2. Logical Equivalence, Translation between English and Propositional Logic, and Consistent/Inconsistent Sets of Propositions (Jan. 11)

Lec 1 Mini Review.

✓ proposition ✓ truth value ✓ compound proposition

✓ logical connectives:

p	$\neg p$
T	F
F	T

p	q	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \to q$	$p \leftrightarrow q$
T	T	T	T	F	T	T
T	F	F	T	T	F	F
F	T	F	T	T	T	F
F	F	F	F	F	T	T

Precedence of Logical Operators. Unless specified otherwise by parentheses, the order of precedence of logical connectives is

1. $\neg$ 2. $\wedge$ 3. $\vee$ 4. $\to$ 5. $\leftrightarrow$ 6. $\oplus$

ex. $\neg p \vee q \to q \wedge r$ means $((\neg p) \vee q) \to (q \wedge r)$

LOGICAL EQUIVALENCE

- Two propositions p and q are called **logically equivalent** if the biconditional statement $p \leftrightarrow q$ is a tautology.
- **Notation:** if p and q are logically equivalent, then we will write
$$p \equiv q.$$
- **Note:** $\equiv$ is not a logical connective; it's just short for "is logically equivalent to".

Example 2.1. Using a truth table and an explanation, prove that $(p \to q) \equiv (\neg p \vee q)$.

p	q	$p \to q$	$\neg p$	$\neg p \vee q$	$(p \to q) \leftrightarrow (\neg p \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Since the biconditional statement $(p \to q) \leftrightarrow (\neg p \vee q)$ is T for all truth assignments, it's a tautology. By definition, this means that $p \to q$ is logically equivalent to $\neg p \vee q$ $\therefore p \to q \equiv \neg p \vee q$

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* These notes are solely for the personal use of students registered in MAT1348.

Example 2.2. Are all tautologies logically equivalent to each other? Explain.

- Suppose P is a tautology, where P is some compound proposition.
- Then, for all truth assignments (of whatever atoms P consists of), the truth value of P is T (by def. of tautology)
- It now follows that the biconditional statement $P \leftrightarrow T$ is of the form $T \leftrightarrow T$ for all truth assignments. $\therefore$ $P \leftrightarrow T$ is a tautology.

$\Rightarrow P \equiv T$.

So any tautology is logically equivalent to T

$\therefore$ all tautologies are logically equivalent to each other.

TRANSLATION BETWEEN ENGLISH AND PROPOSITIONAL LOGIC

For the examples in this section, let us define the following propositional variables:

b : "A bear eats berries."
f : "A bear eats a fish."
r : "A bear is near the river."
s : "A bear sees a fish."

* assume these refer to one particular bear.

Negation.

Common translations in English

$\neg p$: "It is not the case that p."

Ex. $\neg r$: "It is not the case that a bear is near the river."
"A bear is not near the river."

Ex. $\neg s$: "A bear does not see a fish"

Ex "A bear does not eat berries." translation to propositional logic: $\neg b$

Ex "It is not the case that a bear does not eat berries." $\neg(\neg b)$

Conjunction.

Common translations in English

$p \wedge q$ "p and q."

"p but q."

Ex. "$\underbrace{\text{A bear sees a fish}}_{s}$, $\underbrace{\text{but}}_{\wedge}$ $\underbrace{\text{she does not each a fish}}_{\neg f}$." $s \wedge \neg f$

Disjunction.

Common translations in English

$p \vee q$ "p or q."

"p unless q."

* "unless" is not always used as an inclusive or in English but this will be the accepted convention for 1348.

Ex. $\underbrace{\text{A bear eats berries}}_{b}$ $\underbrace{\text{unless}}_{\vee}$ $\underbrace{\text{she sees a fish}}_{s}$. $b \vee s$

Ex. $\underbrace{\text{A bear eats berries}}_{b}$ $\underbrace{\text{or}}_{\vee}$ fish. $b \vee s$

f

Exclusive Or.

Common translations in English

$p \oplus q$ "Either p or q."

"p or q but not both."

* "either...or" is not always used as an exclusive or in English but this will be the accepted convention for 1348.

Ex. "Either $\underbrace{\text{a bear eats fish}}_{f}$ $\underbrace{\text{or}}_{\oplus}$ $\underbrace{\text{she eats berries}}_{b}$." $f \oplus b$

Ex $r \vee s$: "A bear is near the river or she sees a fish, but not both."

* important note on translation: you are only translating between English and propositional logic. The propositions themselves do not need to make sense with respect to the "real world". Do not translate based on context... it is irrelevant to consider how a real bear would behave... Just translate the given propositions.

Conditional Statement $p \to q$

- p is the premise or hypothesis
- q is the conclusion or consequence.

Important $p \to q$ is not logically equivalent to its converse $q \to p$

∴ when translating conditional statements you must be careful not to mix up premise and conclusion.

* in the following chart, each phrase is an English translation of $\boxed{p \to q}$

$p \to q$ Common translations in English

"if p, then q." "if p, q."	"q if p."
"p implies q."	"q is implied by p." "q follows from p."
"p is a sufficient condition for q."	"A sufficient condition for q is p."
"A necessary condition for p is q."	"q is a necessary condition for p."
"p only if q." "whenever it is the case that p, then q." "when p, q."	"q if p" "q whenever p." "q when p."

Ex. "$\underbrace{\text{A bear eats berries}}_{b}$ $\underset{\rightarrow}{\text{only if}}$ $\underbrace{\text{she is not near the river}}_{\neg r}$." $b \to \neg r$

Ex "If $\underbrace{\text{a bear sees a fish}}_{s}$, $\underset{\rightarrow}{\text{then}}$ $\underbrace{\text{she eats a fish}}_{f}$." $s \to f$

Ex. "$\underbrace{\text{A bear eats a fish}}_{f}$ $\underset{\leftarrow}{\text{whenever}}$ $\underbrace{\text{she's near the river.}}_{r}$" $r \to f$

Ex. "$\underbrace{\text{A bear does not eat berries}}_{\neg b}$ $\underset{\leftarrow}{\text{when}}$ $\underbrace{\text{she sees a fish.}}_{s}$" $s \to \neg b$

Ex. "A sufficient condition for $\underbrace{\text{a bear to eat a fish}}_{f}$ is that $\underbrace{\text{she sees a fish.}}_{s}$" $s \to f$

Ex. "A necessary condition for $\underbrace{\text{a bear to eat a fish}}_{f}$ is that $\underbrace{\text{she sees a fish.}}_{s}$" $f \to s$

Biconditional Statement $p \leftrightarrow q$

* Exercise show that $p \leftrightarrow q \equiv (q \to p) \wedge (p \to q)$

So we have translations: 1. (p if q) and (p only if q)

2. (p is necessary for q) and (p is sufficient for q)

Note. $q \to p$ is called the converse of $p \to q$

Thus, a biconditional statement is logically equivalent to the conjunction of a conditional statement and its converse.

Common translations in English

$p \leftrightarrow q$

"p if and only if q."

"if p, then q, and conversely."

"p is necessary and sufficient for q."

Ex. "$\underbrace{\text{A bear eats berries}}_{b}$ $\underset{\leftrightarrow}{\text{if and only if}}$ $\underbrace{\text{she does not see a fish.}}_{\neg s}$" $b \leftrightarrow \neg s$

TRUTH ASSIGNMENTS

- Let P be a compound proposition consisting of propositional variables $p_1, \ldots, p_k$
- An assignment of truth values to each of the variables $p_1, \ldots, p_k$ is called a **truth assignment** of P.

Ex. The compound proposition $P : \quad \neg(p \vee q)$ has 4 possible truth assignments (corresponding to each row of the truth table).

One possible truth assignment of P is $p = T, q = F$.

Example 2.3. Consider the following compound propositions regarding a particular bear:

P_1 : "A bear eats berries unless she is near the river." $b \vee r$
P_2 : "A bear eats fish if and only if she is near the river." $f \leftrightarrow r$
P_3 : "Either a bear eats berries or she eats fish." $b \oplus f$
P_4 : "A bear eats berries and she is near the river." $b \wedge r$

(a) Determine the truth value of each of P_1, P_2, P_3 for each of the following truth assignments

i. $b = \mathbf{T} \quad f = \mathbf{T} \quad r = \mathbf{T}$

P_1: $b \vee r$ is $T \vee T$ so P_1 is T
P_2: $f \leftrightarrow r$ is $T \leftrightarrow T$ so P_2 is T
P_3: $b \oplus f$ is $T \oplus T$ so P_3 is F.

ii. $b = \mathbf{T} \quad f = \mathbf{T} \quad r = \mathbf{F}$

P_1: $b \vee r$ is $T \vee F$ so P_1 is T
P_2: $f \leftrightarrow r$ is $T \leftrightarrow F$ so P_2 is F
P_3: $b \oplus f$ is $T \oplus T$ so P_3 is F.

(b) Construct a complete truth table for the propositions P_1, P_2, P_3, and P_4

			P_1	P_2	P_3	P_4
b	f	r	$b \vee r$	$f \leftrightarrow r$	$b \oplus f$	$b \wedge r$
T	T	T	T	T	F	T
T	T	F	T	F	F	F
T	F	T	T	F	T	T
T	F	F	T	T	T	F
F	T	T	T	T	T	F
F	T	F	F	F	T	F
F	F	T	T	F	F	F
F	F	F	F	T	F	F

(c) Is it possible for P_1, P_2, and P_3 to all be simultaneously true for this bear? Explain.

Yes, the truth assignment b=T, f=F, r=F makes P_1, P_2, and P_3 all have truth value T. (So does the truth assignment b=F, f=T, r=T)

(d) Is it possible for P_1, P_2, P_3, and P_4 to all be simultaneously true for this bear? Explain.

No. The rows of the truth table cover an exhaustive list of all possible truth assignments for the compound propositions P_1, P_2, P_3, and P_4, so we can see that there is no truth assignment (row of table) for which all 4 of P_1, P_2, P_3, and P_4 are true at the same time.

CONSISTENT/INCONSISTENT SET OF PROPOSITIONS.

- A set $\{P_1, P_2, \ldots, P_n\}$ of compound propositions is called **consistent** if there exists at least one truth assignment that makes all the propositions $P_1, \ldots, P_n$ true at the same time.
- Otherwise, for each possible truth assignment, at least one of the propositions P_i is false; in this case, the set $\{P_1, P_2, \ldots, P_n\}$ is called **inconsistent**.

Example 2.4. Is the set $\{P_1, P_2, P_3\}$ of compound propositions defined in Example 2.3 consistent or inconsistent? Explain.

The set $\{P_1, P_2, P_3\}$ is consistent because there exists at least one truth assignment that makes all 3 of P_1, P_2, and P_3 T, namely $b = T$, $f = F$, $r = F$

What about the set $\{P_1, P_2, P_3, P_4\}$?

The set $\{P_1, P_2, P_3, P_4\}$ is inconsistent because there is no truth assigment that makes all 4 of P_1, P_2, P_3, and P_4 T.

A Question to Ponder. Suppose $\{P_1, P_2, \ldots, P_n\}$ is an inconsistent set of propositions. What (if anything) can we conclude about the conjunction $(P_1 \wedge P_2 \wedge \cdots \wedge P_n)$?

- If $\{P_1, P_2, \ldots, P_n\}$ is inconsistent, then for every possible truth assignment, at least one of the propositions $P_i \in \{P_1, P_2, \ldots, P_n\}$ is F.

Q. what happens to a conjunctive clause like $P_1 \wedge P_2 \wedge \cdots \wedge P_n$ when we know at least one of the propositions P_i is F ?

A. The entire conjunction will be F.

∴ for every truth assignment, the conjunction $P_1 \wedge P_2 \wedge \cdots \wedge P_n$ contains at least one F

Thus, if $\{P_1, \ldots, P_n\}$ is an inconsistent set, then the conjunction $P_1 \wedge P_2 \wedge \cdots \wedge P_n$ is a contradiction.

STUDY GUIDE

Important terms and concepts:

logical equivalence
truth assignment
translation between English and propositional logic
consistent vs. inconsistent set of propositions

Rosen Exercises 1.3 # 17–33, 1.1 # 23, 1.2 # 1, 5, 9, 11