4. Logical Equivalences & Valid Arguments (January 18)

Lec 3 Mini Review.

consistent set of propositions✓ The Island of Knights and Knaves puzzles✓
DNF✓ atoms✓ literals✓ conjunctive clauses✓ disjunction of conjunctive clauses✓

Logical Equivalences

Example 4.1. Using only the Laws in the **Table of Logical Equivalences**, prove that

$$P \to Q \equiv \neg Q \to \neg P.$$

$P \to Q \equiv \neg P \vee Q$ (Implication Law)
$\equiv Q \vee \neg P$ (Commutative Law)
$\equiv \neg\neg Q \vee \neg P$ (Double Negation Law)
$\equiv \neg Q \to \neg P$ (Implication Law)

Conditional Statement, Contrapositive, and Converse For a conditional statement $P \to Q$, there are two other conditional statements which are related to $P \to Q$ in important ways:

- $\neg Q \to \neg P$ is called the **contrapositive** of $P \to Q$.
- $Q \to P$ is called the **converse** of $P \to Q$.

- In Example 4.1, we proved that a conditional statement $P \to Q$ is logically equivalent to its contrapositive, namely $\neg Q \to \neg P$.
- This is not the case for a conditional statement and its converse!

Exercise 4.2. Using a truth table, prove that $P \to Q$ is **not** logically equivalent to its converse $Q \to P$. Give all counterexamples, that is, list all truth assignments for which $(P \to Q) \leftrightarrow (Q \to P)$ is **F**.

P	Q	P→Q	Q→P	(P→Q)↔(Q→P)
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Since $(P\to Q)\leftrightarrow(Q\to P)$ is not a tautology, $P\to Q$ is not logically equivalent to its converse $Q\to P$

Counterexamples: when $P=T, Q=F$, or when $P=F, Q=T$, the truth values of $P\to Q$ and $Q\to P$ differ

* These notes are solely for the personal use of students registered in MAT1348.

THE TABLE OF LOGICAL EQUIVALENCES

$P \to Q$ is T when P is F or Q is T	1.	$P \to Q \equiv \neg P \vee Q$	Implication Law
2 ways to think about $\leftrightarrow$	2.	$P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$	Biconditional Laws
	3.	$P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$	
out of P/$\neg$P exactly one is T and other is F	4.	$P \vee \neg P \equiv \mathbf{T}$	Negation Laws
	5.	$P \wedge \neg P \equiv \mathbf{F}$	
	6.	$P \vee \mathbf{F} \equiv P$	Identity Laws
	7.	$P \wedge \mathbf{T} \equiv P$	
	8.	$P \vee \mathbf{T} \equiv \mathbf{T}$	Domination Laws
	9.	$P \wedge \mathbf{F} \equiv \mathbf{F}$	
	10.	$P \vee P \equiv P$	Idempotent Laws
	11.	$P \wedge P \equiv P$	
	12.	$\neg\neg P \equiv P$	Double Negation Law
	13.	$P \vee Q \equiv Q \vee P$	Commutative Laws
	14.	$P \wedge Q \equiv Q \wedge P$	
	15.	$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$	Associative Laws
	16.	$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$	
a bit like factoring	17.	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$	Distributive Laws
	18.	$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$	
a way to switch $\wedge$/$\vee$ with $\neg\neg$	19.	$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$	De Morgan's Laws
	20.	$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$	

Example 4.3. Prove $(x \wedge y) \vee (x \wedge \neg y) \vee (\neg x \wedge y) \equiv x \vee y$

$$(x \wedge y) \vee (x \wedge \neg y) \vee (\neg x \wedge y) \equiv [x \wedge (y \vee \neg y)] \vee (\neg x \wedge y) \quad \text{(distributive law)}$$

$$\equiv [x \wedge T] \vee (\neg x \wedge y) \quad \text{(negation law)}$$

$$\equiv (x) \vee (\neg x \wedge y) \quad \text{(identity law)}$$

$$\equiv (x \vee \neg x) \wedge (x \vee y) \quad \text{(distributive law)}$$

$$\equiv T \wedge (x \vee y) \quad \text{(negation law)}$$

$$\equiv x \vee y \quad \text{(identity law)}$$

$$\therefore (x \wedge y) \vee (x \wedge \neg y) \vee (\neg x \wedge y) \equiv x \vee y$$

Example 4.4. Prove that $(a \wedge \neg b) \wedge (\neg a \vee b)$ is a contradiction

Note contradiction $\equiv F$

$$(a \wedge \neg b) \wedge (\neg a \vee b) \equiv a \wedge (\neg b \wedge (\neg a \vee b)) \quad \text{(associative law)}$$

$$\equiv a \wedge [(\neg b \wedge \neg a) \vee (\neg b \wedge b)] \quad \text{(distributive law)}$$

$$\equiv a \wedge [(\neg b \wedge \neg a) \vee F] \quad \text{(domination law)}$$

$$\equiv a \wedge [\neg b \wedge \neg a] \quad \text{(identity law)}$$

$\equiv a \wedge (\neg a \wedge \neg b)$ (commutative law)

$\equiv (a \wedge \neg a) \wedge \neg b$ (associative law)

$\equiv F \wedge \neg b$ (negation law)

$\equiv F$ (domination law)

$\therefore (a \wedge \neg b) \wedge (\neg a \vee b) \equiv F$ so $(a \wedge \neg b) \wedge (\neg a \vee b)$ is a contradiction.

Example 4.5. Find a DNF for $(p \to q) \vee (\neg(p \vee q) \wedge r)$

$(p \to q) \vee (\neg(p \vee q) \wedge r) \equiv (\neg p \vee q) \vee (\neg(p \vee q) \wedge r)$ (implication law)

$\equiv \neg p \vee q \vee (\neg(p \vee q) \wedge r)$ (associative law)

$\equiv \neg p \vee q \vee ((\neg p \wedge \neg q) \wedge r)$ (De Morgan's law)

$\equiv \underbrace{\neg p}_{\text{conjunctive clause } ✓} \vee \underbrace{q}_{\text{conjunctive clause } ✓} \vee (\underbrace{\neg p \wedge \neg q \wedge r}_{\text{conjunctive clause } ✓})$ (associative law)

$\therefore (p \to q) \vee (\neg(p \vee q) \wedge r) \equiv \neg p \vee q \vee (\neg p \wedge \neg q \wedge r)$ and this is in DNF since it's a disjunction of conjunctive clauses.

Example 4.6. Find a compound proposition that is logically equivalent to $X \wedge Y$ that uses only the logical connectives $\to$ and $\neg$.

$X \wedge Y \equiv \neg\neg(X \wedge Y)$ (double negation law)

$\equiv \neg(\neg X \vee \neg Y)$ (De Morgan's law)

$\equiv \neg(X \to \neg Y)$ (implication law)

Thus, we found a proposition $\neg(X \to \neg Y)$ such that $X \wedge Y \equiv \neg(X \to \neg Y)$ and $\neg(X \to \neg Y)$ uses only the logical connectives $\to$ and $\neg$.

Example 4.7. Find a compound proposition that is logically equivalent to $p \to (q \vee r)$ that uses only the logical connectives $\neg$ and $\wedge$.

$$p \to (q \vee r) \equiv \neg p \vee (q \vee r) \quad \text{(implication law)}$$
$$\equiv \neg\neg[\neg p \vee (q \vee r)] \quad \text{(double negation)}$$
$$\equiv \neg[\neg\neg p \wedge \neg(q \vee r)] \quad \text{(De Morgan's law)}$$
$$\equiv \neg[p \wedge \neg(q \vee r)] \quad \text{(double negation law)}$$
$$\equiv \neg[p \wedge (\neg q \wedge \neg r)] \quad \text{(De Morgan's law)}$$

Thus, $p \to (q \vee r) \equiv \neg[p \wedge (\neg q \wedge \neg r)]$ and $\neg[p \wedge (\neg q \wedge \neg r)]$ uses only the logical connectives $\neg$ and $\wedge$.

Arguments

- An **argument** is a set of propositions in which one (called the **conclusion**) is claimed to follow from the other propositions (called the **premises**). In other words, an argument is a compound proposition of the form

argument: $(P_1 \wedge P_2 \wedge \cdots \wedge P_k) \to C$

where $P_1, P_2, \ldots, P_k$ are the premises; C is the conclusion

$\Rightarrow$ an argument is a conditional statement whose premise is the conjunction of propositions

- Sometimes, arguments are written vertically like this:

$$\begin{array}{rl} & P_1 \\ & P_2 \\ & \vdots \\ & \underline{P_k} \\ \therefore & C \end{array}$$

- An argument $(P_1 \wedge P_2 \wedge \cdots \wedge P_k) \to C$ is called **a valid argument** if the conclusion C is true whenever all the premises $P_1, \ldots, P_k$ are true.
- In other words, $(P_1 \wedge P_2 \wedge \cdots \wedge P_k) \to C$ is a **valid argument** if and only if $(P_1 \wedge P_2 \wedge \cdots \wedge P_k) \to C$ is a tautology.

Example 4.8. Prove that the following argument is valid:

2 premises: P_1: $A \vee B$

P_2: $\neg A$

Conclusion $\therefore C$ $\therefore B$

(this argument is called *Disjunctive Syllogism*)
(it is one of the *Rules of Inference*)

◇ To show this is a <u>valid</u> argument, we must prove that

$[(A \vee B) \wedge (\neg A)] \to B$ is a tautology.

Let's use the Laws of $\equiv$ to show $(A \vee B) \wedge \neg A \to B \equiv T$.

$(A \vee B) \wedge \neg A \to B \equiv \neg((A \vee B) \wedge \neg A) \vee B$ (implication law)

$\equiv (\neg(A \vee B) \vee \neg\neg A) \vee B$ (De Morgan's law)

$\equiv (\neg(A \vee B) \vee A) \vee B$ (double negation)

$\equiv \neg(A \vee B) \vee (A \vee B)$ (associative law)

$\equiv T$ (negation law)

Since the argument is $\equiv T$, it's a tautology $\therefore$ the argument is valid

STUDY GUIDE

Important terms and concepts:

- truth table method for Knights and Knaves puzzles
- DNF (atoms, literals, conjunctive clauses)
- converse versus contrapositive of a conditional statement
- using the Laws in The Table of Logical Equivalences
- valid argument

Exercises	Rosen §1.3 (using Laws) # 21, 23, 29	Supp. Ex. List §1 # 1, 3, 6b, 7c, 8