# Assignment 1, MAT 2355 with solutions

October 6, 2015

This assignment is due Monday, September 28.

**Exercise 1.** For each of the following functions $f$, determine if $f$ is an isometry or not. If it is, prove it. If not, show a counterxample (e.g. two points $P, Q$ such that $d(P,Q) \neq d(f(P), f(Q))$.

- $f_\theta : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$f(x,y,z) = \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

  Where $\theta \in \mathbb{R}$ is a fixed constant.

- $f_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x,y) = \begin{pmatrix} -1/\sqrt{2} & \sqrt{2} \\ 1/\sqrt{2} & -\sqrt{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

- $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x,y) = (x + xy + 3, y^2 + x^2)$.

- $f_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x,y) = \begin{pmatrix} \cos(x \cdot \theta) & -\sin(x \cdot \theta) \\ \sin(x \cdot \theta) & \cos(x \cdot \theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

  Note that $\theta \in \mathbb{R}_{>0}$ is a fixed *positive* constant, while the $x$ in the matrix is the first coordinate of the vector $(x,y)$.

- $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x,y) = (x+3, y+3) - 2\left[(x+3, y+3) \cdot (1/\sqrt{2}, 1/\sqrt{2})\right](1/\sqrt{2}, 1/\sqrt{2})$$

  Note that within the square bracket we have the product of two vectors, which is a number.

Of the functions above that are isometries, which preserve the scalar products $u \cdot v$?

*Solution.* • $f_\theta$ is clearly a linear function, so we only need to check that the matrix associated to it is orthogonal. We have

$$\begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix} =$$

$$\begin{pmatrix} \cos^2(\theta)+\sin^2(\theta) & 0 & -\sin(\theta)\cos(\theta)+\sin(\theta)\cos(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta)\cos(\theta) & 0 & \sin^2(\theta)+\cos^2(\theta) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- here $f$ is linear, but it is not an isometry as $f(0) = 0$ as the matrix representing it is not orthogonal:

$$f(x, y) = \begin{pmatrix} -1/\sqrt{2} & \sqrt{2} \\ 1/\sqrt{2} & -\sqrt{2} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} -1/\sqrt{2} & \sqrt{2} \\ 1/\sqrt{2} & -\sqrt{2} \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ \sqrt{2} & -\sqrt{2} \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} & \sqrt{2} \\ 1/\sqrt{2} & -\sqrt{2} \end{pmatrix} =$$

$$\begin{pmatrix} 1 & -2 \\ -2 & 2 \end{pmatrix}$$

  We should also note that $f$ is not injective as for any vector in the form $v = \left(\lambda\sqrt{2}, \lambda/\sqrt{2}\right)$ we have $f(v) = 0$.

- Here $f$ is clearly not linear, having higher degree terms. We search for two points $P, Q$ such that $\mathrm{d}(f(P), f(Q)) \neq \mathrm{d}(P, Q)$. For example if $P = (0, 0), Q = (1, 0)$ we have $\mathrm{d}(P, Q) = 1$, and $d(f(P), f(Q)) = \mathrm{d}((3, 0), (5, 2)) = \sqrt{(3-5)^2 + (0-2)^2} = \sqrt{4} = 2$. This shows that $f$ is not an isometry.

- Note that while $f$ is described by a matrix, it is still not linear as the elements in the matrix are functions, not real numbers! The function reads

$$f(x, y) = (x\cos(x\theta) - y\sin(x\theta), x\sin(x\theta) + y\cos(x\theta))$$

  Note that if $x = 0$ the function is just the identity $f(0, y) = (0, y)$, while if $y = 0$ it is the rotation $f_\theta(x, 0) = (x\cos(x\theta), x\sin(x\theta))$. So if we choose $x = \pi/2\theta$ we get

$$f_\theta(\pi/2\theta, 0) = (\pi/2\theta\cos(\pi/2), \pi/2\theta\sin(\pi/2)) = (0, \pi/2\theta) = f_\theta(0, \pi/2\theta)$$

  This shows that $f_\theta$ maps the points $P_\theta = (\pi/2\theta, 0)$ and $Q_\theta = (0, \pi/2\theta)$ to the same point. As $\mathrm{d}(P_\theta, Q_\theta) > 0$ and $\mathrm{d}(f_\theta(P_\theta), f_\theta(Q_\theta)) = 0$ we conclude that $f_\theta$ is not an isometry.

  We should note that almost any two fixed points $P, Q$ in $\mathbb{R}^2$ work as a counterexample, but not for all $\theta$, so we need at least two couples of points $\{P, Q\}, \{P', Q'\}$ to make a complete counterexample.

- here $f$ is a reflection. Consider $f' = f - f(o)$. The vector $f(0)$ is equal to

$$(3,3) - 2\left[(3,3)\cdot(1/\sqrt{2},1/\sqrt{2})\right](1/\sqrt{2},1/\sqrt{2}) = (3,3) - 2(6/\sqrt{2})(1/\sqrt{2},1/\sqrt{2}) = \\ (3,3)-(6,6) = (-3,-3)$$

  So we have

$$f'(x,y) = f(x,y) + (3,3) = (x+3-2(x+3+y+3)/2, y+3-2(x+3+y+3)/2) + (3,3) = \\ (x-(x+y), y-(x+y)) = (-y,-x)$$

  So $f'$ is represented by the matrix

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

  And we have

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

  So $f$ is an isometry.

For the last question, recall that we have proven a theorem saying that an isometry fixes scalar products if and only if it sends the origin $(0,\dots,0)$ to itself. So only the first isometry fixes scalar products. □

**Exercise 2.** In the next two problems we will see how to move between several equivalent definitions for a subspace of $\mathbb{R}^n$. Note that each of the questions has multiple (infinitely many) correct answers.

- Consider the line $l$ in $\mathbb{R}^2$ given by $l = \{u \in \mathbb{R}^2 \mid u \cdot (3,4) = 25\}$. Find an equation in the form $ax + by + c = 0$ for $l$ and find vectors $v, w$ such that $l = \{v + \alpha w \mid \alpha \in \mathbb{R}\}$.
- Consider the plane $H$ in $\mathbb{R}^3$ defined by $H = \{(x,y,z) \mid x + 2y + 4z - 2 = 0\}$. Find three vectors $u, v, w$ such that $H = \{u + \alpha v + \beta w \mid \alpha, \beta \in \mathbb{R}\}$. Find a vector $u$ and a number $c$ such that $H = \{v \in \mathbb{R}^3 \mid u \cdot v = c\}$.
- Consider the straight line $r$ in $\mathbb{R}^3$ defined by $H = \{u + \alpha v \mid \alpha \in \mathbb{R}\}$, where $u = (1,1,0), v = (1,0,1)$. Find two equations such that $H = \{(x,y,z) \mid ax + by + cz + d = 0 \quad \textit{and} \quad a'x + b'y + c'z + d' = 0\}$, and two vectors $w, w'$ and two numbers $e, e'$ such that $H = \{t \in \mathbb{R}^3 \mid w \cdot v = e \quad \textit{and} \quad w' \cdot t = e'\}$

*Solution.*

- Let $u = (x,y)$ be a point in $\mathbb{R}^2$. The equation $u \cdot (3,4) = 25$ can be rewritten as $3x + 4y = 25$.

  To explicitly generate the line with as $u + \lambda v$ we just need a solution $u$ of the equation and a solution $v$ of the homogenous equation $3x + 4y = 0$. Then we will have $(u + \lambda v)\cdot(3,4) = u\cdot(3,4) + \lambda v \cdot (3,4) = 25 + 0$, so all the points in this form will satisfy the equation.

  Moreover given two points $u_1, u_2$ such that $u_1 \cdot (3,4) = u_2 \cdot (3,4) = 25$ we have $(u_1 - u_2)\cdot(3,4) = 25 - 25 = 0$ showing that the two sets are equal.

Two vectors $u, v$ as above are for example $u = (3,4), v = (-4,3)$. Note that there are infinitely many possible correct answers to this question.

- First we can write $H = \{(x,y,z) \mid x + 2y + 4z - 2 = 0\}$ as $H = \{v \mid v(1,2,4) = 2\}$. It is just a rewriting of the same equation.

  Now note that if $P, Q$ are points in $H$, and $P = u + \alpha v + \beta w, Q = u + \alpha' v + \beta' w$ then we have $0 = 2 - 2 = (P - Q)(1,2,4) = (\alpha - \alpha')v \cdot (1,2,4) + (\beta - \beta')w \cdot (1,2,4)$. This is always satisfied if $v$ and $w$ are perpendicular to $(1,2,4)$.

  Moreover if $v$ and $w$ are perpendicular to $(1,2,4)$ and $u \in H$ then $u + \alpha v + \beta w$ always belongs to $H$. So we just need $u \in H$ and $v, w$ perpendicular to $(1,2,4)$ and linearly independent. Then we can choose for example $u = (2,0,0)$, $v = (2,-1,0)$ and $w = (0,2,-2)$. Note that it is important that $v, w$ are linearly independent or they will not generate the whole plane.

- Finding two equations is exactly the same as finding two vectors perpendicular to $v$. This is true as if $w \cdot v = 0$ then for all points in the form $u + \alpha v$ we will have $(u + \alpha v) \cdot w = u \cdot w$. These vectors should also not be proprtional or we will get twice the same equation.

  Two suitable choices are $w = (0,1,0)$ and $w' = (1,0,-1)$. We will then get the equations $t \cdot w = (0,1,0) \cdot (0,1,0) = 1$ and $t \cdot w' = (1,0,-1) \cdot (0,1,0) = 0$. We can rewrite these two equations as $y = 1$ and $x - z = 0$.

$\square$

**Exercise 3.**

- Let $u, v$ be fixed distinct vectors in $\mathbb{R}^3$. Show that the set $H = \{w \in \mathbb{R}^3 \mid \|u - w\| = \|v - w\|\}$ is a plane.

- Let $u, v, w$ be fixed distinct vectors in $\mathbb{R}^3$, not belonging to the same line. Show that the set $l = \{w' \in \mathbb{R}^3 \mid \|u - w'\| = \|v - w'\| = \|w - w'\|\}$ is a straight line.

- Show that the vector $u + (v - u)/2$ belongs to the plane $H = \{w \in \mathbb{R}^3 \mid \|u - w\| = \|v - w\|\}$.

- Show that if $w$ is perpendicular to $u - v$ then $w + u + (v - u)/2$ belongs to $H = \{w \in \mathbb{R}^3 \mid \|u - w\| = \|v - w\|\}$. Use this to write $H$ as $u + (v - u)/2 + \{w \in \mathbb{R}^3 \mid (u - v) \cdot w = 0\}$.

*Solution.*

- The equation $\|u - w\| = \|v - w\|$ is equivalent to $\|u - w\|^2 = \|v - w\|^2$ which can be rewritten as $(u - w) \cdot (u - w) = (v - w) \cdot (v - w) \Leftrightarrow w^2 + u^2 - 2w \cdot u = w^2 + v^2 - 2w \cdot v \Leftrightarrow w \cdot (u - v) = (u^2 - v^2)/2$, which is the equation for a plane in $\mathbb{R}^3$.

- The equation $\|u - w'\| = \|v - w'\| = \|w - w'\|$ is equivalent to asking that both $\|u - w'\| = \|v - w'\|$ and $\|v - w'\| = \|w - w'\|$ are true simultaneously. We know that that these two equations define planes $H, H'$, so the points that satisfy both equations correspond to the intersection $H \cap H'$.

  We need to show that the two planes $H, H'$ are not parallel. To see this, note that a normal vector to $H$ is $u - v$, and a normal vector to $H'$ is $v - w$. We have proven this in the previous exercise.

  Saying that $H$ and $H'$ are not parallel is like saying that their normal vectors are not proportional, which in this case is exactly the same as saying that the vectors $u, v, w$ are not aligned.

- We have $u-(u+(v-u)/2) = -(v-u)/2, v-(u+(v-u)/2) = (v-u)/2$, so both $\|u-(u+(v-u)/2)\|$ and $\|v-(u+(v-u)/2)\|$ are equal to $\|(v-u)/2\|$.

- In the first part of the exercise we have seen that an equation for $H$ is $w\cdot(u-v) = (u^2-v^2)/2$, so given two points $P, Q \in H$ we have $(P-Q)\cdot(u-v) = (u^2-v^2)/2-(u^2-v^2)/2 = 0$. This shows that $(u-v)$ is perpendicular to $H$.

  Now take $u+(v-u)/2$. Given any $w$ perpendicular to $u-v$ we have $(w+u+(v-u)/2)\cdot(u-v) = w\cdot(u-v)+(u+(v-u)/2)(u-v) = 0+u^2-uv+uv/2-u^2/2-v^2+uv/2 = (u^2-v^2)/2$ showing that $w+u+(v-u)/2 \in H$. As the vectors perpendicular to $u-v$ form a plane the set $u+(v-u)/2+\{w \mid w\cdot(u-v) = 0\}$ is a plane contained in $H$ and thus must be equal to $H$.

□

**Exercise 4** (bonus exercise)**.** Let $u, v, w \in \mathbb{R}^3$ be the vectors $(1,0,0), (0,1/2,\sqrt{3}/2), (0,-\sqrt{3}/2,1/2)$. Let $O$ be the origin $(0,0,0)$ in $\mathbb{R}^3$.

- Show that if $f, g$ are isometries such that $f(u) = g(u), f(v) = g(v), f(w) = g(w)$ and $f(O) = g(O)$ then $f = g$.

- Give an example where $f(u) = g(u), f(v) = g(v), f(w) = g(w)$ but $f$ is not equal to $g$. (Suggestion: consider the reflection through the plane where $f(u), f(v)$ and $f(w)$ lie).

- Show that if $f(u) = u, f(v) = v$ and $f(O) = O$ then necessarily $f$ is the identity map or the reflection through the plane generated by $u$ and $v$.

*Solution.*

- We can write $f(v) = Av+b, g(v) = Bv+c$. As $b = f(0), c = g(0)$ we have $b = c$. Then we can write $f(v)-g(v) = Av-Bv$. We know that $Av = f(v)-f(O)$, $Bv = g(v)-g(O)$. Then $A$ and $B$ send the vectors $(1,0,0), (0,1/2,\sqrt{3}/2), (0,-\sqrt{3}/2,1/2)$ to the same vectors. We can now conclude in two different ways.

  The first way is to just say that two linear functions $A, B : \mathbb{R}^n \to \mathbb{R}^n$ that are equal on a given base $v_1, \ldots, v_n$ must be equal.

  The second way, more explicit, is to write $e_1 = (1,0,0)$, $e_2 = 1/2(0,1/2,\sqrt{3}/2)-\sqrt{3}/2(0,-\sqrt{3}/2,1/2)$, $e_3 = \sqrt{3}/2(0,1/2,\sqrt{3}/2)+1/2(0,-\sqrt{3}/2,1/2)$. Then by linearity we can say that $A(e_1) = B(e_1), A(e_2) = B(e_2), A(e_3) = B(e_3)$ and these are exactly the first, second and third column of the two matrices.

- Note that as an isometry fixes angles and distances the three points $f(u), f(v), f(w)$ are distinct and do not lie on the same line. Then there is a unique plane $H$ passing through all of them. It follows that the reflection $R_H$ satisfies $R_H(f(u)) = f(u), R_H(f(v)) = v, R_H(f(w)) = f(w)$.

  Then $f$ and $R_H \circ f$ are the same on $u, v, w$ but they are not the same map. An explicit example is given by the identity map and the reflection $R_H$, where $H$ is the plane $H = \{(1,0,0)+\alpha(1,-1/2,-\sqrt{3}/2)+\beta(1,\sqrt{3}/2,-1/2)\}$.

- The set $u, v, w$ is an orthonormal base of $\mathbb{R}^3$. As $f(O) = O, f(u) = u, f(v) = v$ the isometry $f$ is linear and we must have $f(w)\cdot f(v) = 0$, $f(w)\cdot f(u) = 0$ and $\|f(w)\| = 1$.

The first two equations define a line that is exactly the line $\{\alpha w \mid \alpha \in \mathbb{R}\}$. The last equation implies that $1 = f(w)^2 = (\alpha w)^2 = \alpha^2 w^2 = \alpha^2$, so we can conclude that $f(w)$ is equal to plus or minus $w$. Then we can use the first point of the problem to conclude.

□