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Notes

How to solve Linear Equations?

1. Determine if the equation is in the following form:
$$y' + P(x)y = Q(x)$$
2. Find the integration factor:
$$\rho = e^{\int P(x)\,dx}$$
3. Multiply both sides by the integration factor in Step 1
4. Use the following technique:
$$y'\rho + P(x)\rho = Q(x)\rho$$
$$(y\rho)' = Q(x)\rho$$
$$\int y\rho\,dy = \int Q(x)\,dx$$
5. Isolate "y"
6. Done

How to solve an exact equation?

1. Determine M(x,y) and N(x,y)
$$M(x,y)dx + N(x,y)dy = 0$$
2. Take the partial derivative of M(x,y) in terms of "y" and the partial derivative of N(x,y) in terms of "x".

   Tip: Notice how I'm taking the M(x,y) in terms of "y" when "dx" is in front of M(x,y) and

   N(x,y) in terms of "x" when "dy" is in front of N(x,y).

   Opposites.
3. Determine if both partial derivatives are equal. If they are equal, they are "exact equations" If they are not, then they are called "non-exact equations." Covered after STEP 8.
$$\frac{\partial \mathrm{M}}{\partial \mathrm{y}} = \frac{\partial \mathrm{N}}{\partial \mathrm{x}}$$
4. Write down the following equation:
$$M(x,y) = \frac{\partial \mathrm{f(x,y)}}{\partial \mathrm{x}}$$
5. Integrate both sides.
$$\int M(x,y)\,dx = \int \frac{\partial \mathrm{f(x,y)}}{\partial \mathrm{x}}\,dx$$

   Which will give you the following equation
$$\int M(x,y)\,dx = f(x,y) + h(y)$$

$$f(x,y) = \int M(x,y)\,dx - h(y)$$

6. Although we may find the integral of M(x,y), we still need to find h(y). To do so, we do the following:

$$N(x,y) = \frac{\partial \mathrm{f(x,y)}}{\partial \mathrm{y}}$$

But since we already have almost found our f(x,y) in step 5, we only need to plug in that function

$$N(x,y) = \frac{\partial(\mathrm{M}_{\mathrm{integrated}} - h(y)}{\partial \mathrm{y}}$$

Which will lead to

$$N(x,y) = \frac{\partial\left(\mathrm{M}_{\mathrm{integrated}}\right)}{\partial \mathrm{y}} - h'(y)$$

$$h'(y) = M_{integrated\ and\ partially\ derived\ by\ y} - N(x,y)$$

The only thing left to do is to find "h" by integrating both sides in terms of y.

$$\int h'(y)\,dy = \int M_{integrated\ and\ partially\ derived\ by\ y} - N(x,y)\,dy$$

$$h(y) = h_{found} = \int M_{integrated\ and\ partially\ derived\ by\ y} - N(x,y)\,dy$$

Which you can finally plug back to f(x,y) all the way to step 5.

7. Make f(x,y) = 0 and isolate y.
8. Done.
9. IF the equation was "non-exact", you have to make it exact using the following method:

$$result = \frac{M_{partially\ derived\ by\ y} - N_{partially\ derived\ by\ x}}{N(x,y)}$$

OR

$$result = \frac{N_{partially\ derived\ by\ x} - M_{partially\ derived\ by\ y}}{M(x,y)}$$

Only one of these will work. The one you're looking for however, is the one that only contains EITHER X OR Y.

10. Multiply the equation in STEP 1 by the following integrating factor:

$$e^{\int result\ dx} \quad or \quad e^{\int result\ dy} \qquad (depending\ on\ your\ \mathrm{result})$$

How to solve a homogenous differential equation?

1. You must first figure out whether or not the differential equation is homogenous
   By replacing "x" with "tx" and "y" with "ty", such as in the following example:

$$(x^2 + y^2)dx + 2x^2dy = 0$$
$$((xt)^2 + (yt)^2)dx + 2(xt)^2dy = 0$$
$$(t^2x^2 + t^2y^2)dx + 2t^2x^2dy = 0$$
$$t^2(x^2 + y^2)dx + t^2(2x^2)dy) = 0$$

   Since M and N are both homogenous equation of the same degree, the differential equation is homogenous.

2. Since this is a homogenous differential equation, we can do the following substitution:

$$y = ux$$
$$dy = udx + xdu$$

3. Add the substitution back in Step 1.
4. Separate the "du" component in one side and "dx" components on the other.
5. Integrate both sides.
6. Return the substitution using:

$$u = \frac{y}{x}$$

7. Isolate "y"
8. Done.

How to solve a Bernoulli equation?

1. Determine whether the equation is in the following form:

$$y' + P(x)y = f(x)y^n$$

2. Use the following substitution:

$$u = y^{1-n} \quad \rightarrow \quad isolate\ y\ and\ take\ its\ derivative\ y'$$

3. Substitute back to Step 1.

Checking for linear independence for two solutions

1. If you have two solution, f1(x) and f2(x), they may either linearly independent or linearly dependent. If they are linearly dependent, it would look like two vectors with same direction. While linear independency would mean they are like two vectors with different directions. In order to check for linear independency, the following method is used:

$$W(x) = \begin{vmatrix} f1(x) & f2(x) \\ f1(x)' & f2(x)' \end{vmatrix}$$

$$If\ W(x) = 0 \rightarrow \ Linear\ dependance$$
$$if\ w(x) \neq 0 \rightarrow Linear\ Independance$$

How to check for solution uniqueness?

1. Make sure that $a_n(x)\ and\ g(x)$ are continuous on an interval.
$$a_5(x)y''''' + a_4(x)y'''' + a_3(x)y''' + a_2(x)y'' + a_1(x)y + a_0(x)a_0 = g(x)$$
2. Example:
$$x^2y'' + 5xy' + y = 5x^5 \rightarrow \quad y'' + \frac{5y'}{x} + \frac{y}{x^2} = 5x^3\ (standard\ form)$$
Is not continuous at x=0 at interval (-inf,inf) because $a_2(x) = 0$ and $a_1(x) = 0$.
But is continuous at every other x on interval I.

How to solve a homogenous linear equation with constant coefficients?

1. To solve an equation of the following form:
$$ay'' + by' + cy = 0$$
$$Choose\ y = e^{mx}$$
2. You get
$$e^{mx}(am^2 + bm + c) = 0$$
3. Since e^{mx} can never be 0, we must find "m" such that it the function will equal to zero. Use the Quadratic Formula
$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad find \qquad m_1 \qquad and \qquad m_2$$
4. Determine the case:

CASE 1

$$m_1 \neq m_2$$
$$y = c_1e^{m_1} + c_2e^{m_2}$$

CASE 2

$$m_1 = m_2$$

Finding the second solution

$$y_{p_2} = y_1(x)\int \frac{e^{-\int P(x)dx}}{y_1^2(x)}dx = e^{m_1x}\int \frac{e^{-\int -2m_1dx}}{e^{(m_1x)^2}}dx = xe^{mx}$$
$$y = c_1e^{m_1} + c_2xe^{m_1}$$

CASE 3

$$m = a \pm \beta i$$
$$y = e^{ax}(c_1\cos(\beta x) + c_2\sin(\beta x))$$

How to solve an equation with undetermined coefficients?

1. Determine if the equation is of the following form:
$$ay'' + by' + cy = g(x)$$
2. Solve the homogenous linear equation
$$ay'' + by' + cy = 0$$
3. Look for the form of $g(x)$ and determine the form of its particular solution

$g(x)$	$Form\ of\ y_p$
$1\ (Constant)$	A
$5x + 7$	$Ax + B$
$3x^2 - 2$	$Ax^2 + Bx + C$
$x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
$\sin 4x$	$Acos\ 4x + Bsin\ 4x$
$\cos 4x$	$Acos\ 4x + Bsin\ 4x$
e^{5x}	Ae^{5x}
$(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
$e^{3x} \sin 4x$	$Ae^{3x} \cos 4x + Be^{3x} \sin 4x$
$5x^2 \sin 4x$	$(Ax^2 + Bx + C) \cos 4x + (Ex^2 + Fx + G) \sin 4x$
$xe^{3x} \cos 4x$	$(Ax + B)e^{3x} \cos 4x + (Cx + E)e^{3x} \sin 4x$

If you have more than one g(x), than write the form for each g(x). Example:

$$\mathrm{g(x) = x + sin(x)}$$

$$\mathrm{y_p} = Ax + B + Ccos\ x + Dsin\ x$$

4. Determine y_p, y_p', y_p''
But beware, if the solution of the homogeneous linear equation contains a function that is also included in the y_p form, then you must add x, x^2, x^3 or so on. Example:
$$If\ the\ solution\ of\ the\ homogenous\ linear\ equation\ is\ e^{3x}$$
$$And\ g(x) = e^{3x}$$
$$Then\ y_p\ cannot\ be\ e^{3x}, it\ will\ be\ y_p = xe^{3x}$$

$$If\ the\ solution\ of\ the\ homogenous\ linear\ equation\ is\ x^7e^{3x}$$
$$And\ g(x) = x^7e^{3x}$$
$$Then\ y_p\ cannot\ be\ x^7e^{3x}, it\ will\ be\ y_p = x^8e^{3x}$$

5. Plug y_p, y_p', y_p'' respectively inside the homogenous linear equation in Step 1, while keeping g(x) to the right.
6. Determine the coefficient values by factoring out such that you get the g(x) form.

$$You'll\ start\ with\ something\ like\ Asinx + Bsinx + Ce^{3x} = sinx + e^{3x}$$
$$But\ you\ have\ to\ factor\ out\ terms\ such\ as\ (A + B)sinx + Ce^{3x} = sinx + e^{3x}$$

7. Determine the coefficient values by equation the left side to the right side. Write the actual y_p by moving all found coefficient values to step 4.
8. You have found the solution, you now only need to put the general and particular solution together:
$$y = y_{general\ from\ step\ 2} + y_{p\ from\ step\ 7}$$
9. Done

How to solve with Variation of Parameters?

1. Determine the left side homogenous linear equation for the equation in STANDARD FORM
$$y'' + P(x)y' + Q(x)y = g(x)$$
Solve
$$y'' + P(x)y' + Q(x)y = 0$$
2. Determine if you can't find the form of the particular solution of the equation with undetermined coefficients.
3. Separate the components of the general solution into y_1 and y_2 (or more)
4. Find the result of the following determinants:
$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$
$$W_1 = \begin{vmatrix} 0 & y_2 \\ g(x) & y_2' \end{vmatrix}$$
$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$
5. Determine $u_1'\ and\ u_2'$
$$u_1' = \frac{W_1}{W}$$
$$u_2' = \frac{W_2}{W}$$
6. Determine $u_1\ and\ u_2$
$$u_1 = \int u_1{}' = \int \frac{W_1}{W}$$
$$u_2 = \int u_2{}' = \int \frac{W_2}{W}$$
7. Put the particular solution together
$$y_p = u_1 y_1 + u_2 y_2$$
8. Determine the solution
$$y = y_g + y_p$$
9. Done

How to solve Cauchy-Euler?

1. Determine if you have the equation in the following form
$$ax^2y'' + bxy + y = 0$$
2. Substitute
$$y = x^m \qquad y' = mx^{m-1} \qquad y'' = m(m-1)x^{m-2}$$
3. Find all m after the substitution

Case 1

$$m_1 \neq m_2$$
$$y = c_1x^{m_1} + c_2x^{m_2}$$

Case 2

$$m_1 = m_2$$
$$y = c_1x^{m_1} + c_2x^{m_1}\ln(\mathrm{x})$$

Case 3

$$m = complex \quad \rightarrow \quad a + i\beta$$
$$y = x^a(c_1\cos(\beta\ln x) + c_2\sin(\beta\ln x)$$

How to solve the combination?

1. Determine if you have the equation in the following form
$$ax^2y'' + bxy + y = g(x)$$
2. Use Cauchy-Euler first, then use Variation of Parameters

Reduction of order

Case 1 (y is missing in function)

$$u = y' \qquad \frac{du}{dx} = y'' \qquad substitute$$

Case 2 (x is missing in function)

$$u = y' \qquad u\frac{du}{dy} = y'' \qquad \text{substitute}$$

How to solve Spring Problems?

IMPORTANT!!!

TAKE NOTE OF THE SIGN CONVENTION

Experiment it on the initial values for position and velocity.

ALSO IMPORTANT!!!

Find the constants of the general solution AFTER having found the general solution and particular solution put together.

Case 1 – Free Undamped Motion

$$k: spring\ coefficient, \qquad m: mass$$

$$x'' + \left(\frac{k}{m}\right)x = 0 \qquad w^2 = \frac{k}{m}$$

$$x'' + w^2 x = 0$$

Case 2 – Free Damped Motion

$$B: damping\ coefficient, \qquad m: mass \qquad k: spring\ coefficient$$

$$x'' + \left(\frac{B}{m}\right)x' + \left(\frac{k}{m}\right)x = 0$$

$$x'' + 2\lambda\, x' + w^2 x = 0$$

CASE 1 : **Overdamped**

$$\lambda^2 - w^2 > 0$$

CASE 2: **Critically Damped**

$$\lambda^2 - w^2 = 0$$

CASE 3: **Underdamped**

$$\lambda^2 - w^2 < 0$$

Case 3 – Driven Motion with Damping

$$x'' + \left(\frac{B}{m}\right)x' + \left(\frac{k}{m}\right)x = g(x)$$

$$x'' + 2\lambda\, x' + w^2 x = g(x)$$

How to solve Circuit Problems?

1. It's the same as the spring problems, except

$$L\, q'' + R\, q' + \frac{1}{C}\, q = E(t)$$

How to find the amplitude and phase shift?

1. If you find both constants in

$$y = e^{at}(c_1 cosBt + c_2 sinBt)$$

2. Then

$$Amplitude = A = \sqrt{c_1^2 + c_2^2}$$

3. Phase shift

$$\emptyset = \arctan\left(\frac{c_1}{c_2}\right)$$

4. Which can then be used to convert the equation in step 1 to

$$y = A\, e^{at} \sin(60t + \emptyset)$$

Linear Model – Boundry Value Problem

Deflection of a Beam

$$EI \frac{d^4 y}{dx^5} = w(x) = w_0$$

Solve it

$$y(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + \frac{w_0}{24EI} x^4$$

BVP (Boundry Value Problem) most important part

CASE 1: Embedded Beam

$$y(0) = 0, \quad y'(0) = 0 \qquad y(L) = 0, \quad y'(L) = 0$$

CASE 2: Supported on both ends

$$y(0) = 0, \quad y''(0) \qquad y(L) = 0, y''(L) = 0$$

CASE 3: Free on one end but embedded on other end

$$y(0) = 0. y'(0) \qquad y''(L) = 0, y'''(L) = 0$$

RELATIONS:

$$y \ : \quad Attachment$$

$$y' \ : \quad Deflection$$

$$y'' : \quad Relative\ Bending$$

$$y''' : \quad Shear\ Force$$

Nontrivial solutions of a BVP

$$y'' + \lambda y = 0 \quad y(0) = 0, \; y(L) = 0$$

CASE 1:

$$\lambda = 0$$

$$y = 0$$

CASE 2:

$$\lambda < 0 \;, \qquad \lambda = -a^2$$

$$y = 0$$

CASE 3:

$$\lambda > 0$$

$$\lambda_n = a^2 = \left(\frac{n\pi}{L}\right)^2 \quad n = 1,2,3 \ldots$$

$$y = c_2 \sin\left(\frac{n\pi x}{L}\right)$$

Euler Load

$$EIy'' + Py = 0, \quad y(0) = 0, \quad y(L) = 0$$

$$\lambda_n = \frac{P_n}{EL} \;\rightarrow\; P_n = \lambda EI \;\rightarrow\; P_n = \left(\frac{n\pi}{L}\right)^2 EI \rightarrow \; P_n = \frac{n^2\pi^2 EI}{L^2}$$

With P_n as the critical load, the force required to deflect the beam a certain way

$$y = c_2 \sin(\frac{n\pi}{L}x)$$

How to solve Power Series?

1. Write

$$y = \sum_{n=0}^{inf} c_n x^n \qquad y' = \sum_{n=1}^{inf} c_n n x^{n-1} \qquad y'' = \sum_{n=2}^{inf} c_n n(n-1)x^{n-2}$$

2. Substitute in equation. Example:
   Although the example is from the book, the steps are much more comprehensive here)

$$y'' + xy = 0$$

$$\sum_{n=2}^{inf} c_n n(n-1)x^{n-2} + x\sum_{n=0}^{inf} c_n x^n = 0$$

$$\sum_{n=2}^{inf} c_n n(n-1)x^{n-2} + \sum_{n=0}^{inf} c_n x^{n+1} = 0$$

3. Find a way such that $x^{something}$ does not equal to 1, which means "something" should not equal to zero. We will remove a term from the first sum by making n=2.

$$c_2 + \sum_{n=3}^{inf} c_n n(n-1)x^{n-2} + \sum_{n=0}^{inf} c_n x^{n+1} = 0$$

4. Substitute k=n-2 for the first term and k=n+1 for the second sum. Notice where these terms are coming from. (Hint: from $x^{something}$ of both sums)

$$c_2 + \sum_{k=1}^{inf} c_{k-2}(k+2)(k+1)x^k + \sum_{k=1}^{inf} c_{k+1}x^k = 0$$

5. Factorize x^k

$$c_2 + \sum_{k=1}^{inf} [\, c_{k-2}(k+2)(k+1) + c_{k+1}\,]x^k = 0$$

6. Find what makes the equation zero:

$$c_2 = 0$$

$$c_{k-2} = -\frac{c_{k+1}}{(k+2)(k+1)}$$

7. Write the terms:

$$k = 1, \qquad c_3 = -\frac{c_0}{2 \cdot 3}$$

$$k = 2, \qquad c_4 = -\frac{c_1}{3 \cdot 4}$$

$$k = 3, \qquad c_5 = -\frac{c_2}{4 \cdot 5} = 0 \quad \leftarrow c_2 \text{ is zero}$$

$$k = 4, \qquad c_6 = -\frac{c_3}{5 \cdot 6} = \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} c_0$$

$$k = 5, \qquad c_7 = -\frac{c_4}{6 \cdot 7} = \frac{1}{3 \cdot 4 \cdot 6 \cdot 7} c_1$$

$$k = 6, \qquad c_8 = -\frac{c_5}{7 \cdot 8} = 0 \quad \leftarrow c_5 \text{ is zero}$$

$$k = 7, \qquad c_9 = -\frac{c_6}{8 \cdot 9} = -\frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} c_0$$

$$k = 8, \qquad c_{10} = -\frac{c_7}{9 \cdot 10} = -\frac{1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} c_1$$

$$k = 9, \qquad c_{11} = -\frac{c_8}{10 \cdot 11} = 0 \quad \leftarrow c_8 \text{ is zero}$$

8. Although the previous step is long and tedious, it is important to note a few things:
    a. Only c_0 and c_1 are kept
    b. The previously noted $c_2 = 0$ is kept and others are inheriting it.
9. Move all the found terms in a solution form:

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \ldots$$

Notice the connecting $c_n x^n$

Remember that in step 7 that you have made everything in terms of c_0 and c_1, this is important because it helps group everything neatly:

$$y = c_0 y_1(x) + c_1 y_2(x)$$

Where y_0 and y_1 are the solutions determined from the sum patterns.

$$y_1(x) = 1 + \frac{1}{2 \cdot 3} x^3 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} x^6 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} x^9 + \cdots = 1 + \sum_{k=1}^{\infty} \frac{1}{2 \cdot 3 \cdots (3k - 1)(3k)} x^{3k}$$

$$y_2(x) = x + \frac{1}{3 \cdot 4} x^4 + \frac{1}{3 \cdot 4 \cdot 6 \cdot 7} x^7 + \frac{1}{2 \cdot 3 \cdot 6 \cdot 7 \cdot 9 \cdot 10} x^{10} + \cdots = x + \sum_{k=1}^{\infty} \frac{1}{3 \cdot 4 \cdots (3k)(3k + 1)} x^{3k+1}.$$

10. Done!

How to Solve a System of Linear Equations?

Case 1 : Distinct Eigenvalues

1. Write down the system of linear equations.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

2. Find the following determinant

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda)-(b)(c)$$

3. If

$$\lambda_1 \neq \lambda_2$$

Determine the eigenvectors. How?

Eigenvector 1:

$$\begin{bmatrix} a-\lambda_1 & b \\ c & d-\lambda_1 \end{bmatrix} = \begin{matrix} 0 \\ 0 \end{matrix}$$

Since one way or another the first row will cancel the second row, we can assume

$$(a-\lambda_1)k_1 + b\,k_2 = 0$$

$$(a-\lambda_1)k_1 = -b\,k_2$$

$$By\ logic{:}\quad k_1 = -b \qquad k_2 = (a-\lambda_1)$$

$$So\ the\ eigenvector\ is$$

$$V_1 = \begin{pmatrix} -b \\ a-\lambda_1 \end{pmatrix}$$

4. Do the same for the other eigenvectors, but with λ_2
5. Solution:

$$y = c_1 V_1 e^{\lambda_1 t} + c_2 V_2 e^{\lambda_2 t}$$

Case 2: Repeated Eigenvalues

1. Write down the system of linear equations.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

2. Find the following determinant

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda)-(b)(c)$$

3. If

$$\lambda_1 = \lambda_2$$

Then determine the eigenvectors. How?

4. Do the same thing as the previous example. But as for the second eigenvector, do the following:

$$\begin{bmatrix} a-\lambda_1 & b \\ c & d-\lambda \end{bmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

Where p_1 and p_2 are the first determined eigenvectors.

Again, since the first row will eventually cancel the second row…

$$(a-\lambda_1)k_1 + b\,k_2 = p_1$$

Find k_1 and k_2 to determine the second eigenvector

$$V_2 = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

5. Determine the solution:

$$y = c_1 V_1 e^{\lambda_1 t} + c_2 (V_1 t + V_2) e^{\lambda_1 t}$$

Case 3: Complex Eigenvalues

1. Write down the system of linear equations.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

2. Find the following determinant

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - (b)(c)$$

If the eigenvalue will eventually give you a complex number of the form

$$\lambda = \boldsymbol{\alpha} + \boldsymbol{\beta i}$$

Determine the eigenvector

$$\big(a - (\boldsymbol{\alpha} + \boldsymbol{\beta i})\big)k_1 + b\,k_2 = 0$$
$$\big(a - (\boldsymbol{\alpha} + \boldsymbol{\beta i})\big)k_1 = -b\,k_2$$

Find k_1 and k_2

$$V_1 = \begin{pmatrix} p_1 \\ p_2 + \boldsymbol{\beta} i \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} + i \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$$

3. Find the solution

$$y = c_1 \left(\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} cos\boldsymbol{\beta} t - \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} sin\boldsymbol{\beta} t \right) e^{\boldsymbol{\alpha t}} + c_2 \left(\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} cos\boldsymbol{\beta} t + \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} sin\boldsymbol{\beta} t \right) e^{\boldsymbol{\alpha t}}$$

NOTICE THE SIGN PLACEMENTS AND THE EIGENVECTORS PLACEMENTS

The different forms of equations you might encounter:

$$y' + P(x)y = Q(x)$$

$$M(x,y)dx + N(x,y)dy = 0$$

$$y' + P(x)y = f(x)y^n$$

$$y'' + by' + cy = g(x)$$

$$ax^2y'' + bxy + y = 0$$

$$ax^2y'' + bxy + y = g(x)$$

$$y'' + P(x)y' + Q(x)y = g(x)$$

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

$$X' = AX$$