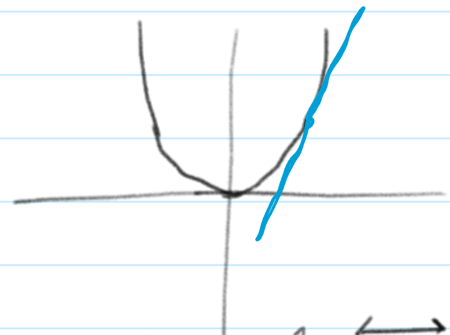
Lecture 04 (2.1-2.2)

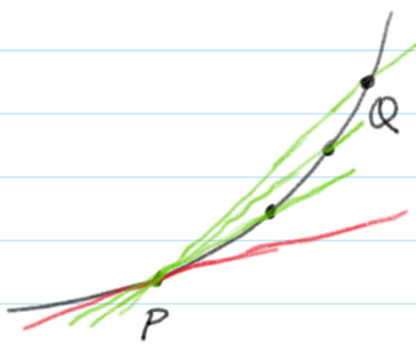
2018年9月8日　　下午 03:40

Take a function, say $f(x) = x^2$. We like to compute the slope of a <u>tangent line</u>.

<u>tangent line</u>: a line which touches the graph of f at exactly one point



Idea:



The slope of $\overleftrightarrow{PQ}$ gets closer to the slope of the tangent as Q gets closer to P.

"gets closer and closer" $\longrightarrow$ concept of <u>limits</u>

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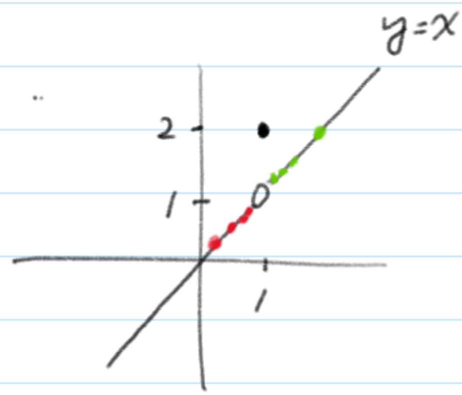
<u>2.1</u> Limits and Continuity

$$f(x) = \begin{cases} x, & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$$



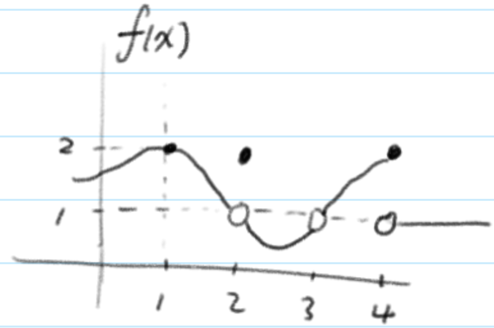
Then $f(1) = 2$.
However, when x approaches 1, $f(x)$ approaches 1

If $f(x)$ approaches L as x gets close to (but not equal to) c <u>from both sides</u>, we write

$$\lim_{x \to c} f(x) = L$$

- $\lim_{x \to c} f(x)$ is the number that describes the behavior of f near, <u>but not at</u>, the point $x = c$.

<u>Example</u> Find



a) $\lim_{x \to 1} f(x)$ b) $\lim_{x \to 2} f(x)$

c) $\lim_{x \to 3} f(x)$ d) $\lim_{x \to 4} f(x)$

<u>Sol.</u> a) 2 b) 1 c) 1

d) x approches 4 from the left

$\longrightarrow$ $f(x)$ goes to 2

right

$\longrightarrow$ $f(x)$ goes to 1

They don't agree, so the limit <u>does not exist</u>

<u>Example</u> Find the following limits.

a) $\lim_{x \to 0} \frac{x^2+x}{x}$ b) $\lim_{x \to 0} \frac{|x|}{x}$

<u>Sol.</u>

a) $\lim \frac{x^2+x}{} = \lim \frac{x(x+1)}{}$

Sol.

a) $\lim_{x\to 0} \frac{x^2+x}{x} = \lim_{x\to 0} \frac{x(x+1)}{x}$

$= \lim_{x\to 0} (x+1)$ (We can cancel x because we never touch the point $x=0$)

$= 1$

b) When x approaches 0 from the left,

$\frac{|x|}{x} = \frac{-x}{x} = -1.$

" " " " right,

$\frac{|x|}{x} = \frac{x}{x} = 1$

So the limit does not exist.

One-sided limits

If $f(x) \to L$ when $x \to c$ from the left

we write

$$\lim_{x\to c^-} f(x) = L$$

If " " " " from the right

we write

$$\lim_{x\to c^+} f(x) = L$$

Example

$$\lim_{x\to 0^-} \frac{|x|}{x} = -1, \quad \lim_{x\to 0^+} \frac{|x|}{x} = 1.$$

✡ $\lim_{x\to c} f(x)$ exists precisely when we have

✡ $\lim_{x \to c} f(x)$ exists precisely when we have

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$$

* * *

A function f is <u>continuous</u> at the point $x = c$ if

$$\lim_{x \to c} f(x) = f(c)$$

(intuitively, can draw the graph without picking up the pencil)

The following functions are continuous at every point in their <u>domain</u>.

- rational functions (including all polynomials)
- radical (square root, cubic root, n^{th}-root)
- exponential functions
- logarithmic functions
- any sum, product, composition of continuous functions.

<u>Example</u> a) $\lim_{x \to 2} (x^3 - 4x) = 2^3 - 4(2) = 0$

b) $\lim_{x \to 2} \log_3 \left(\frac{5x-4}{2}\right) = \log_3 \left(\frac{10-4}{2}\right)$

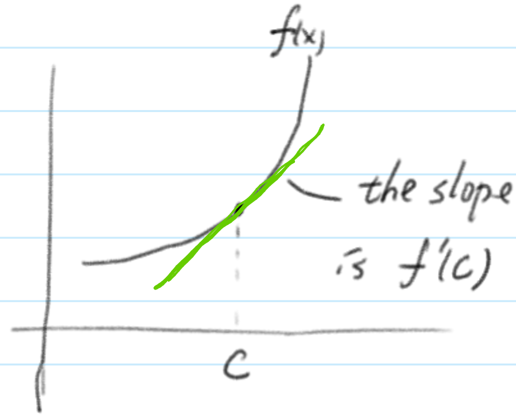
$$= \log_3(3) = 1.$$

2.2. Derivative

Let f be a function. We write $f'(c)$ for

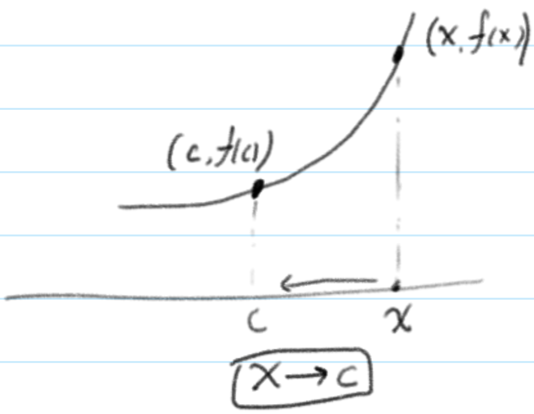
Let f be a function. We write $f'(c)$ for the slope of the tangent line at c

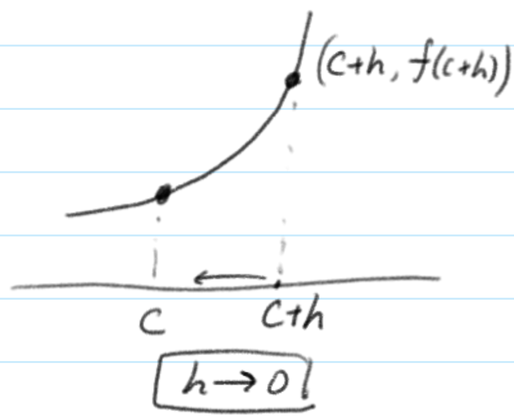
☆ f' is a function, and
$f'(c)$ is a number.



The slope is computed by

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{(c+h) - c}$$





When viewing $f'(x)$ as a function, we have

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

<u>Example</u> $f(x) = x^2$. Find the slope of the tangent line at $(x, f(x))$

<u>Sol.</u>

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{(x^2+2hx+h^2)-x^2}{h}$$

$$= \lim_{h\to 0} \frac{2hx+h^2}{h}$$

$$= \lim_{h\to 0} (2x+h) = 2x. \quad ※$$

Other notations for f'

$$\dot{f} \text{ (Newton)} \qquad \frac{df}{dx} \text{ (Leibniz)}$$

Note $\frac{df}{dx}$ is merely a notation for the derivative.
It is not a fraction.

Example (piecewise defined function)

$$f(x) = \begin{cases} x^2, & x > 2 \\ a, & x = 2 \\ x+b, & x < 2 \end{cases}$$

Find the values of a and b such that f is continuous at $x=2$.

Sol.

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2 = 4$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x+b) = 2+b$$

$$f(2) = a$$

We need $\lim_{x \to 2} f(x)$ exists and equals to $f(2)$:

$$4 = 2 + b \quad \text{(so the limit exists)}$$

and

$$4 = a \quad \text{(limit equals to } f(2)\text{)}$$

$$\boxed{a = 4, \quad b = 2}$$