MAT 3320 Assignment #6 Solutions

1. $u_{tt} = u_{rr} + \frac{1}{r} u_r$, $u(4,t) = 0$, $u(r,0) = 6 J_0(\frac{\alpha_7}{4} r)$, $u_t(r,0) = 0$

so $c = 1$, $R = 4$, $f(r) = 6 J_0(\frac{\alpha_7}{4} r)$, $g(r) = 0$

then the solution has the form

$$u(r,t) = \sum_{m=1}^{\infty} \left(a_m \cos(\alpha_m t/4) + b_m \sin(\alpha_m t/4)\right) J_0\left(\frac{\alpha_m}{4} r\right)$$

$$u(r,0) = \sum_{m=1}^{\infty} a_m J_0\left(\frac{\alpha_m}{4} r\right) = f(r) = 6 J_0\left(\frac{\alpha_7}{4} r\right)$$

so $a_7 = 6$, all others are 0

$g(r) = 0 \Rightarrow$ all $b_m = 0$

$\therefore$ the solution is $\boxed{u(r,t) = 6 \cos(\alpha_7 t/4) J_0\left(\frac{\alpha_7}{4} r\right)}$

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2. $u_t = u_{rr} + \frac{1}{r} u_r$, $u(2,t) = 0$, $u(r,0) = \sqrt{2} J_0(\frac{\alpha_3}{2} r) + J_0(\frac{\alpha_5}{2} r)$

so $c = 1$, $R = 2$ and $f(r) = \sqrt{2} J_0(\frac{\alpha_3}{2} r) + J_0(\frac{\alpha_5}{2} r)$

the solution has the form

$$u(r,t) = \sum_{m=1}^{\infty} A_m J_0\left(\frac{\alpha_m}{2} r\right) e^{-\alpha_m^2 t/4}$$

$$u(r,0) = \sum_{m=1}^{\infty} A_m J_0\left(\frac{\alpha_m}{2} r\right) = \sqrt{2} J_0\left(\frac{\alpha_3}{2} r\right) + J_0\left(\frac{\alpha_5}{2} r\right)$$

so $A_3 = \sqrt{2}$, $A_5 = 1$ and all others are 0

$\therefore$ the solution is

$$\boxed{u(r,t) = \sqrt{2} J_0\left(\frac{\alpha_3}{2} r\right) e^{-\alpha_3^2 t/4} + J_0\left(\frac{\alpha_5}{2} r\right) e^{-\alpha_5^2 t/4}}$$

3. $u_t = 4\left(u_{rr} + \frac{1}{r} u_r\right)$, $u_r(3,t) = 0$, $u(r,0) = 2 + J_0\left(\frac{\beta_2}{3} r\right)$

we have $c = 2$, $R = 3$ and $f(r) = 2 + J_0\left(\frac{\beta_2}{3} r\right)$

the solution has the form

$$u(r,t) = a_1 + \sum_{m=2}^{\infty} a_m J_0\left(\frac{\beta_m}{3} r\right) e^{-4\beta_m^2 t/9}$$

$$u(r,0) = a_1 + \sum_{m=2}^{\infty} a_m J_0\left(\frac{\beta_m}{3} r\right) = 2 + J_0\left(\frac{\beta_2}{3} r\right)$$

so $a_1 = 2$, $a_2 = 1$ and all others are 0

$\therefore$ the solution is $\boxed{u(r,t) = 2 + J_0\left(\frac{\beta_2}{3} r\right) e^{-4\beta_2^2 t/9}}$

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4. $R = 2$, $u(r,\theta) = \sum_{n=0}^{\infty} r^n \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right]$

$$u^*(r,\theta) = \sum_{n=0}^{\infty} r^{-n} \left[a_n^* \cos(n\theta) + b_n^* \sin(n\theta)\right]$$

then $u(2,\theta) = \sum_{n=0}^{\infty} 2^n \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right]$

$$= 2\cos(\theta) - 5\sin(3\theta)$$

$$= u^*(2,\theta) = \sum_{n=0}^{\infty} 2^{-n} \left[a_n^* \cos(n\theta) + b_n^* \sin(n\theta)\right]$$

so $2^1 a_1 = 2 = 2^{-1} a_1^* \Rightarrow a_1 = 1, \; a_1^* = 4$

$2^3 b_3 = -5 = 2^{-3} b_3^* \Rightarrow b_3 = -5/8, \; b_3^* = -40$

all other coefficients are 0

$\therefore$ the solutions are

$$\boxed{u(r,\theta) = r\cos(\theta) - \frac{5}{8} r^3 \sin(3\theta)}$$

and $\boxed{u^*(r,\theta) = \frac{4}{r}\cos(\theta) - \frac{40}{r^3}\sin(3\theta)}$