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## 1. Propositional Logic ( January 8 )

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### WHY MATHEMATICS AND LOGIC ARE AMAZING!

- Mathematics is a form of art. The masterpieces of math are elegant arguments (proofs) that tell us some fragment of truth.
- Unlike other sciences where results are most often obtained from experimental data and observation, mathematical results are “discovered” when someone is able to prove that something must be true.
- For a proof to be correct, the underlying steps must be a valid logical argument and the language we use must be well-defined and exact.
- Starting from some premises, a proof must follow the rules of logic, step by step, until the conclusion is reached. If each step of the proof follows logically from the previous step, then we know beyond doubt that the conclusion must be true whenever the premises are true.
- It is useful to know when something is true! Many scientific results are based on underlying mathematical models → physics, chemistry, biology, engineering...
- Logic itself is especially important in Computer Science; the language of computers is Boolean algebra. Using logic, we are able to translate our intentions so that a machine made of electrical circuits does our bidding.

### THE ROUGH PLAN FOR MAT1348

- We will begin by studying Propositional Logic.
- We will work our way towards writing and understanding correct mathematical proofs.
- We will study the basics of set theory, functions, relations, counting, and induction — these are all fundamental tools for math and computer science.
- We will end with an introduction to graph theory, which is an essential tool for modelling many discrete structures.

### Warnings!

- This course is very fast-paced.
- You will be given **many** new and precise definitions and techniques to use.
- Having precise (lawyer-like?) language is very important — when we communicate (in the form of written mathematical proofs, computer programs, and otherwise), we want to be certain we understand each other exactly.

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## BASIC BUILDING BLOCKS OF LOGIC

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- A **proposition** is a declarative sentence that is either **true** or **false** but not both.
- If a proposition is true, then its **truth value** is TRUE, denoted T.
- If a proposition is false, then its **truth value** is FALSE, denoted F.

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**Example 1.1.** For each of the following sentences, determine whether or not it is a proposition. If it is a proposition, what is its truth value? If it is not a proposition, explain.

- (a) Ottawa is the capital of Canada. ← a proposition ✓ It's T.
- (b)  $1 + 2 = 3$ . ← a proposition ✓ It's T.
- (c)  $2 + 2 = 3$ . ← a proposition ✓ It's F.
- (d)  $1 + x = 3$ . ← not a proposition unless we know what  $x$  is.
- (e) There exists an integer  $x$  such that  $1 + x = 3$ . ← a proposition ✓ It's T.
- (f) For all integers  $x$ , the equation  $1 + x = 3$  is true. ← a proposition ✓ It's F.
- (g) Eat your vegetables. ← not a proposition since it's not a declarative statement
- (h) What time is it? ← not a proposition since it's not a declarative statement
- (i) It is winter now. ← a proposition (if it is clear when "now" is) and it's T.
- (j) You are a student in MAT1348.  
↳ a proposition (if it is clear who "you" is) and it's T
- (k) All books in the library are yellow.  
↳ a proposition ✓ It's F (at uOttawa library)

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### Propositional variables (a.k.a atoms) and compound propositions.

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- For short, we will use variables to represent propositions.
- The variables we use to represent propositions are called **propositional variables** or "**atoms**" for short.

Ex.  $p$  : "Tigers are furry animals."

Ex.  $q$  : "Elephants are wise."

- A **compound proposition** is formed by combining one or more atoms using **logical connectives**.

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## LOGICAL CONNECTIVES

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**Negation.** Let  $p$  be a proposition.

- The negation of  $p$ , denoted  $\neg p$ , read "not  $p$ " is the proposition

"It is not the case that  $p$ ."

Truth table:

$p$	$\neg p$
T	F
F	T

- The truth value of  $\neg p$  is the opposite of the truth value of  $p$ .

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**Conjunction.** Let  $p$  and  $q$  be propositions.

- The conjunction of  $p$  and  $q$ , denoted  $p \wedge q$ , read " $p$  and  $q$ ", is the proposition

" $p$  and  $q$ "

Truth table:

$p$	$q$	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- $p \wedge q$  is T when both  $p$  and  $q$  are T; otherwise,  $p \wedge q$  is F.

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**Disjunction.** Let  $p$  and  $q$  be propositions.

- The disjunction of  $p$  and  $q$ , denoted  $p \vee q$ , read " $p$  or  $q$ ", is the proposition " $p$  or  $q$ "

Truth table:

$p$	$q$	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- $p \vee q$  is F when both  $p$  and  $q$  are F; otherwise,  $p \vee q$  is T.

$\Rightarrow$  "or" is the inclusive or i.e. "and/or"

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**Exclusive Or.** Let  $p$  and  $q$  be propositions.

- The exclusive or of  $p$  and  $q$ , denoted  $p \oplus q$ , read " $p$  x-or  $q$ ", is the proposition

" $p$  or  $q$  but not both"

Truth Table:

$p$	$q$	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

- $p \oplus q$  is T when exactly one of  $p$  and  $q$  is T and the other is F; otherwise,  $p \oplus q$  is F.

**Conditional Statement / Implication.** Let  $p$  and  $q$  be propositions.

- The conditional statement "if  $p$ , then  $q$ ", is denoted  $p \rightarrow q$ , read " $p$  implies  $q$ ", is the proposition "If  $p$ , then  $q$ ."

$p \rightarrow q$   
 $p$  is the premise  $\leftarrow$   $q$  is the conclusion

- $p \rightarrow q$  is F when  $p$  is T and  $q$  is F; otherwise,  $p \rightarrow q$  is T.

- $p \rightarrow q$  is like a promise or bargain. If  $p$  is T, then  $q$  is promised to be T. The only way a promise is broken is when  $p$  is T but  $q$  is F.

Note:  $[F \rightarrow (\text{anything})]$  is T.  
 $[(\text{anything}) \rightarrow T]$  is T.

Truth Table:

$p$	$q$	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

**Biconditional Statement.** Let  $p$  and  $q$  be propositions.

- The biconditional statement " $p$  if and only if  $q$ " is denoted  $p \leftrightarrow q$ , read " $p$  if and only if  $q$ ".

- $p \leftrightarrow q$  is T when  $p$  and  $q$  have the same truth value (both T or both F); otherwise,  $p \leftrightarrow q$  is F.

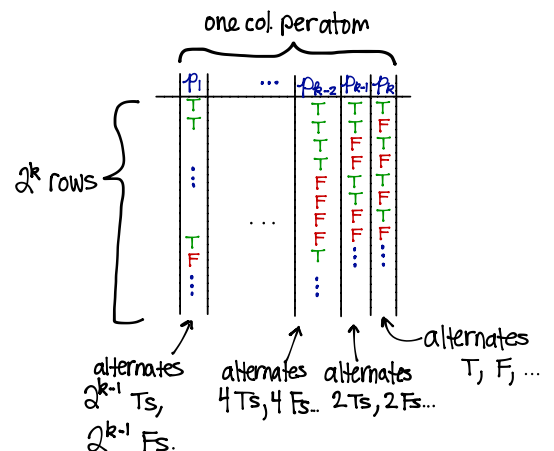
Truth Table:

$p$	$q$	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

## TRUTH TABLES

When making a truth table for a compound proposition consisting of atoms  $p_1, p_2, \dots, p_k$ , we will always use the following convention:

- $2^k$  rows for  $k$  atoms  $p_1, \dots, p_k$
- truth values always Ts and Fs (not Os/1s)
- Each atom has its own column.
- For rightmost atom's column, we alternate T, F, T, F, ... from top to bottom of column
- Moving one column to the left, the next atom's column alternates twice as many Ts and twice as many Fs as the column on its immediate right



**Example 1.2.** Write a truth table for each of the following compound propositions:

- a.  $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$
- b.  $(p \oplus q) \rightarrow \neg q$
- c.  $(p \rightarrow q) \wedge (q \rightarrow p)$
- d.  $\neg(\neg p \vee q)$
- e.  $(p \wedge \neg p) \leftrightarrow (p \vee \neg p)$
- f.  $q \leftrightarrow (p \leftrightarrow r)$

a)	$p$	$q$	$p \rightarrow q$	$\neg p \vee q$	$(p \rightarrow q) \leftrightarrow (\neg p \vee q)$
	T	T	T	T	T
	T	F	F	F	T
	F	T	T	T	T
	F	F	T	T	T
b)	$p$	$q$	$p \oplus q$	$\neg q$	$(p \oplus q) \rightarrow \neg q$
	T	T	F	F	T
	T	F	T	T	T
	F	T	T	F	F
	F	F	F	T	T
c)	$p$	$q$	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
	T	T	T	T	T
	T	F	F	T	F
	F	T	T	F	F
	F	F	T	T	T
d)	$p$	$q$	$\neg p$	$\neg p \vee q$	$\neg(\neg p \vee q)$
	T	T	F	T	F
	T	F	F	F	T
	F	T	T	T	F
	F	F	T	T	F
e)	$p$	$\neg p$	$p \wedge \neg p$	$p \vee \neg p$	$(p \wedge \neg p) \leftrightarrow (p \vee \neg p)$
	T	F	F	T	F
	F	T	F	T	F

