^2_1 \\ f^2_2 \\ f^2_3 \end{Bmatrix}.$$

On an element basis, with $\{U^e\}^T = [u_i \quad v_i \quad p_i \quad u_j \quad v_j \quad p_j \quad u_k \quad v_k \quad p_k]$,

$$
\begin{bmatrix}
[K^{11}_{11}] & [K^{12}_{11} & [K^{13}_{11}] & [K^{11}_{12}] & [K^{12}_{12} & [K^{13}_{12}] & [K^{11}_{13}] & [K^{12}_{13} & [K^{13}_{13}] \\
[K^{21}_{11}] & [K^{22}_{11}] & [K^{23}_{11}] & [K^{21}_{12}] & [K^{22}_{12}] & [K^{23}_{12}] & [K^{21}_{13}] & [K^{22}_{13}] & [K^{23}_{13}] \\
[K^{31}_{11}] & [K^{32}_{11}] & [K^{33}_{11}] & [K^{31}_{12}] & [K^{32}_{12}] & [K^{33}_{12}] & [K^{31}_{13}] & [K^{32}_{13}] & [K^{33}_{13}] \\
[K^{11}_{21}] & [K^{12}_{21} & [K^{13}_{21}] & [K^{11}_{22}] & [K^{12}_{22} & [K^{13}_{22}] & [K^{11}_{23}] & [K^{12}_{23} & [K^{13}_{23}] \\
[K^{21}_{21}] & [K^{22}_{21}] & [K^{23}_{21}] & [K^{21}_{22}] & [K^{22}_{22}] & [K^{23}_{22}] & [K^{21}_{23}] & [K^{22}_{23}] & [K^{23}_{23}] \\
[K^{31}_{21}] & [K^{32}_{21}] & [K^{33}_{21}] & [K^{31}_{22}] & [K^{32}_{22}] & [K^{33}_{22}] & [K^{31}_{23}] & [K^{32}_{23}] & [K^{33}_{23}] \\
[K^{11}_{31}] & [K^{12}_{31} & [K^{13}_{31}] & [K^{11}_{22}] & [K^{12}_{22} & [K^{13}_{22}] & [K^{11}_{33}] & [K^{12}_{33} & [K^{13}_{33}] \\
[K^{21}_{31}] & [K^{22}_{31}] & [K^{23}_{31}] & [K^{21}_{22}] & [K^{22}_{22}] & [K^{23}_{22}] & [K^{21}_{33}] & [K^{22}_{33}] & [K^{23}_{33}] \\
[K^{31}_{31}] & [K^{32}_{31}] & [K^{33}_{31}] & [K^{31}_{22}] & [K^{32}_{22}] & [K^{33}_{22}] & [K^{31}_{33}] & [K^{32}_{33}] & [K^{33}_{33}]
\end{bmatrix}
\begin{Bmatrix} u_i \\ v_i \\ p_i \\ u_j \\ v_j \\ p_j \\ u_k \\ v_k \\ p_k \end{Bmatrix}
= \begin{Bmatrix} f^1_1 \\ f^2_1 \\ f^3_1 \\ f^1_2 \\ f^2_2 \\ f^3_2 \\ f^1_3 \\ f^2_3 \\ f^3_3 \end{Bmatrix}.
$$

The assemblage is now possible such that the vector of nodal unknowns is

$$\{U\}^T = [u_1 \quad v_1 \quad p_1 \quad \cdots \quad u_n \quad v_n \quad p_n]\,.$$

Note that the matrix is not symmetric, and that the system is nonlinear (matrix depends on $\{U\}$).

Chapter 10

Transient and dynamic analyses

Learning Outcomes:
By the end of reading and attending the lecture of this chapter, the student should be able to:
-Explain the finite element formulation of dynamic structural or transient thermal problems
-Distinguish betwen consistent and lumped mass matrices
-Explain two- and three-point recurrence schemes as solution methods
-Explain the link between finite elment formulation and modla analysis
Evidence of Learning:
A student has achieved the learning outcomes when he or she is able to to reproduce the knowledge listed above.
Teaching and Learning Activities:
-At-home reading before class
-Lecturing
-In-class practice problems
-At-home reading/reviewing after class
-At-home practice problems

Transient and dynamic refer both to unsteady, time-dependent or propagation problems. Transient is mostly used for non structural analyses, while dynamic is used to describe structural problems. So far, we have only dealt with discretizations in space, e.g. $\phi(x,y) = [N(x,y)]\{\phi^e\}$, where $\{\phi^e\}$ is composed of constants. In time-dependent problems, time t is important. We could use a space and time discretization such that $\phi(x,y,t) = [N(x,y,t)]\{\phi^e\}$, but this increases the number of dimensions by one. This complexity can be avoided by using partial discretization, where the vector of nodal unknowns is a function of time: $\phi(x,y,t) = [N(x,y)]\{\phi^e(t)\}$. We will no longer have $[K^e]\{\phi^e\} = \{f^e\}$, additional terms will appear.

10.1 Dynamic structural analysis

Using d'Alembert's principle, the elemental inertial and damping forces $\{df_I\}$ and $\{df_D\}$ are given by

$\{df_I\} = -\rho\{\ddot{u}\}dV$ and $\{df_D\} = -[\mu]\{\dot{u}\}dV$, where ρ is the mass density, $[\mu]$ is the viscous matrix, $\{\ddot{u}\}$ the acceleration vector and $\{\dot{u}\}$ the velocity vector, and the minus sign describes that these are forces that resist displacements.

Then, the principle of virtual work in dynamics, for one element, is
$\int_{V^e} \{\delta\varepsilon\}^T \{\sigma\}\, dV =$
$\int_{V^e} \{\delta u\}^T \{b\}\, dV + \int_{A^e} \{\delta u\}^T \{s\}\, dA + \sum_{p=1}^{p=N} \{\delta u\}^T \{f_p\}$
$- \int_{V^e} \{\delta u\}^T \rho\{\ddot{u}\}dV - \int_{V^e} \{\delta u\}^T [\mu]\{\dot{u}\}dV,$
$\forall \{\delta u\}, \{\delta\varepsilon\}$

Note that, with $\{u\} = [N]\{U^e\}$,
$\{\dot{u}\} = \frac{d}{dt}\{u\} = \frac{d}{dt}([N]\{U^e\}) = [N]\frac{d}{dt}\{U^e\} = [N]\{\dot{U}^e\}$, and similarly,
$\{\ddot{u}\} = [N]\{\ddot{U}^e\}$.

One finally obtains $[M^e]\{\ddot{U}^e\} + [D^e]\{\dot{U}^e\} + [K^e]\{U^e\} = \{f^e\}$, where $[K^e]$ and $\{f^e\}$ are the same as in statics, and $[M^e]$ is the element mass matrix, while $[D^e]$ is the element damping matrix.

$[M^e] = \int_{V^e} [N]^T \rho [N] dV$
$[D^e] = \int_{V^e} [N]^T [\mu][N] dV$

The viscous matrix $[\mu]$ is not known in general. After assemblage, $[D]$ is usually taken to be of the form

$[D] = \alpha[M] + \beta[K]$, where α, β are experimentally determined constants (Rayleigh damping). In any event, the assemblage system of equations can be written as

$[M]\{\ddot{U}\}+[D]\{\dot{U}\}+[K]\{U\}=\{f\}$ once boundary conditions are applied.

Example 19 *determine $[M^e]$ for a 2-node, 1-D element*

$[M^e]=\int_{V^e}[N]^T\rho[N]dV=\int_{l^e}\left\{\begin{array}{c}L_i\\ L_j\end{array}\right\}\rho\,[L_i \quad L_j]\,Adx$

$[M^e]=\int_{l^e}\left[\begin{array}{cc}L_i^2 & L_iL_j\\ L_jL_i & L_j^2\end{array}\right]\rho Adx$

$[M^e]=\frac{\rho AL}{6}\left[\begin{array}{cc}2 & 1\\ 1 & 2\end{array}\right]$

10.2 Transient thermal analysis

If an energy storage mechanism is included into the steady state heat transfer equation, the 1-D governing equation becomes

$\rho cA\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}(kA\frac{\partial T}{\partial x})-hP(T-T_a)-\varepsilon\sigma P(T^4-T_r^4)+QA$, where

ρ : mass density

c : specific heat (at constant volume or constant pressure).

In 2-D,

$\rho c\bar{t}\frac{\partial T}{\partial t}=$

$\frac{\partial}{\partial x}(k\bar{t}\frac{\partial T}{\partial x})+\frac{\partial}{\partial y}(k\bar{t}\frac{\partial T}{\partial y})-h(T-T_a)-\varepsilon\sigma(T^4-T_r^4)+q_s+Q\bar{t}$, where

$\bar{t}$: thickness of the 2-D region (not to be confused with time t).

In 3-D,

$\rho c\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x})+\frac{\partial}{\partial y}(k\frac{\partial T}{\partial y})+\frac{\partial}{\partial z}(k\frac{\partial T}{\partial z})+Q$.

For axisymmetric problems,

$\rho c\frac{\partial T}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}(rk\frac{\partial T}{\partial r})+\frac{\partial}{\partial z}(k\frac{\partial T}{\partial z})+Q$.

Using partial discretization, $T=[N]\{U^e(t)\}$, where the shape function matrix $[N]$ is unchanged compared to the steady state case,

and $\{U^e(t)\}^T=[T_1(t) \quad \cdots \quad T_n(t)]$, where $T_i(t)$ is the temperature of node i at time t, and n is the number of nodes defining element e.

Applying the Galerkin method to the above equation in the 2-D case yields

$\int_{A^e}[N]^T\left[\begin{array}{c}\rho c\bar{t}\frac{\partial T}{\partial t}-\frac{\partial}{\partial x}(k\bar{t}\frac{\partial T}{\partial x})-\frac{\partial}{\partial y}(k\bar{t}\frac{\partial T}{\partial y})+\\ h(T-T_a)+\varepsilon\sigma(T^4-T_r^4)-q_s-Q\bar{t}\end{array}\right]dxdy=\{0\}$, from which is derived

$[C^e]=\int_{A^e}[N]^T\rho c\bar{t}[N]dxdy$ in 2-D,

$[C^e]=\int_{l^e}[N]^T\rho cA[N]dx$ in 1-D,

$[C^e]=\int_{V^e}[N]^T\rho c[N]dxdydz$ in 3-D, and

$[C^e] = \int_{A^e} 2\pi[N]^T \rho c r [N] dr dz$ for axisymmetric problems.

Example 20 *determine $[C^e]$ for a 3-node triangular element in 2-D*

$$[C^e] = \int_{A^e} [N]^T \rho c \bar{t} [N] dx dy =$$

$$[C^e] = \int_{A^e} \left\{ \begin{array}{c} L_i \\ L_j \\ L_k \end{array} \right\} \rho c \bar{t} [L_i \quad L_j \quad L_k] dx dy$$

$$[C^e] = \int_{A^e} \left[\begin{array}{ccc} L_i^2 & L_i L_j & L_i L_k \\ L_j L_i & L_j^2 & L_j L_k \\ L_k L_i & L_k L_j & L_k^2 \end{array} \right] \rho c \bar{t} dx dy$$

$$[C^e] = \frac{\rho c \bar{t} A}{12} \left[\begin{array}{ccc} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{array} \right]$$, where A is the element area.

Assemblage of the matrices is done as usual, and the initial and boundary conditions must be imposed before solving.

10.3 Lumped versus consistent matrices

Consistent matrices are those obtained directly from their definitions for mass and capacitance matrices. They are not necessarily diagonal which, as will be seen in the next section, may make it more difficult to solve the problem. Therefore, consistent mass and capacitance matrices can be *made* into diagonal lumped mass and capacitance matrices, *but note that this not done by mathematically correct diagonalization procedures.* There are different ways of doing it. A rule of thumb consists in summing the entries in a given row in the consistent matrix, dividing the result by the total of all entries, and allocating this result to the diagonal entry of the row under consideration.

Example 21 $[M^e] = \rho A L \left[\begin{array}{cc} \frac{2}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{6} \end{array} \right]$ *becomes* $[M^e] = \rho A L \left[\begin{array}{cc} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{array} \right]$.

Advantage: in addition to simplifying the solution process (see below), initial oscillations of the solution have been reported not to occur if lumped matrix is used (Fig. 10.1).

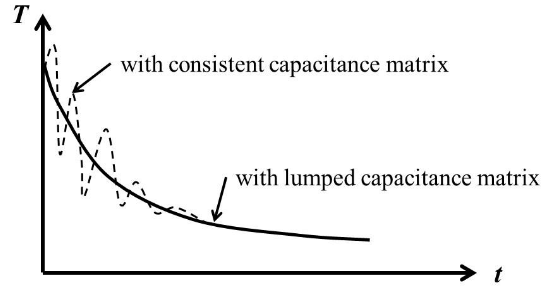

Figure 10.1: Comparison of transient solutions, with consistent capacitance matrix versus lumped capacitance matrix.

10.4 Solution methods

10.4.1 Two-point recurrence scheme

Let us consider equation $[C]\{\dot{U}\} + [K]\{U\} = \{f\}$.

Temporal elements can be introduced to discretize the time domain, and the weighted-residual method can be used from time t to $t+\Delta t$, or t_i to t_{i+1} : $\int_{t_i}^{t_{i+1}} W \left[[C]\{\dot{U}\} + [K]\{U\} - \{f\} \right] dt = \{0\}$, where the weighting function W is a scalar.

With the help of the shape functions, we can represent vector $\{U\}$ by $\{U\} = N_i\{U_i\} + N_{i+1}\{U_{i+1}\}$, where the shape functions are linear in time (Fig. 10.2).

Note that $\{U\} = \{U_i\}$ at $t = t_i$ and $\{U\} = \{U_{i+1}\}$ at $t = t_{i+1}$.

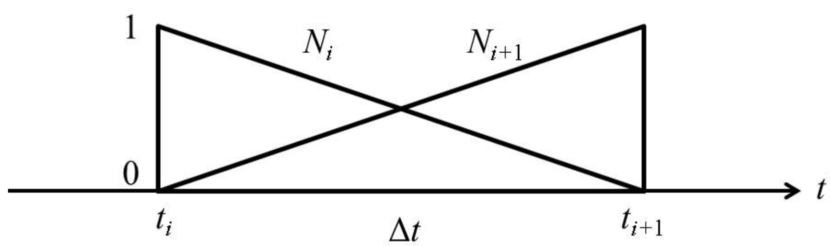

Figure 10.2: Linear timewise interpolation between time t_i and t_{i+1}.

For convenience, let us write $N_i = 1 - \xi$ and $N_{i+1} = \xi$, where $\xi = \frac{t-t_i}{\Delta t}$ with $\Delta t = t_{i+1} - t_i$. Then, $0 \leq \xi \leq 1$, and

$\{U\} = (1-\xi)\{U_i\} + \xi\{U_{i+1}\}$, from which it follows that

$\{\dot{U}\} = \frac{d}{dt}\{U\} = \frac{d}{d\xi}\{U\}\frac{d\xi}{dt}$

$\{\dot{U}\} = \frac{d}{d\xi}[(1-\xi)\{U_i\} + \xi\{U_{i+1}\}]\frac{d\xi}{dt}$, or

$\{\dot{U}\} = -\frac{1}{\Delta t}\{U_i\} + \frac{1}{\Delta t}\{U_{i+1}\}$.

Similarly,

$\{f\} = (1-\xi)\{f_i\}+\xi\{f_{i+1}\}$, therefore the weighted residual method yields, with $dt = \Delta t\, d\xi$

$$\int_0^1 W \left\{ \begin{array}{c} [C]\left[-\frac{1}{\Delta t}\{U_i\} + \frac{1}{\Delta t}\{U_{i+1}\}\right] \\ +[K]\left[(1-\xi)\{U_i\} + \xi\{U_{i+1}\}\right] \\ -\left[(1-\xi)\{f_i\} + \xi\{f_{i+1}\}\right] \end{array} \right\} \Delta t\, d\xi = \{0\}, \text{ or}$$

$$\left[[C]\int_0^1 W d\xi + [K]\Delta t \int_0^1 W\xi d\xi\right]\{U_{i+1}\} =$$
$$\left[[C]\int_0^1 W d\xi - [K]\Delta t \int_0^1 W(1-\xi) d\xi\right]\{U_i\}$$
$$+\left[\Delta t \int_0^1 W(1-\xi)d\xi\right]\{f_i\} + \left[\Delta t\int_0^1 W\xi d\xi\right]\{f_{i+1}\}.$$

Dividing both sides by $\int_0^1 W d\xi$ gives:

$$[[C] + \theta[K]\Delta t\,]\,\{U_{i+1}\} =$$
$$[[C] - (1-\theta)[K]\Delta t\,]\,\{U_i\} + [(1-\theta)\{f_i\} + \theta\{f_{i+1}\}]\,\Delta t\,, \text{ where}$$

$$\theta = \frac{\int_0^1 W\xi d\xi}{\int_0^1 W d\xi}.$$

Depending on the choice of weighting function W, θ varies.

Point collocation

at time $t_i : \theta = 0$ (forward difference, or Euler's method)
at time $\frac{t_i+t_{i+1}}{2} : \theta = \frac{1}{2}$ (central difference, or Crank-Nicholson's method)
at time $t_{i+1} : \theta = 1$ (backward difference)

If $\theta = 0$, $[C]\,\{U_{i+1}\} = [[C] - [K]\Delta t\,]\,\{U_i\} + \{f_i\}\Delta t$
known at any time t, including $t = 0$

- If lumped (diagonal) $[C]$ is used, $\{U_{i+1}\}$ can be explicitly calculated: explicit method.

- If consistent $[C]$ is used, $[C]^{-1}$ must be calculated: implicit method.
- Potential problems with stability and accuracy of solution.

If $\theta = 1$, $\left[[C] + [K]\Delta t\right]\{U_{i+1}\} =$ $[C]\{U_i\} + \{f_{i+1}\}\Delta t$
known at any time t,
because $\{f_{i+1}\}$ is the forcing function

- Implicit solution because $[K]$ cannot be lumped.
- Method always stable, but accuracy deteriorates if Δt increases.

If $\theta = \frac{1}{2}$,

$$\left[[C] + [K]\frac{\Delta t}{2}\right]\{U_{i+1}\} = \left[[C] - [K]\frac{\Delta t}{2}\right]\{U_i\} + \frac{\Delta t}{2}\left[\{f_i\} + \{f_{i+1}\}\right]$$

known at any time t

- Implicit solution.
- Stability problems if Δt above critical value.
- Accuracy decreases as Δt decreases.
- Most accurate method of the three.

Subdomain collocation

with $W = 1$ from t_i to t_{i+1} : $\theta = \frac{1}{2}$.

Galerkin method

with $W = N_i = 1 - \xi : \theta = \frac{1}{3}$
with $W = N_{i+1} = \xi : \theta = \frac{2}{3}$

If $\theta = \frac{2}{3}$, the solution is more accurate than if $\theta = 1$, and more stable than if $\theta = \frac{1}{2}$.

Note: if $[C]$ or $[K]$ is temperature dependent, the equations are nonlinear and require an iterative solution such that $\{U\}^{j+1} = (1-\theta)\{U_i\}^j + \theta\{U_{i+1}\}^j$, where j is the iteration number. Exception: when $\theta = 0$, no iteration is necessary, because all is known at t.

10.4.2 Three-point recurrence scheme

Let us consider equation $[M]\{\ddot{U}\} + [D]\{\dot{U}\} + [K]\{U\} = \{f\}$, and let t_{i-1}, t_i and t_{i+1} respectively represent $t - \Delta t$, t, and $t + \Delta t$. The weighted residual method requires

$$\int_{t_i-1}^{t_{i+1}} W \left[\begin{array}{c} [M]\{\ddot{U}\} + [D]\{\dot{U}\} \\ +[K]\{U\} - \{f\} \end{array} \right] dt = \{0\}.$$

Assuming $N_{i-1} = -\frac{1}{2}r(1-r)$ with $r = \frac{t-t_i}{\Delta t}$, $\Delta t = t_{i+1} - t_i = t_i - t_{i-1}$
$N_i = (1+r)(1-r)$
$N_{i+1} = \frac{1}{2}r(1+r)$

Then, for $t_{i-1} \leq t \leq t_{i+1}$, $-1 \leq r \leq 1$, and $dr = \frac{dt}{\Delta t}$.

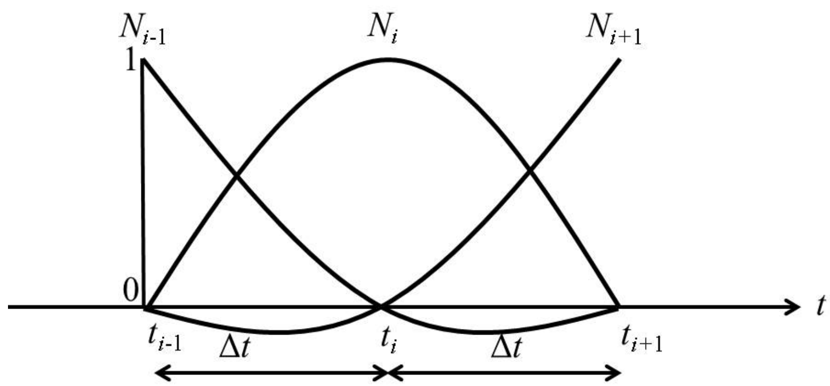

Figure 10.3: Quadratic interpolation between t_{i-1}, t_i and t_{i+1}.

In a fashion similar to what was done with the two-point scheme, we get:

$[[M] + \gamma[D]\Delta t + \lambda[K]\Delta t^2]\{U_{i+1}\} =$
$[2[M] - (1-2\gamma)[D]\Delta t - (\frac{1}{2} - 2\lambda + \gamma)[K]\Delta t^2]\{U_i\}$
$+[-[M] + (1-\gamma)[D]\Delta t - (\frac{1}{2} + \lambda - \gamma)[K]\Delta t^2]\{U_{i-1}\}$
$-(\frac{1}{2} + \lambda - \gamma)\Delta t^2]\{f_{i-1}\} + (\frac{1}{2} - 2\lambda + \gamma)\Delta t^2]\{f_i\} - \lambda\Delta t^2]\{f_{i+1}\}$,

where $\lambda = \dfrac{\int_{-1}^{+1} \frac{1}{2}Wr(1+r)dr}{\int_{-1}^{+1} W dr}$, $\gamma = \dfrac{\int_{-1}^{+1} W(\frac{1}{2}+r)dr}{\int_{-1}^{+1} W dr}$.

This is because $\{\dot{U}\} = \frac{1}{\Delta t}[(-\frac{1}{2} + r)\{U_{i-1}\} - 2r\{U_i\} + (\frac{1}{2} + r)\{U_{i+1}\}]$,
$\{\ddot{U}\} = \frac{1}{\Delta t^2}[\{U_{i-1}\} - 2\{U_i\} + \{U_{i+1}\}]$, and
$\{f\} = N_{i-1}\{f_{i-1}\} + N_i\{f_i\} + N_{i+1}\{f_{i+1}\}$.

W		λ	γ
Point collocation	at t_{i-1}	0	$-\frac{1}{2}$
	at t_i	0	$\frac{1}{2}$
	at t_{i+1}	1	$\frac{3}{2}$
Subdomain collocation		$\frac{1}{6}$	$\frac{1}{2}$
Galerkin, based	on N_{i-1}	$-\frac{1}{5}$	$-\frac{1}{2}$
	on N_i	$\frac{1}{10}$	$\frac{1}{2}$
	on N_{i+1}	$\frac{4}{5}$	$\frac{3}{2}$

10.4.3 Initial conditions

In the three-point scheme, the values of $\{U_i\}$ and $\{U_{i-1}\}$ are needed to determine $\{U_{i+1}\}$. In particular, initial conditions (i.e. information at time t_0 and for which $i = 1$) are needed to start the solution scheme.

Let us write $[M]\{\dot{V}\} + [D]\{V\} + [K]\{U\} = \{f\}$ with $\{\dot{U}\} = \{V\}$. The initial displacements and velocities are noted $\{U_0\}$ and $\{V_0\}$. We need $\{U_1\}$.

Euler starting method (less accurate)

Using the forward difference scheme or Euler's method, $\frac{\{U_i\}-\{U_{i-1}\}}{\Delta t} = \{V_{i-1}\}$. For $i = 1$, we get $\{U_1\} = \{U_0\} + \{V_0\}\Delta t$.

Crank-Nicholson starting method (more accurate)

Using the central difference scheme or Crank-Nicholson's method,

$[M]\frac{\{V_i\}-\{V_{i-1}\}}{\Delta t} + [D]\frac{\{V_i\}+\{V_{i-1}\}}{2} + [K]\frac{\{U_i\}+\{U_{i-1}\}}{2} = \frac{1}{2}(\{f_i\} + \{f_{i-1}\})$, with $\frac{\{U_i\}-\{U_{i-1}\}}{\Delta t} = \frac{\{V_i\}+\{V_{i-1}\}}{2}$, from which $\{V_i\}$ is determined.

For $i = 1$, we get $\{U_1\} = \left[2\frac{[M]}{\Delta t} + [D] + [K]\frac{\Delta t}{2}\right]^{-1}$

$\times \left\{ \left[2\frac{[M]}{\Delta t} + [D] - [K]\frac{\Delta t}{2}\right] \{U_0\} + 2[M]\{V_0\} + \frac{\Delta t}{2}\{f_0\} + \frac{\Delta t}{2}\{f_1\} \right\}.$

Once $\{U_0\}$ and $\{U_1\}$ are known, the three point scheme can proceed with any of the methods in Section 10.4.2.

10.5 Introduction to modal analysis

Many times in dynamic structural analysis, the nodal displacements are not really needed. Instead, the natural frequencies of the sustained vibrations are of interest. Natural frequencies occur when the forcing function $\{f(t)\}$ is identically zero, and damping is negligible. Therefore, let us consider $[M]\{\ddot{U}\} + [K]\{U\} = \{0\}$, and represent $\{U\}$ as $\{U\} = \{\overline{U}\}e^{j\omega t}$ $(j^2 = -1)$, where $\{\overline{U}\}$ is the eigenvector corresponding to natural frequency ω. Then, $\{\ddot{U}\} = (j\omega)^2\{\overline{U}\}e^{j\omega t} = -\omega^2\{\overline{U}\}e^{j\omega t}$. The initial equation becomes $(-\omega^2[M] + [K])\{\overline{U}\}e^{j\omega t} = \{0\}$. Since only non-trivial solutions for $\{\overline{U}\}$ are sought, the determinant $\det(-\omega^2[M] + [K])$ must be zero, guaranteeing that matrix $-\omega^2[M] + [K]$ is singular. $\det(-\omega^2[M] + [K]) = 0$ can be solved for the natural frequencies ω of the structure. If N nodes are used in the discretization of the structure, and each node has m d.o.f., we get mN natural frequencies (vs. an infinite number for a continuous structure, i.e. not discrete). The frequencies obtained on the low end of the spectrum are quite accurate if enough elements are used. Eigenvectors (i.e. such that $(-\omega^2[M] + [K])\{\overline{U_j}\} = \{0\}$) are usually normalized such that $\{\overline{U_j}\}^T[M]\{\overline{U_j}\} = 1$. They represent mode shapes. They are also used to construct solutions to problems with damping and/or forcing functions (mode superposition).