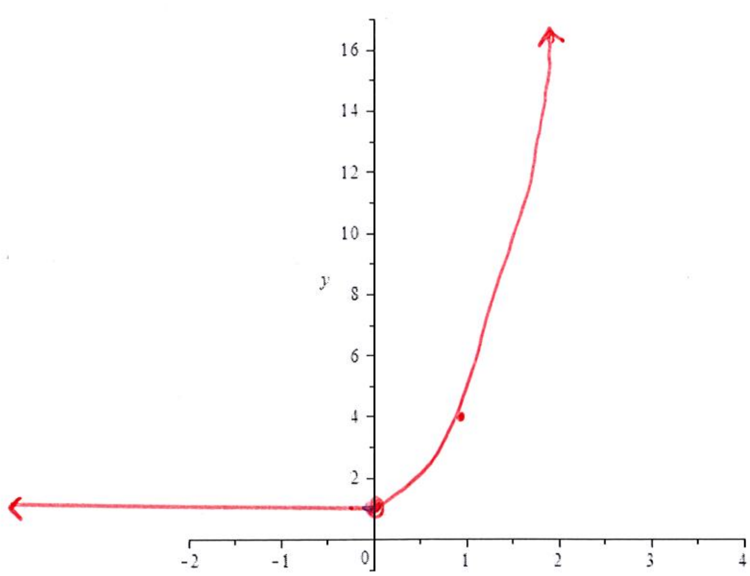
2 Inequalities

2.1 Piecewise Functions

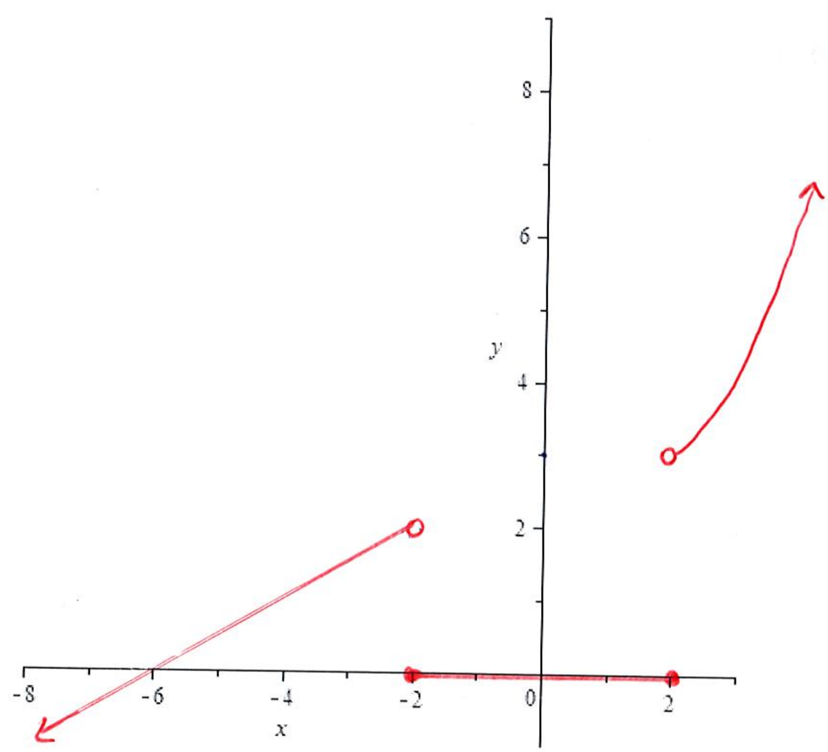
Piecewise or "branch" functions typically feature one or more points at which the function changes from one form to another. To graph a piecewise function, simply graph each **p** iece and then restrict it to its designated **d** omain . Pay special attention when plotting the '**b** reaking **p** oints '; closed circle **i** nclude the point, open circle **e** xclude the point.

Example 1. *Graph the function given by* $y = \begin{cases} 1, & \textit{if } x \leq 0 \\ 4^x, & \textit{if } x > 0. \end{cases}$



Example 2. *Graph the piecewise function given by*

$$y = \begin{cases} \frac{1}{2}x + 3, & \textit{if } x < -2 \\ 0, & \textit{if } -2 \le x \le 2 \\ x^2 - 1, & \textit{if } x > 2. \end{cases}$$

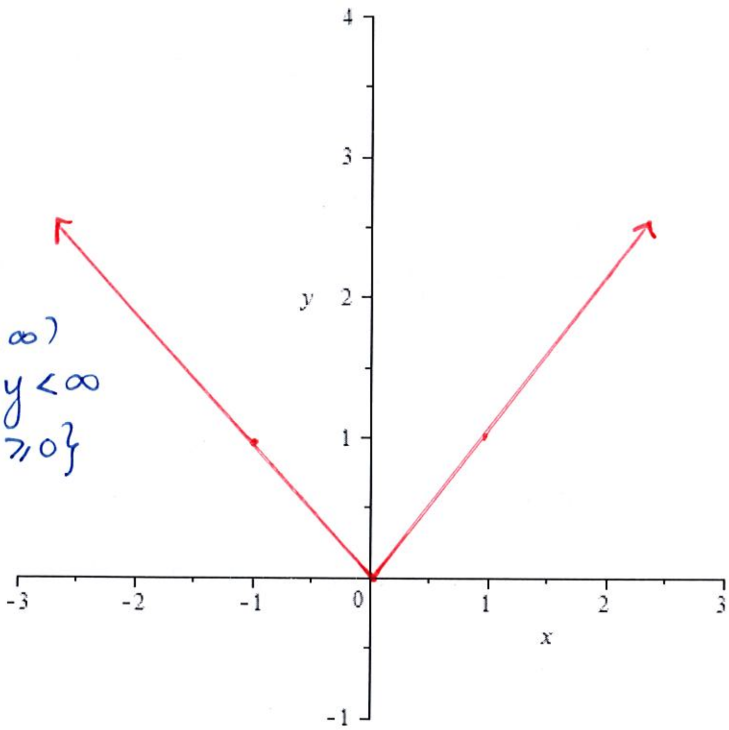


2.2 The Absolute Value Function

A very special — and *very* common— piecewise function is the absolute value function. Be familiar (and comfortable!) with it.

Basic Form:

$$y = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$$



Domain = $\mathbb{R}$ $(-\infty, \infty)$ $-\infty < x < \infty$ Range = $0 \leq y < \infty$ $[0, \infty)$ $\{y \in \mathbb{R} : y \geq 0\}$ in

Obeys adjustment rules:

$y = c|k(x - p)| + q$

Properties of Absolute Value

1. $|ab| =$ $|a| \cdot |b|$
2. $\left|\frac{a}{b}\right| =$ $\frac{|a|}{|b|}$

$|a| \leq \text{stuff} \iff -\text{stuff} \leq a \leq \text{stuff}$

3. If $|a| \leq b$, then $-b \leq a \leq b$

4. If $|a| \geq b$, then $a \geq b$ or $a \leq -b$

5. (Triangle Inequality) $|a + b|$ $\leq |a| + |b|$

Proof. (Triangle Inequality)

Prove $|a+b| \leq |a| + |b|$

We know:

$$
\begin{array}{rl}
 & -|a| \leq a \leq |a| \\
+ & -|b| \leq b \leq |b| \\
\hline
 & -|a| - |b| \leq a+b \leq |a| + |b| \\
 & -(|a|+|b|) \leq a+b \leq |a| + |b|
\end{array}
$$

looks like $-\text{stuff} \leq a+b \leq \text{stuff}$

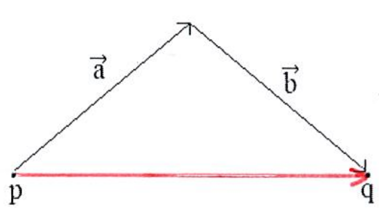
By property 3

$$|a+b| \leq |a| + |b|$$

□

Why do we call this the *Triangle* Inequality?

Imagine adding 2 vectors together to get a third vector:



Recall: The **s**hortest distance between two points p and q is a **s**traight **l**ine.

The idea is the same!

2.3 Notation

We wish to extend our skills with equation-solving to deal with

inequalities . These are expressions separated by the

relations $\geq$, $\leq$, $>$, or $<$. For example:

$$3x + 2 \leq 1 \qquad x^3 > -7 \qquad \ln(x) + e^x - \sqrt{x} > x$$

When solving equations we may get a **s**ingle answer, or a

number of answers that satisfy the equation. Consider

$3x - 5 = 1 \implies 3x = 6 \implies x = 2$

Only **o**ne value satisfies this equation. But if we consider

$x^2 - 1 = 3 \implies x^2 = 4 \implies \sqrt{x^2} = \sqrt{4} \implies \sqrt{x^2} = 2 \implies |x| = 2$

$x = \pm 2$

More **t**han **o**ne value satisfies this equation.

On the other hand, it is usually the case that we get a **r**ange of

values that satisfy an inequality!

There are a few ways to express our answers notationally. One way is using **i**nequalities, for example

$4 < x \leq 9$ $\qquad -2 \leq x$ $\qquad 0 < x < \infty$

where the symbols $\leq$ and $\geq$ indicate inclusion of an endpoint, and $<$ and $>$ indicate exclusion of an endpoint. A second notation is **i**nterval **n**otation, for example

$(4, 9]$ $\qquad [-2, \infty)$ $\qquad (0, \infty)$

where a square (or closed) bracket [or] indicates inclusion of an endpoint, and a round (or open) bracket (or) indicates exclusion of an endpoint.

Example 3. *Write each of the following in interval notation.*

(a) $2 \leq x < 7$ $\qquad$ *(b)* $-3 > x > 0$ $\qquad$ *(c)* $x < 9$

(a) $[2, 7)$

(b) this doesn't make sense

x can't be greater than zero and less than −3 at the same time

(c) $(-\infty, 9)$

Example 4. *Write each of the following using inequalities.*

"in"

(a) $x \in [3,6)$ (b) $x \in (-\infty,-1]$ (c) $x \in (-2,4)$

$3 \le x < 6$

$x \le -1$

or

$-\infty < x \le -1$

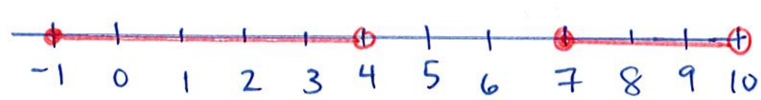
$-2 < x < 4$

✍: Notice that numbers are written increasing order from left to right. That is, we would write $[2,4)$ not $[4,2)$. The same is true when using inequalities.

✍: In interval notation, the infinity symbol (∞) is *always* accompanied by round brackets.

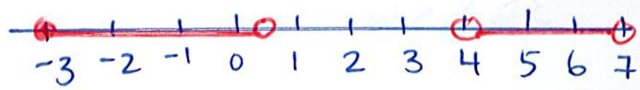
It is possible to have ranges of values that are **d**isjoint .

We use the **u**nion symbol, $\cup$, to include all of the values in any of the **d**isjoint ranges; for example $[-1,4) \cup [7,10)$ means $-1 \le x < 4$ or $7 \le x < 10$.



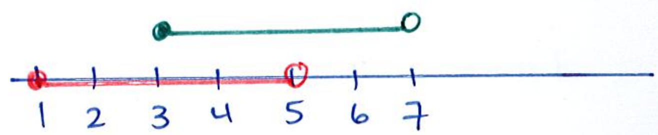
Example 5. *Express each of the following in interval notation, eliminating redundancies where possible.*

(a) $-3 \leq x < \frac{1}{2}$ *or* $4 < x < 7$



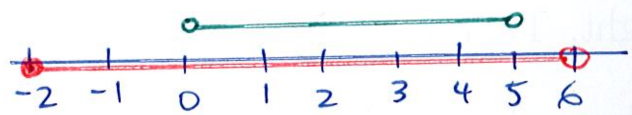
$x \in [-3, \frac{1}{2}) \cup (4, 7)$

(b) $1 \leq x < 5$ *or* $3 \leq x < 7$



$x \in [1, 7)$

(c) $x \in [-2, 6) \cup (0, 5)$

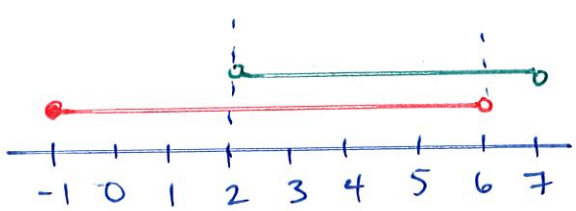


$x \in [-2, 6)$

One final symbol may come in handy is the

intersection symbol, $\cap$. This set operator allows only the values that are **c**ommon between intervals; for example

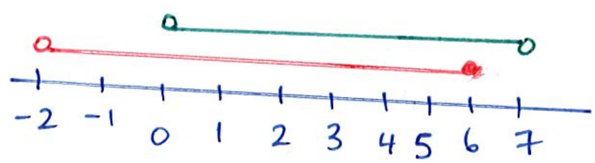
$[-1, 6) \cap (2, 7) =$



$x \in (2, 6)$

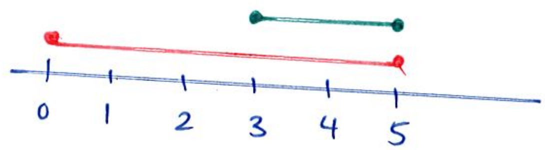
Example 6. *Express each of the following in interval notation, eliminating redundancies where possible.*

(a) $-2 < x \leq 6$ $\overset{\cap}{\textit{and}}$ $0 < x < 7$



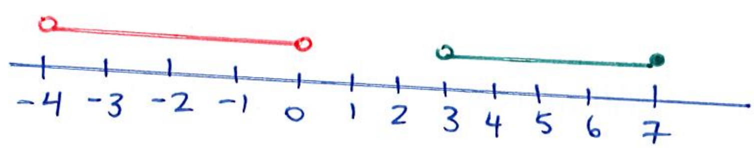
$x \in (0, 6]$

(b) $x \in [0, 5] \cap [3, 5]$



$x \in [3, 5]$

(c) $-4 < x < 0$ $\overset{\cap}{\textit{and}}$ $3 < x \leq 7$



$x \in \emptyset$ ← empty set.

nothing in common

2.4 Solving Inequalities

When solving inequalities, there are a few rules that we must follow:

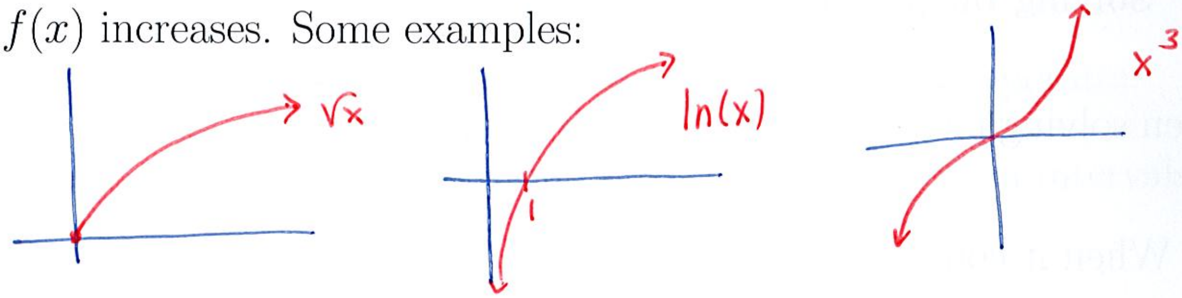
1. When it comes to addition, subtraction, multiplication, and division: What you **d**o **t**o **o**ne **s**ide of the inequality, you must **d**o **t**o **t**he **o**ther.

2. If you **m**ultiply or **d**ivide by a **n**egative quantity, you must **f**lip the inequality.

3. If both sides are positive or both sides are negative then you can take the **r** eciprocal of both sides but you must **f** lip the inequality.

4. If you want to apply a function to both sides of the inequality, the function being applied must be **m** onotone **i** ncreasing in order to guarantee that the inequality is preserved.

What is a monotone increasing function? A function $f(x)$ is monotone increasing if as x increases, the height of the function $f(x)$ increases. Some examples:



Example 7. *Find all values of x that satisfy the following.*

(a) $-6x + 7 \geq 8x$

$-6x - 8x \geq -7$

$-14x \geq -7$

flip! ÷ neg!

$x \leq \frac{7}{14}$

$x \leq \frac{1}{2}$

$x \in (-\infty, \frac{1}{2}]$

(b) $-\frac{5}{2} < 4 - 2x \leq 1$

$-\frac{5}{2} - 4 < -2x \leq 1 - 4$

$-\frac{13}{2} < -2x \leq -3$

$\frac{13}{4} > x \geq \frac{3}{2}$

$x \in [\frac{3}{2}, \frac{13}{4})$

$(c)\ 5x^3 + 27 > -13$

$5x^3 > -40$

$x^3 > -8$

$\sqrt[3]{x^3} > \sqrt[3]{-8}$

$x > -2$

$x \in (-2, \infty)$

I want to apply $\sqrt[3]{\ }$. M.I?

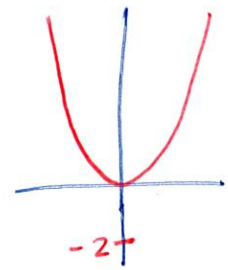
yes!

$(d)\ 3x^2 + 2 < -4$

$3x^2 < -6$

$x^2 < -2$

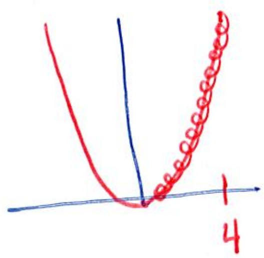
this is never true



$x \in \phi$

$(e)\ \sqrt{x-1} > 4$

we want to square both sides, but x^2 is NOT M.I..



since we are working on a part of the function that is M.I. we can do this.

$(\sqrt{x-1})^2 > 4^2$

$x - 1 > 16$

$x > 17$

$x \in (17, \infty)$

$(f)\ \log_2(3x) \leq -3$