PART A (35 marks)

Use the following information for questions 1, 2 and 3.

Let $\mathbf{u} = (1, 0, 3)$ and $\mathbf{v} = (0, 5, 0)$.

1. [1 mark] Find the length (magnitude) of the vector $\mathbf{u} + \mathbf{v}$.

A: $\sqrt{35}$	B: 9	C: $\sqrt{10} + \sqrt{25}$	D: 0	E: 10

Solution:

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (1, 0, 3) + (0, 5, 0) = (1 + 0, 0 + 5, 3 + 0) = (1, 5, 3) \\ \text{so } \|\mathbf{u} + \mathbf{v}\| &= \|(1, 5, 3)\| = \sqrt{1^2 + 5^2 + 3^2} = \sqrt{1 + 25 + 9} = \sqrt{35} \end{aligned}$$

2. [1 mark] Find $\mathbf{u} \bullet \mathbf{v}$.

A: $(-15, 0, 5)$	B: 0	C: 16	D: 20	E: $(0, 0, 0)$

Solution:

$$\mathbf{u} \bullet \mathbf{v} = (1, 0, 3) \bullet (0, 5, 0) = 1(0) + 0(5) + 3(0) = 0 + 0 + 0 = 0$$

3. [1 mark] Find $\mathbf{u} \times \mathbf{v}$.

A: 0	B: $(0, 0, 0)$	C: $(5, -15, 0)$	D: $(15, 0, -5)$	E: $(-15, 0, 5)$

Solution:

$$\mathbf{u} \times \mathbf{v} = (1, 0, 3) \times (0, 5, 0) = (0(0) - 5(3), 3(0) - 0(1), 1(5) - 0(0)) = (0 - 15, 0 - 0, 5 - 0) = (-15, 0, 5)$$

4. [1 mark] Which one of the following is an equation of the line in $\Re^2$ which passes through the point $P(5, 7)$ and is perpendicular to the vector $\mathbf{v} = (2, 3)$?

A: $(x, y) = (5, 7) + t(2, 3)$	B: $(x, y) = (5, 7) + t(3, -2)$	C: $(x, y) = (2, 3) + t(5, 7)$
D: $(x, y) = (2, 3) + t(7, -5)$	E: $(x, y) = (5, 7) + t(0, 0)$	

Solution: The answer choices are all point-parallel form equations of lines. To write a point-parallel form equation, we need a point which is on the line and a vector which is parallel to the line. We know that the vector $\mathbf{v} = (2, 3)$ is perpendicular to the line, so the vector $(3, -2)$, which is orthogonal to $\mathbf{v}$, is parallel to the line. Using this vector, and the point we know is on the line, we get the point-parallel form equation $(x, y) = (5, 7) + t(3, -2)$.

5. [1 mark] Consider the following vectors in $\Re^3$: $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{v} = \mathbf{i} - \mathbf{j}$. Which of the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{k}$ are parallel to both the plane $3x + 2y = 5$ and the plane $x + y = 2$ (in $\Re^3$)?

A: $\mathbf{u}$ only	B: $\mathbf{v}$ only	C: $\mathbf{k}$ only	D: $\mathbf{u}$ and $\mathbf{v}$ but not $\mathbf{k}$	E: all of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{k}$

Solution: First, since we are in $\Re^3$, each plane has 0 as the coefficient of z in the given standard form equation. We know that in a standard form equation of a plane, the coefficients of x, y and z are the components (in that order) of a normal vector for the plane. So the first plane mentioned has normal vector $(3, 2, 0)$ and the second has normal vector $(1, 1, 0)$.

We can tell that a vector is parallel to a plane if the dot product of that vector with a normal vector for the plane is 0. We need to check dot products of each of the vectors we're asked about with normals for both planes. Since $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$ and $\mathbf{k} = (0,0,1)$ in $\Re^3$, we see that

$$\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} = 2(1,0,0) - 3(0,1,0) = (2,-3,0) \qquad \text{and} \qquad \mathbf{v} = (1,0,0) - (0,1,0) = (1,-1,0)$$

Therefore we get

$$\begin{aligned} (2,-3,0) \bullet (3,2,0) &= 2(3) + (-3)(2) + 0(0) = 0 \quad \text{but} \quad (2,-3,0) \bullet (1,1,0) = 2(1) + (-3)(1) + 0(0) = -1 \\ (1,-1,0) \bullet (1,1,0) &= 1(1) + (-1)(1) + 0(0) = 0 \quad \text{but} \quad (1,-1,0) \bullet (3,2,0) = 1(3) + (-1)(2) + 0(0) = 1 \\ (0,0,1) \bullet (3,2,0) &= 0(3) + 0(2) + 1(0) = 0 \quad \text{and} \quad (0,0,1) \bullet (1,1,0) = 0(1) + 0(1) + 1(0) = 0 \end{aligned}$$

We see that $\mathbf{k}$ is the only one of the 3 vectors which is orthogonal to both normals, so $\mathbf{k}$ is the only one of the 3 vectors which is parallel to both planes. (Each of $\mathbf{u}$ and $\mathbf{v}$ is parallel to one of the planes, but not to the other.)

6. [1 mark] Which one of the following is an equation of the line in $\Re^2$ which passes through the point $P(4,2)$ and is parallel to the line $x - y = 12$?

A: $(x,y) = (4,2) + t(1,-1)$	B: $(x,y) = (4,2) + 12(1,-1)$	C: $(x,y) = (1,-1) + t(2,-4)$
D: $(x,y) = (4,2) + t(1,1)$	E: $(x,y) = (1,1) + 12(4,2)$	

Solution: The line $x - y = 12$ has normal vector $\mathbf{n} = (1,-1)$, so this is a vector which is perpendicular to that line. The answer choices are all point-parallel form equations, and since we want a line which is parallel to $x - y = 12$, we need a vector which is parallel to that line, and hence is orthogonal to $\mathbf{n}$. We know that $\mathbf{u} = (1,1)$ is orthogonal to $\mathbf{n} = (1,-1)$, so we can $\mathbf{u}$, along with the given point, to write a point-parallel form equation of the line: $(x,y) = (4,2) + t(1,1)$.

7. [1 mark] Find the distance from the point $P(3,1,-1)$ to the plane $3x + y - z = 2$.

A: 11	B: 9	C: 0	D: $\sqrt{11}$	E: $\dfrac{9}{\sqrt{11}}$

Solution: The distance from a point P to a plane with normal vector $\mathbf{n}$ is given by

$$\text{distance} = \frac{|\mathbf{n} \bullet (\mathbf{q} - \mathbf{p})|}{\|\mathbf{n}\|} = \frac{|(\mathbf{n} \bullet \mathbf{q}) - (\mathbf{n} \bullet \mathbf{p})|}{\|\mathbf{n}\|}$$

where $\mathbf{q}$ is a vector from the origin to any point Q which lies on the plane. And since any point Q which lies on the plane must satisfy the equation of the plane, any vector $\mathbf{q}$ to such a point must have $\mathbf{n} \bullet \mathbf{q} = 2$ (the RHS of the equation of the plane). The plane $3x + y - z = 2$ has normal vector $\mathbf{n} = (3,1,-1)$, so we see that

$$\mathbf{n} \bullet \mathbf{p} = (3,1,-1) \bullet (3,1,-1) = 3(3) + 1(1) + (-1)(-1) = 9 + 1 + 1 = 11$$

Also, since $\mathbf{p} = \mathbf{n}$ in this case, finding the magnitude of $\mathbf{n}$ involves exactly the same arithmetic, but under a square root sign:

$$\|\mathbf{n}\| = \|(3,1,-1)\| = \sqrt{3^2 + 1^2 + (-1)^2} = \sqrt{9+1+1} = \sqrt{11}$$

Therefore we see that the distance from the point $P(3,1,-1)$ to the plane $3x + y - z = 2$ is

$$\frac{|2 - 11|}{\sqrt{11}} = \frac{|-9|}{\sqrt{11}} = \frac{9}{\sqrt{11}}$$

8. [1 mark] Which of the matrices shown below is/are in row-reduced echelon form?

$$M_1 = \begin{bmatrix} 1 & 0 & 5 & 5 \\ 0 & 1 & 8 & 10 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad M_3 = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad M_4 = \begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 7 & 8 \end{bmatrix}$$

A: M_3 only.	B: M_1 and M_4 only.	C: M_2 and M_3 only.
D: M_2, M_3 and M_4 only.	E: None of the matrices.	

Solution: In order to be in row-reduced echelon form, there are 4 characteristics that a matrix must have: (1) in any row which has any non-zero entries, the first non-zero entry must be a 1 (a "leading one"); (2) any column which contains the leading one for some row must not contain any other non-zero entries (i.e. must have only 0s above and below that leading one); (3) each leading one must be further to the right than the leading ones in all rows above it; and (4) any row which does not have any non-zero entries must be lower in the matrix than all rows which do have non-zero entries.

Matrices M_1 and M_4 have all 4 of these characteristics. But matrix M_2 does not have characteristic (3), and matrix M_3 does not have characteristic (2).

9. [1 mark] Which of the following describes the region that represents the intersection of the planes $x + 2y + 3z = 4$, $2x + y + 4z = 1$ and $-x - 2y - 3z = 4$?

A: No points.	B: Only 1 point.	C: Only 3 points.	D: A line.	E: A plane.

Solution: The points in the region we're interested in must satisfy the equations of all 3 planes, so we need to find all solutions to the system of linear equations consisting of the equations of the given planes. We write the augmented matrix for the system and row reduce. First, we need to get 0s below row 1's leading one, so we subtract 2 times row 1 from row 2, and we add row 1 to row 3:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 1 \\ -1 & -2 & -3 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -7 \\ 0 & 0 & 0 & 8 \end{array}\right]$$

Although the matrix is not yet in row-reduced echelon form, we see that the third row of the new matrix represents an equation which says $0x + 0y + 0z = 8$. Since this equation cannot be satisfied by any values of x, y and z, the system is inconsistent and has no solutions. That is, no point lies on all 3 planes, so there are no points in the region that represents the intersection of the 3 planes.

10. [1 mark] Find all solutions to the system of linear equations represented by the augmented matrix shown here:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

A: The system has no solutions.	B: $(3, 4, 7)$ only.	C: $(3 - t, 4 - t, 7 - t)$ for any real t.
D: $(-4, -3, 7)$ only.	E: None of A, B, C or D.	

Solution: First, we finish bringing the matrix to row-reduced echelon form. The leading one for row 3 is in column 3, which still has other non-zero entries, so we must get rid of those, by subtracting row 3 from each of rows 1 and 2:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 7 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

The augmented matrix is now in row-reduced echelon form, and we see that the 3 rows correspond to the 3 equations $x = -4$, $y = -3$ and $z = 7$ (assuming that those are the names of the unknowns) and so $(-4, -3, 7)$ is the only solution to the system represented by the given augmented matrix.

11. [1 mark] Find all solutions to the system of linear equations represented by the augmented matrix shown here:

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 2 & 1 \\ 0 & 1 & 3 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

A: $(1, 4, 0, 0)$ only.	B: $(1, 0, 2, 2)$ and $(0, 1, 3, 1)$ only.
C: $(1 - 4t, 4 - 4t, t, t)$ for any real t.	D: The system has no solutions.
E: $(1 - 2s - 2t, 4 - 3s - t, s, t)$ for any real s and t.	

Solution: This time, the given matrix is already in row-reduced echelon form. We simply need to interpret what it is telling us. We see that row 1's leading 1 is in column 1, so row 1 is telling us about the first unknown. Similarly, row 2's leading one is in column 2, so that row is telling us about the second unknown. But neither column 3 nor column 4 contains the leading one for any row, so each of the third and the fourth unknown is free to have any value. We represent that by setting each of these unknowns equal to a different parameter.

Assuming that the unknowns are called x_1, x_2, x_3 and x_4 (in that order), we set $x_3 = s$ and $x_4 = t$. And now we look at what those first 2 rows are telling us about the values of x_1 and x_2. Row 1 corresponds to the equation $x_1 + 2x_3 + 2x_4 = 1$, which rearranges to $x_1 = 1 - 2x_3 - 2x_4$, so we need $x_1 = 1 - 2s - 2t$. And row 2 corresponds to the equation $x_2 + 3x_3 + x_4 = 4$, which rearranges to $x_2 = 4 - 3x_3 - x_4$, so we need $x_2 = 4 - 3s - t$. That is, we see that the solutions to the system are all points of the form

$$(x_1, x_2, x_3, x_4) = (1 - 2s - 2t, 4 - 3s - t, s, t)$$

12. [1 mark] Let A be a 2×2 matrix, B be a 3×2 matrix, and C be a 4×3 matrix. Which one of the following matrix operations **is** defined?

A: AB^TC^T	B: $AB^T + C$	C: $CB + A$	D: BC^T	E: AB

Solution: Saying that a matrix is $m \times n$ tells us that it has m rows and n columns. And the transpose of a matrix has the number of rows and number of columns switched. That is, if M is an $m \times n$ matrix, then M^T is an $n \times m$ matrix. We can only add two matrices together if they have the same dimensions, i.e. have the same number of rows and also have the same number of columns. And we can only multiply two matrices together if the number of columns in the first matrix is the same as the number of rows in the second matrix. The result of the multiplication is a matrix which has the same number of rows as the first matrix in the product, and has the same number of columns as the second matrix in the product.

Since A has 2 columns, while B has 3 rows, the product AB is not defined. Similarly, B has 2 columns, while C^T has 3 rows, so BC^T is not defined. We do have 3 columns in C and 3 rows in B, so CB is defined, but it is a 4×2 matrix and cannot be added to the 2×2 matrix A. Likewise, A has 2 columns, and B^T has 2 rows, so AB^T is defined, and is a 2×3 matrix, which means it cannot be added to the 4×3 matrix C. However, since AB^T has 3 columns, which is the same as the number of rows in C^T, the product AB^TC^T **is** defined (and is a 2×4 matrix).

13. [1 mark] If A and B are both 5×2 matrices, while C is a 5×3 matrix and D is a 2×7 matrix, what are the dimensions of $C^T(A+B)D$?

A: 5×7	B: 3×5	C: 7×3	D: 5×2	E: 3×7

Solution: C^T is a 3×5 matrix and $A+B$ is a 5×2 matrix, so their product, $C^T(A+B)$, is a 3×2 matrix. And when we multiply this by the 2×7 matrix D, the result is a 3×7 matrix.

14. [1 mark] Find A^2 if $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$.

A: $\begin{bmatrix} 1 & 8 \\ 0 & 9 \end{bmatrix}$	B: $\begin{bmatrix} 1 & 4 \\ 0 & 9 \end{bmatrix}$	C: $\begin{bmatrix} 2 & 4 \\ 0 & 6 \end{bmatrix}$	D: $\begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}$	E: $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

Solution: A^2 means A times A. The product of two matrices (when it is defined) is the matrix whose (i,j)-entry is the dot product of the vector whose components are the entries of row i of the first matrix with the vector whose components are the entries of column j of the second matrix. This gives

$$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} (1,2) \bullet (1,0) & (1,2) \bullet (2,3) \\ (0,3) \bullet (1,0) & (0,3) \bullet (2,3) \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 0 & 9 \end{bmatrix}$$

15. [1 mark] Find a_{22} if $A = [a_{ij}] = BCD$, where B, C and D are the matrices shown here:

$$B = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 1 \\ 4 & 0 \\ 1 & 1 \end{bmatrix}$$

A: 2	B: 10	C: 1	D: 5	E: A is not defined

Solution: Since B has 1 column, and C has 1 row, the product BC is defined. It has 2 rows (like B) and 3 columns (like C), and since D has 3 rows (the same as the number of columns of BC), then $A = BCD = (BC)D$ **is** defined. We could find the whole of the matrix A (which is 2×2), by first finding BC and then $(BC)D$, but since we're only asked for the $(2,2)$-entry of A that would be much more work than is really needed. a_{22} will be given by "dotting" the second row of BC with the second column of D. And the second row of BC is obtained by "dotting" the second row of B with each of the columns of C. We see that the second row of BC is $\begin{bmatrix} 2(4) & 2(0) & 2(1) \end{bmatrix} = \begin{bmatrix} 8 & 0 & 2 \end{bmatrix}$, so dotting this with the second column of D we get $a_{22} = (8,0,2) \bullet (1,0,1) = 8+0+2 = 10$.

16. [1 mark] Let $A^{-1} = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 5 \\ 6 & 0 & 0 \end{bmatrix}$ be the inverse of $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Find the solution to the system of linear equations shown here:

$$\begin{array}{ccccccc} ax & + & by & + & cz & = & 2 \\ dx & + & ey & + & fz & = & 4 \\ gx & + & hy & + & iz & = & 7 \end{array}$$

A: $(35,12,12)$	B: $(6,20,42)$	C: The system has no solution.	D: $(12,12,35)$	E: $(12,35,12)$

Solution: We use the method of inverses: $\mathbf{x} = A^{-1}\mathbf{b}$, where $\mathbf{x}$ is the 3×1 column vector of unknowns and $\mathbf{b}$ is the 3×1 vector of right-hand side values.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 5 \\ 6 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 0+12+0 \\ 0+0+35 \\ 12+0+0 \end{bmatrix} = \begin{bmatrix} 12 \\ 35 \\ 12 \end{bmatrix}$$

That is, the unique solution to the system is $(x, y, z) = (12, 35, 12)$.

17. [1 mark] Find the rank of the augmented matrix $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ -3 & -1 & -6 & 1 \end{array}\right]$.

A: 0	B: 1	C: 2	D: 3	E: 4

Solution: The rank of a matrix is the number of non-zero rows, i.e. the number of leading ones, in the row-reduced echelon form of the matrix. We don't necessarily have to bring the matrix entirely to row-reduced echelon form; we just need to get close enough to row-reduced echelon form to see how many rows will contain one or more non-zero entries (and thus have a leading one). We start by subtracting row 1 from row 2, and adding 3 times row 1 to row 3:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ -3 & -1 & -6 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 4 \end{array}\right]$$

At this point, we can see that row 1 and row 2 have (and will continue to have) leading ones. We would next need to get a 0 in the second position in row 3 (below row 2's leading one), which is accomplished by subtracting 2 times row 2 from row 3. Doing so will not eliminate the 4 in row 3, so row 3 will still have a non-zero entry. That is, row 3 will be replaced by $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 4 \end{array}\right]$ and to have the augmented matrix be fully in row-reduced echelon form we would need to multiply row 3 by $\frac{1}{4}$ so that the first non-zero entry in that row will be a leading one, and then get rid of the non-zeros above the leading ones for rows 2 and 3. But without carrying that out, we can already tell that the row-reduced echelon form of the augmented matrix will have a non-zero in every row (i.e. will have 3 leading ones), so the rank of the augmented matrix is 3.

If you thought the rank was 2, you were looking only at the coefficient matrix part of the augmented matrix. That is, considering the augmented matrix as $[A|\mathbf{b}]$ for a system of linear equations $A\mathbf{x} = \mathbf{b}$, the rank of A is 2, but the rank of $[A|\mathbf{b}]$ is 3.

18. [1 mark] Which one of the following statements is **false**?

(*i*) If A is an $n \times n$ matrix with rank n, then A must be invertible.

(*ii*) If A is a square invertible matrix, then the linear system $A\mathbf{x} = \mathbf{0}$ must have a unique solution.

(*iii*) Any homogeneous system of 3 linear equations in 4 unknowns must have non-trivial solutions.

(*iv*) If A is an $n \times n$ matrix with rank $n-1$, then the linear system $A\mathbf{x} = \mathbf{b}$ must have infinitely many solutions.

A: (*i*)	B: (*ii*)	C: (*iii*)	D: (*iv*)	E: None of them.

Solution: It is not true that $A\mathbf{x} = \mathbf{b}$, with A being an $n \times n$ matrix with rank $n-1$, must have infinitely many solutions. It may have, but it is also possible that the system could be inconsistent, and have no solution. That is, we know that the last row in the row-reduced echelon form of A will contain only 0s, and column n will not contain the leading one for any row, but the rank of $[A|\mathbf{b}]$ might be n, because we do not know that there will be a 0 in the bottom position in the extra column.

The other 3 statements are all things which you should know are true.

19. [1 mark] If A is a 6×4 matrix, what is the largest possible value of the rank of A?

A: 4	B: 6	C: 10	D: 24	E: 2

Solution: The row-reduced echelon form of a matrix cannot have more leading ones than there are columns, since each leading one must be in a different column. Therefore the largest possible rank for a matrix with 6 rows and 4 columns is 4.

20. [1 mark] Consider the linear system $A\mathbf{x} = \mathbf{b}$ where A is a 12×12 invertible matrix. Which one of the following statements is **false**?

A: The rank of $[A \mid \mathbf{b}]$ is 12.	B: The system may have no solution.
C: The row-reduced echelon form of A is an identity matrix.	D: The rank of A is 12.
E: If $B = A^{-1}$ then B is also a 12×12 invertible matrix.	

Solution: Since A is invertible, we know that A^{-1} exists (and is also a 12×12 matrix), which means that $AA^{-1} = A^{-1}A = I$. Therefore if $B = A^{-1}$ we have $AB = BA = I$, which means that $A = B^{-1}$ and so B is also a 12×12 invertible matrix. Also, since A is invertible, row-reducing $[A|I]$ to find the inverse must result in $[I|A^{-1}]$. But when we do this, the columns to the left of the line are the row-reduced echelon form of A, so this must be the 12×12 identity matrix. This means that the ranks of both A and $[A|\mathbf{b}]$ must be 12. And it also means that when we row-reduce $[A|\mathbf{b}]$, the resulting row-reduced echelon form matrix does have a leading one in every row, to the left of the line, so it is not possible to have a row that has only 0s to the left of the line with a non-zero to the right of the line, which we would have to get in order for the system to have no solution. So the system cannot "have no solution".

21. [1 mark] Find the 2, 3-minor of $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$.

A: 9	B: −9	C: −4	D: 3	E: −3

Solution: The i, j-minor of A is the determinant of the submatrix A_{ij}, i.e. of the matrix obtained by deleting row i and column j from A. We see that in this case the 2, 1-minor of A is

$$\det A_{23} = \det \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix} = 2(0) - 3(1) = -3$$

22. [1 mark] Find $\det A$, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

A: 0	B: 1	C: 2	D: 3	E: 4

Solution: The easiest way to find the determinant of this matrix is by row-reducing. In fact, we don't even need to carry that out. We would subtract row 1 from each of the other rows, which would leave us with an upper triangular matrix with ones all along the main diagonal. Since replacing a row by that row plus a multiple of another row does not change the value of the determinant, and the determinant of an upper triangular matrix is the product of the entries along the main diagonal, then $\det A = 1^3 = 1$. (Or expanding along row 1 gives $\det A = +1(4-1) - 1(2-1) + 1(1-2) = +3 - 1 + (-1) = 1$.)

23. [1 mark] Find $\det \begin{bmatrix} 10 & 20 & 30 \\ 20 & 50 & 70 \\ 30 & 60 & 80 \end{bmatrix}$.

A: 0	B: 10	C: -10	D: 1000	E: -1000

Solution: Again, the easiest approach is to row-reduce to an upper triangular matrix. This time, we probably should actually do that. We subtract 2 times row 1 from row 2, and subtract 3 times row 1 from row 3 (which does not change the value of the determinant, and gives an upper triangular matrix):

$$\det \begin{bmatrix} 10 & 20 & 30 \\ 20 & 50 & 70 \\ 30 & 60 & 80 \end{bmatrix} = \det \begin{bmatrix} 10 & 20 & 30 \\ 0 & 10 & 10 \\ 0 & 0 & -10 \end{bmatrix} = 10(10)(-10) = -1000$$

24. [1 mark] Find $\det \begin{bmatrix} 2 & 0 & 0 \\ 5 & 5 & 0 \\ -1 & -1 & -1 \end{bmatrix}$.

A: 2	B: 5	C: -10	D: -1	E: -50

Solution: This matrix is lower triangular, so its determinant is simply the product of the entries along the main diagonal: $2(5)(-1) = -10$.

25. [1 mark] Find $\det \begin{bmatrix} 11 & 22 & 33 \\ 22 & 44 & 66 \\ 33 & 66 & 99 \end{bmatrix}$.

A: 0	B: 11	C: 11^3	D: 6	E: 66

Solution: Looking at the matrix, we see that row 2 is 2 times row 1 (and row 3 is 3 times row 1). Whenever a matrix contains one row that is a multiple of another row, its determinant is 0. (Factoring 11 out of each row does give 11^3 as the multiplier, but the value it is multiplying is 0.)

26. [1 mark] Let $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Find $\det(A - AA^{-1})$.

A: 2	B: 4	C: -2	D: -4	E: 0

Solution: Notice that (by which ever means you care to use to find it) $\det A = 2 \neq 0$ so A is invertible. Since $AA^{-1} = I$, we find the matrix whose determinant we need to find by subtracting the 3×3 identity matrix from A, which means subtracting 1 from each entry on the main diagonal of A (and subtracting 0 from each of the other entries). We see that

$$\det(A - AA^{-1}) = \det(A - I) = \det \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Now, adding row 1 to row 2 and also to row 3, we see that

$$\det(A - AA^{-1}) = \det \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

If we interchange rows 2 and 3 we get an upper triangular matrix. Doing this interchange will change the sign of the determinant. We get

$$\det(A - AA^{-1}) = -\det \begin{bmatrix} -1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = -[(-1)(2)(2)] = -(-4) = 4$$

27. [1 mark] Let I be the identity matrix with 2016 rows and 2016 columns. Find $\det(-3I)$.

A: $-3(2016)$	B: $3(2016)$	C: 3^{2016}	D: $-\left(3^{2016}\right)$	E: 2016^{-3}

Solution: We know that $\det(cI) = c^n(\det I)$, where here we have $c = -3$ and $n = 2016$. And of course $\det I = 1$. Notice that 2016 is an even number, so the even number of negatives all cancel each other out when 2016 copies of -3 are multiplied together. That is, $(-3)^{2016} = (-1\times 3)^{2016} = (-1)^{2016}(3^{2016}) = 3^{2016}$, because $(-1)^{2016} = 1$.

28. [1 mark] Let A be a 3×3 invertible matrix with $\det A = 2$. Find $\det(-2AA^TA^{-1})$.

A: -32	B: -4	C: 4	D: -16	E: 16

Solution: We use $\det(cM) = c^n(\det M)$ again (with $n = 3$ this time), along with the fact that the determinant of the product of matrices is the product of the determinants of the matrices.

$$\det(-2AA^TA^{-1}) = (-2)^3 \det(AA^TA^{-1}) = (-2)^3[(\det A)(\det A^T)(\det A^{-1})]$$

But we know that $\det A^T = \det A$, and that $\det A^{-1} = \frac{1}{\det A}$. And of course $(-2)^3 = -8$. So now, using the fact that $\det A = 2$ we see that

$$\det(-2AA^TA^{-1}) = (-2)^3[(\det A)(\det A^T)(\det A^{-1})] = (-8)(2)(2)\left(\frac{1}{2}\right) = (-8)(2) = -16$$

29. [1 mark] If $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 3$, find $\det \begin{bmatrix} a & 5b & c \\ -2d & -10e & -2f \\ g & 5h & i \end{bmatrix}$.

A: -30	B: -6	C: -10	D: 6	E: 30

Solution: To get from the matrix whose determinant we need to find to the matrix whose determinant we know, we can factor -2 out of row 2 and then factor 5 out of column 2. We get

$$\det\begin{bmatrix} a & 5b & c \\ -2d & -10e & -2f \\ g & 5h & i \end{bmatrix} = (-2)\det\begin{bmatrix} a & 5b & c \\ d & 5e & f \\ g & 5h & i \end{bmatrix} = (-2)(5)\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = (-10)(3) = -30$$

30. [1 mark] If $\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 3$, find $\det\begin{bmatrix} a+2g & b+2h & c+2i \\ d & e & f \\ d+a & e+b & f+c \end{bmatrix}$.

A: -3	B: 3	C: -6	D: 6	E: 0

Solution: First, we see that the numbers in row 3 are sums which have as one term in each sum the corresponding number in row 2. So we start by subtracting row 2 from row 3, which has no effect on the value of the derivative, so:

$$\det\begin{bmatrix} a+2g & b+2h & c+2i \\ d & e & f \\ d+a & e+b & f+c \end{bmatrix} = \det\begin{bmatrix} a+2g & b+2h & c+2i \\ d & e & f \\ a & b & c \end{bmatrix}$$

Now, we see that the numbers in row 1 are sums involving the corresponding numbers in (the new) row 3, so we subtract row 3 from row 1, which again does not affect the value of the derivative:

$$\det\begin{bmatrix} a+2h & b+2h & c+2i \\ d & e & f \\ a & b & c \end{bmatrix} = \det\begin{bmatrix} 2g & 2h & 2i \\ d & e & f \\ a & b & c \end{bmatrix}$$

If we factor 2 out of (the current) row 1, and then interchange that row and the last row, we will have the matrix whose determinant we know is 3. But when we interchange 2 rows, it changes the sign of the determinant, so we see that the answer is

$$\det\begin{bmatrix} 2g & 2h & 2i \\ d & e & f \\ a & b & c \end{bmatrix} = 2\det\begin{bmatrix} g & h & i \\ d & e & f \\ a & b & c \end{bmatrix} = -2\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = (-2)(3) = -6$$

31. [1 mark] The system of linear equations $A\mathbf{x} = \mathbf{b}$, has 3 equations in 3 unknowns, with $\det A = 0$. Which one of the following statements is **true**?

A: The system must have a unique solution.	B: The system always has at least one solution.
C: The system cannot have infinitely many solutions.	D: The system might not have any solutions.
E: None of A, B, C or D.	

Solution: We know that a matrix A is invertible if and only if $\det A \neq 0$, and that the system $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if A is invertible. So knowing that $\det A = 0$ tells us that the system cannot have a unique solution. But this leaves two possibilities: it might be inconsistent and have no solution, or it might have infinitely many solutions. Therefore the only one of the statements (answer choices) which is true is the one that says that the system might not have any solutions.

32. [1 mark] Find the $(2, 3)$-entry of Adj A, where $A = \begin{bmatrix} 1 & 3 & -1 \\ 1 & -1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$.

A: -2	B: 2	C: -3	D: 3	E: 1

Solution: The $(2, 3)$-entry of Adj A is the $3, 2$-cofactor of A, which is

$$(-1)^{3+2} \det A_{32} = (-1)^5 \det \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} = (-1)[2 - (-1)] = -(2 + 1) = -3$$

33. [1 mark] Consider a system of linear equations $A\mathbf{x} = \mathbf{b}$, where A is a $12 \times n$ matrix with $\det A = 0$. Which one of the following statements is **false**?

A: n must be 12.	B: A has no inverse.
C: The rank of A is less than 12.	D: Adj A is not defined.
E: Cramer's Rule cannot be used to solve the system.	

Solution: Since Adj A is defined for *every* square matrix A, the statement which is false is the one which says that Adj A is not defined.

For the others, we are told the value of $\det A$, so $\det A$ must be defined. But only square matrices have determinants, so A must be a square matrix and therefore $n = 12$. And Cramer's Rule can only be used when $\det A \neq 0$, so it cannot be used for this system. As well, knowing that $\det A = 0$ also tells us that A has no inverse and that the rank of A cannot be 12 and must therefore be less than 12.

34. [1 mark] Find the $(2, 3)$-entry of A^{-1} if it is known that $\det A = -2$ and that Adj A is the matrix shown here:

$$\begin{bmatrix} 2 & 2 & -1 \\ 6 & 4 & 1 \\ 2 & 2 & -2 \end{bmatrix}$$

A: -2	B: 1	C: -1	D: $\frac{1}{2}$	E: $-\frac{1}{2}$

Solution: We use the adjoint form of the inverse, i.e. the fact that $A^{-1} = \frac{1}{\det A}(\text{Adj } A)$. So we can find the value of the $(2, 3)$-entry of A^{-1} by multiplying the $(2, 3)$-entry of Adj A by $\frac{1}{\det A} = \frac{1}{-2} = -\frac{1}{2}$. In row 2 and column 3, we see that the $(2, 3)$-entry of Adj A is 1, so the $(2, 3)$-entry of A^{-1} is $-\frac{1}{2} \times 1 = -\frac{1}{2}$.

35. [1 mark] Consider any 3×3 matrix A which is invertible. Find $\det \left[\frac{1}{\det A}(A \text{ Adj } A)\right]$.

A: 0	B: -1	C: 1	D: -3	E: 3

Solution: A constant multiplier in a matrix product can be applied to any matrix in the product, and $\frac{1}{\det A}$ is just a constant. So we see that

$$\frac{1}{\det A}(A \text{ Adj } A) = A\left[\frac{1}{\det A}(\text{ Adj } A)\right]$$

But we also know that $\frac{1}{\det A}(\text{ Adj } A) = A^{-1}$, so we get

$$\det \left[\frac{1}{\det A}(A \text{ Adj } A)\right] = \det \left[A\left(\frac{1}{\det A}(\text{ Adj } A)\right)\right] = \det \left[A\left(A^{-1}\right)\right] = \det I = 1$$

PART B (15 marks)

36. [3 marks] Find the point of intersection of the line $(x, y, z) = (3, 3, 1) + t(2, 1, 1)$ with the plane $2x + y + z = 7$.

Solution: We write parametric equations of the line: $x = 3 + 2t$, $y = 3 + t$ and $z = 1 + t$. Now, we use these to substitute for x, y and z in the equation of the plane:

$$\begin{aligned} 2x + y + z = 7 \quad &\Rightarrow \quad 2(3 + 2t) + (3 + t) + (1 + t) = 7 \\ &\Rightarrow \quad 6 + 4t + 3 + t + 1 + t = 7 \\ &\Rightarrow \quad 4t + t + t = 7 - (6 + 3 + 1) \\ &\Rightarrow \quad 6t = 7 - 10 = -3 \Rightarrow t = \frac{-3}{6} = -\frac{1}{2} \end{aligned}$$

Now, we use this value of t in the equation of the line to get the point of intersection:

$$(x, y, z) = (3, 3, 1) + \left(-\frac{1}{2}\right)(2, 1, 1) = \left(3 - 1, 3 - \frac{1}{2}, 1 - \frac{1}{2}\right) = \left(2, \frac{5}{2}, \frac{1}{2}\right)$$

37. [3 marks] Find a standard form equation of the plane which passes through the point $P(1, 0, 2)$ and is parallel to both of the lines shown below:

$$(x, y, z) = (15, 7, 0) + s(2, 2, 1) \qquad \text{and} \qquad (x, y, z) = (4, 22, 3) + t(1, 3, 4)$$

Solution: We recognize these as point-parallel forms of equations of the lines, so we know that the vector $\mathbf{u} = (2, 2, 1)$ is parallel to the first line and $\mathbf{v} = (1, 3, 4)$ is parallel to the second line. Since the plane whose equation we're looking for is parallel to both lines, then any normal vector for the plane must be perpendicular to both lines and therefore is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. That is, any vector that is orthogonal to both these vectors is a normal vector for the plane. And we know that the cross product of 2 vectors is orthogonal to both those vectors, so we can use $\mathbf{n} = \mathbf{u} \times \mathbf{v}$ as a normal for the plane. We get

$$\mathbf{n} = (2, 2, 1) \times (1, 3, 4) = (2(4) - 3(1), 1(1) - 4(2), 2(3) - 1(2)) = (8 - 3, 1 - 8, 6 - 2) = (5, -7, 4)$$

The components of $\mathbf{n}$ are the coefficients of x, y and z on the left-hand side of the standard form equation of the plane, and the constant on the right-hand side can be found as $\mathbf{n} \bullet \mathbf{p}$ where $\mathbf{p}$ is the vector whose endpoint is P for a point P which is on the plane, in this case the point $P(1, 0, 2)$. We get

$$\mathbf{n} \bullet \mathbf{p} = (5, -7, 4) \bullet (1, 0, 2) = 5(1) + (-7)(0) + 4(2) = 5 + 0 + 8 = 13$$

Therefore a standard form equation for the plane which passes through P and is parallel to both of the given lines is $5x - 7y + 4z = 13$.

38. [2 marks] If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$, find $A^T - 2B$.

Solution: To find the transpose of A, we interchange the rows and columns. And for $2B$ we multiply each entry of B by 2. So we get

$$A^T - 2B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 2 & -2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 - 2 & 3 - (-2) \\ 2 - 0 & 4 - 4 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 2 & 0 \end{bmatrix}$$

39. [2 marks] Find the determinant of the matrix A shown below **by expanding along column 2**.

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 2 \\ -2 & 1 & 3 \end{bmatrix}$$

Solution:

$$\begin{aligned} \det A &= (-1)^{1+2}a_{12}(\det A_{12}) + (-1)^{2+2}a_{22}(\det A_{22}) + (-1)^{3+2}a_{32}(\det A_{32}) \\ &= (-1)^3(2)\det\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} + (-1)^4(0)\det\begin{bmatrix} 2 & 1 \\ -2 & 3 \end{bmatrix} + (-1)^5(1)\det\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\ &= -2[1(3) - (-2)(2)] + 0 - 1[2(2) - 1(1)] = -2(3+4) - (4-1) = -2(7) - 3 = -14 - 3 = -17 \end{aligned}$$

40. [3 marks] **Use Cramer's Rule** to find the value of y in the unique solution to the system of linear equations shown below.

$$\begin{array}{rcrcrcr} x & + & 2y & & & = & 1 \\ 2x & + & 5y & - & z & = & -1 \\ x & - & y & + & z & = & 0 \end{array}$$

Solution: We have the system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & -1 \\ 1 & -1 & 1 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

The value of y in the unique solution to this system is given by $y = \frac{\det A(2)}{\det A}$, where $A(2)$ is the matrix obtained by replacing column 2 of A by the entries in $\mathbf{b}$.

We can find $\det A$ by easily enough by expanding along row 1:

$$\det A = \det\begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & -1 \\ 1 & -1 & 1 \end{bmatrix} = +1[5(1) - (-1)(-1)] - 2[2(1) - 1(-1)] + 0 = (5-1) - 2(2+1) = 4 - 6 = -2$$

(Finding $\det A$ by any other method was fine and should still have given you $\det A = -2$.) Now we form the $A(2)$ matrix, and we can again easily find the determinant by expanding along row 1 (or any other method):

$$\begin{aligned} \det A(2) = \det\begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix} &= +1[(-1)(1) - 0(-1)] - 1[2(1) - 1(-1)] + 0 = (-1-0) - [2 - (-1)] \\ &= -1 - (2+1) = -1 - 3 = -4 \end{aligned}$$

Therefore we see that the value of y in the unique solution to the system is

$$y = \frac{\det A(2)}{\det A} = \frac{-4}{-2} = 2$$

41. [2 marks] Let I be the 3×3 identity matrix. Find $\det\left(2I^{-1} - 5I^T\right)$.

Solution: The transpose of I is just I, and the inverse of I is also I. So we see that

$$\det\left(2I^{-1} - 5I^T\right) = \det(2I - 5I) = \det(-3I) = (-3)^3 \det I = -27(1) = -27$$