

ENGR-213 FINAL EXAM (SAMPLE)

SOLUTIONS

This is just a sample; it gives an idea of the structure and the level of the exam.

1. Solve these equations using an appropriate substitution.

$$(a) \quad \frac{dy}{dx} = 1/(e^{x+2y} + e^{-x-2y}) - 1/2 ;$$

$$\text{Solution: Substitution } x+2y=z; \quad dx+2dy=dz; \quad dy=\frac{dz-dx}{2};$$

$$dz-dx=(2/(e^z+e^{-z})-1)dx; \quad dz=\frac{2}{e^z+e^{-z}}dx; \quad \int(e^z+e^{-z})dz=\int 2dx+C;$$

$$e^z-e^{-z}=2x+C; \quad \text{Answer: } e^{x+2y}-e^{-x-2y}=2x+C.$$

$$(b) \quad 4x^2 \frac{dy}{dx} = 4y^2 + x^2 .$$

$$\text{Solution: } \frac{dy}{dx} = \frac{4y^2+x^2}{4x^2} = \frac{y^2}{x^2} + \frac{1}{4}; \quad \text{Homogeneous equation.}$$

$$\text{Substitution: } y=ux; \quad dy=xdu+udx; \quad xdu+udx=(u^2+\frac{1}{4})dx;$$

$$xdu=(u^2-u+\frac{1}{4})dx; \quad \frac{du}{(u-1/2)^2}=\frac{dx}{x}; \quad \frac{\int du}{(u-1/2)^2}=\frac{\int dx}{x};$$

$$-\frac{1}{u-1/2}=\ln x+C; \quad u=\frac{1}{2}-\frac{1}{\ln x+C};$$

$$\text{Answer: } \frac{y}{x}=\frac{1}{2}-\frac{1}{\ln x+C}.$$

2. Reduce to an exact equation and solve it:

$$(a) \quad \frac{dy}{dx} = \frac{2x \sin 2y + (2 \sin y)/x^2}{2x^2 \cos x - (2 \cos y)/x};$$

Sorry, wrong equation.

$$(b) \quad (2x^2 y \sin y - x y^3 \sin x) dx + (x^3 \sin y + x^3 y \cos y + 3x y^2 \cos x) dy = 0.$$

$$\text{Solution: } M = 2x^2 y \sin y - x y^3 \sin x; \quad N = x^3 \sin y + x^3 y \cos y + 3x y^2 \cos x;$$

$$\frac{\partial M}{\partial y} = 2x^2 \sin y + 2x^2 y \cos y - 3x y^2 \sin x; \quad \frac{\partial N}{\partial x} = 3x^2 \sin y + 3x^2 y \cos y + 3y^2 \cos x - 3x y^2 \sin x;$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -x^2 \sin y - x^2 y \cos y - 3y^2 \cos x;$$

$$\frac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)}{N} = -\frac{1}{x};$$

Hence, the integrating factor $\mu = e^{\int \frac{-dx}{x}} = \frac{1}{x}$; after multiplication by μ we have the exact equation

$$(2x y \sin y - y^3 \sin x) dx + (x^2 \sin y + x^2 y \cos y + 3y^2 \cos x) dy = 0;$$

$$f(x, y) = \int (2x y \sin y - y^3 \sin x) dx + g(y) = x^2 y \sin y + y^3 \cos x + g(y);$$

$$\frac{\partial f}{\partial y} = x^2 \sin y + x^2 y \cos y + 3y^2 \cos x + g'(y) = x^2 \sin y + x^2 y \cos y + 3y^2 \cos x;$$

$$g'(y) = 0; \quad g(y) = \text{const}; \quad f(x, y) = x^2 y \sin y + y^3 \cos x;$$

$$\text{Answer: } x^2 y \sin y + y^3 \cos y = C.$$

3. Solve the equation using the integrating factor:

$$(x+1)\frac{dy}{dx}+xy=e^{-x} .$$

$$\text{Solution: } \frac{dy}{dx}+\frac{x}{x+1}y=\frac{e^{-x}}{x+1};$$

$$\text{Integrating factor } m=e^{\int \frac{x}{x+1}dx}=e^{\int (1-1/(x+1))dx}=e^x \cdot e^{-\ln(x+1)}=\frac{e^x}{x+1};$$

$$\frac{d}{dx}\left(\frac{e^x}{x+1}y\right)=\frac{1}{(x+1)^2}; \quad \frac{e^x}{x+1}y=-\frac{1}{x+1}+C;$$

$$\text{Answer: } y=-e^{-x}+Ce^{-x}(x+1).$$

4. Solve the Bernoulli equation

$$\frac{dy}{dx}-\frac{1}{x}y+3y^2=0 .$$

$$\text{Solution: Substitution } y=\frac{1}{z}; \quad \frac{dy}{dx}=-\frac{1}{z^2}\frac{dz}{dx};$$

$$\frac{-1}{z^2}\frac{dz}{dx}-\frac{1}{xz}+\frac{3}{z^2}=0; \quad \frac{dz}{dx}+\frac{1}{x}z=3;$$

$$x\frac{dz}{dx}+z=3x; \quad \frac{d}{dx}(xz)=3x; \quad xz=\frac{3}{2}x^2+C;$$

$$z=\frac{3}{2}x+\frac{C}{x}; \quad \text{Answer: } y=\frac{1}{z}=\frac{1}{\frac{3}{2}x+\frac{C}{x}}.$$

5.

(a) Find the general solution of the equation

$$x^2 y'' - 3xy' + 2y = x^2 + \frac{1}{x};$$

Solution: This is a Cauchy-Euler equation. The characteristic equation:

$$m(m-1) - 3m + 2 = 0; \quad m^2 - 4m + 2 = 0; \quad m_1 = 2 + \sqrt{2}, \quad m_2 = 2 - \sqrt{2}.$$

$$y_c = c_1 x^{2+\sqrt{2}} + c_2 x^{2-\sqrt{2}};$$

$$y_p = ax^2 + \frac{b}{x} \quad (\text{no resonances});$$

$$y_p' = 2ax - \frac{b}{x^2}; \quad y_p'' = 2a + \frac{2b}{x^3}; \quad x^2 y'' - 3xy' + 2y = 2ax^2 + \frac{2b}{x} - 6ax^2 + \frac{3b}{x} + 2ax^2 + \frac{2b}{x}$$

$$= -2ax^2 + \frac{7b}{x}; \quad a = -\frac{1}{2}, \quad b = \frac{1}{7}.$$

$$\text{General solution: } y = -\frac{1}{2}x^2 + \frac{1}{7x} + c_1 x^{2+\sqrt{2}} + c_2 x^{2-\sqrt{2}}.$$

(b) Solve the initial value problem

$$y'' - 3y' + 2y = 3e^{2t}; \quad y(0) = 2; \quad y'(0) = 1.$$

Solution: Characteristic equation: $\lambda^2 - 3\lambda + 2 = 0$; $\lambda_1 = 1$, $\lambda_2 = 2$.

$y_c = c_1 e^t + c_2 e^{2t}$; the right hand side is resonant, so the particular solution $y_p = ate^{2t}$.
Substituting into the equation, we find $a = 3$, and the general solution is

$$y = 3te^{2t} + c_1 e^t + c_2 e^{2t}. \quad \text{Then the initial conditions give us } y(0) = c_1 + c_2 = 2, \\ y'(0) = 3 + c_1 + 2c_2 = 1; \quad \text{from here we have } c_1 = -3, \quad c_2 = 4.$$

$$\text{Answer: } y = 3te^{2t} - 3e^t + 4e^{2t}.$$

6. Find the general solution of the equation

$$y''+2y'+2y=e^{-t}\sin t+e^{-2t}\cos 2t$$

Solution. Characteristic equation: $\lambda^2+2\lambda+2=0$; $\lambda=-1\pm i$. So, the general solution of the homogeneous equation is $y_c=e^{-t}(c_1\cos t+c_2\sin t)$. The first term in the right hand side is resonant, and the second one is not. So, the particular solution of the nonhomogeneous equation has the form

$y_p=te^{-t}(d_1\cos t+d_2\sin t)+e^{-2t}(d_3\cos 2t+d_4\sin 2t)$. Substituting this into the equation and comparing with the right hand side, we find that

$$d_1=-\frac{1}{2}, d_2=\frac{1}{2}, d_3=-\frac{1}{5}, d_4=\frac{1}{10}, \text{ and the answer is}$$

$$y=te^{-t}\left(\frac{-1}{2}\cos t+\frac{1}{2}\sin t\right)+e^{-2t}\left(\frac{-1}{10}\cos 2t-\frac{1}{5}\sin 2t\right)+c_1e^{-t}\cos t+c_2e^{-t}\sin 2t .$$

7. A mass $m=10\text{kg}$ is suspended on a spring and stretches it by $h=2\text{cm}$. Find the period of oscillation (the friction and air resistance are negligible; the gravity constant $g=9.81\text{m/sec}^2$).

Solution. The spring constant $k=\frac{F}{h}=\frac{mg}{h}=4905 \text{ (N/m)}$;

The frequency $\omega=\sqrt{\frac{k}{m}}$; the period $T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{m}{k}}=2\pi\sqrt{\frac{10}{4905}}=0.2837 \text{ (sec)}$.

Answer: $T=0.2837 \text{ sec}$.

8. Solve the equation using the variation of parameters:

$$y''-2y'-8y=x$$

Solution. Characteristic equation: $\lambda^2 - 2\lambda - 8 = 0$; $\lambda = \frac{2 \pm \sqrt{4 + 32}}{2}$; $\lambda_1 = -2$, $\lambda_2 = 4$.

Solutions of the homogeneous equation: $y_1 = e^{-2x}$, $y_2 = e^{4x}$. The Wronskian

$$W = \begin{vmatrix} e^{-2x} & e^{4x} \\ -2e^{-2x} & 4e^{4x} \end{vmatrix} = 6e^{2x};$$

$$y = u_1 y_1 + u_2 y_2;$$

$$u_1' = \frac{-x e^{4x}}{e^{2x}} = -\frac{1}{6} x e^{2x}; \quad u_2' = \frac{x e^{-2x}}{6 e^{2x}} = \frac{1}{6} x e^{-4x};$$

$$u_1 = \int -\frac{1}{6} x e^{2x} dx = -\frac{1}{6} \left(\frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} \right) + c_1;$$

$$u_2 = \frac{1}{6} \int x e^{-4x} dx = \frac{1}{6} \left(-\frac{1}{4} x e^{-4x} - \frac{1}{16} e^{-4x} \right) + c_2;$$

$$y = e^{-2x} \left(-\frac{1}{12} x e^{2x} + \frac{1}{24} e^{2x} + c_1 \right) + e^{4x} \left(-\frac{1}{24} x e^{-4x} - \frac{1}{96} e^{-4x} + c_2 \right) = -\frac{1}{8} x + \frac{1}{32} + c_1 e^{-2x} + c_2 e^{4x}.$$

9.

(a) Solve the initial value problem for the system

$$\frac{dx_1}{dt} = x_1 - 2x_2$$

$$\frac{dx_2}{dt} = x_1 + 3x_2$$

Initial condition: $x_1(0) = 0$, $x_2(0) = -1$.

Solution. Solve this system by elimination. From the second equation,

$$x_1 = \frac{dx_2}{dt} - 3x_2. \quad \text{Substitute into the first equation:}$$

$$\frac{d^2 x_2}{dt^2} - 3 \frac{dx_2}{dt} - \frac{dx_2}{dt} + 3x_2 + 2x_2 = 0 ;$$

$$x_2'' - 4x_2' + 5x_2 = 0 ; \quad \lambda^2 - 4\lambda + 5 = 0; \quad \lambda = 2 \pm i .$$

$x_2 = e^{2t}(c_1 \cos t + c_2 \sin t)$. Substitute into the equation for x_1 :

$$x_1 = \frac{dx_2}{dt} - 3x_2 = e^{2t} [(-c_1 + c_2) \cos t + (-c_1 - c_2) \sin t] .$$

Substitute into the initial conditions:

$$x_1(0) = c_1 + c_2 = 0; \quad c_1 = -c_2 ;$$

$$x_2(0) = c_1 = 1; \quad c_1 = 1, \quad c_2 = -1 .$$

Answer: $x_1 = e^{2t}(-2 \sin t)$; $x_2 = e^{2t}(\cos t + \sin t)$.

(b) Find a general solution of the system

$$\frac{dx}{dt} = y + t$$

$$\frac{dy}{dt} = 4x - t$$

Solution. We solve this problem by diagonalization.

$$A = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} ; \text{ the system has the form } \frac{dX}{dt} = AX + F \text{ where } X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad F(t) = \begin{pmatrix} t \\ -t \end{pmatrix} .$$

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ 4 & -\lambda \end{pmatrix} = \lambda^2 - 4 = 0; \quad \lambda_1 = 2, \quad \lambda_2 = -2 .$$

$$A - \lambda_1 I = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}; \text{ the first eigenvector } K_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix};$$

$$A - \lambda_2 I = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}; \text{ the second eigenvector } K_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

$$U = (K_1 \ K_2) = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}; \quad U^{-1} = -\frac{1}{4} \begin{pmatrix} -2 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{pmatrix};$$

The diagonal matrix of eigenvalues $D = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$; so, $A = U D U^{-1}$.

Our system has the form $\frac{dX}{dt} - AX = F$; multiplying it by U^{-1} , we have

$$\frac{dC}{dt} - DC = U^{-1}F \quad \text{where } C = U^{-1}X = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \text{ Now,}$$

$$U^{-1}F = \begin{pmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{pmatrix} \begin{pmatrix} t \\ -t \end{pmatrix} = \begin{pmatrix} t/4 \\ 3t/4 \end{pmatrix}, \text{ and the system for } c_1, c_2 \text{ is}$$

$$\frac{dc_1}{dt} - 2c_1 = t/4;$$

$$\frac{dc_2}{dt} + 2c_2 = 3t/4.$$

These two equations are independent from one another. Solving them by the undefined coefficients, we find

$c_1 = -t/8 - 1/16 + d_1 e^{2t}$; $c_2 = 3t/8 - 3/16 + d_2 e^{-2t}$, and from the equality $X = UC$ we find

$$x = c_1(t) + c_2(t) = t/4 - 1/4 + d_1 e^{2t} + d_2 e^{-2t};$$

$$y = 2c_1 - 2c_2 = -t + 1/4 + 2d_1 e^{2t} - 2d_2 e^{-2t}.$$

10. Find the power series solution of the equation $(x^2 + 1)y'' + y = 0$; find the convergence radius of the series.

Solution. Let us look for the solution in the form $y = \sum_{n=0}^{\infty} c_n x^n$. Then

$y' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$. Substituting into the equation gives:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0.$$

Making index shift in the second term ($k=n-2$; $n=k+2$), and renaming n by k in the first and the third terms, we get the equation

$$\sum_{k=2}^{\infty} k(k-1)c_k x^k + \sum_{k=0}^{\infty} (k+1)(k+2)c_{k+2} x^k + \sum_{k=0}^{\infty} c_k x^k = 0,$$

or

$$c_0 + c_1 x + 2c_2 + 2 \cdot 3c_3 x + \sum_{k=2}^{\infty} [(k(k-1)+1)c_k + (k+1)(k+2)c_{k+2}] x^k = 0.$$

The coefficients c_0 and c_1 are arbitrary (they are the initial conditions: $c_0 = y(0)$, $c_1 = y'(0)$). The further coefficients are found by equating the coefficients at x^k to zero for all k :

$$k=0: \quad 2c_2 + c_0 = 0.$$

$$k=1: \quad 2 \cdot 3c_3 + (1 \cdot 2 + 1)c_2 = 0$$

$$k=2: \quad 3 \cdot 4c_4 + (1 \cdot 2 + 1)c_2 = 0$$

$$k=3: \quad 4 \cdot 5c_5 + (2 \cdot 3 + 1)c_3 = 0$$

.....

$$k: \quad (k+1)(k+2)c_{k+2} + (k \cdot (k-1))c_k = 0$$

From here we have the following chain of equalities:

$$c_2 = -\frac{c_0}{2}; \quad c_4 = -\frac{1 \cdot 2 + 1}{3 \cdot 4} c_2; \quad c_6 = -\frac{3 \cdot 4 + 1}{5 \cdot 6} c_4; \dots; \quad c_{2k+2} = -\frac{(2k-1) \cdot 2k + 1}{(2k+1)(2k+2)} c_{2k}; \dots$$

$$c_3 = -\frac{1}{2 \cdot 3} c_1; \quad c_5 = -\frac{2 \cdot 3 + 1}{4 \cdot 5} c_3; \quad c_7 = -\frac{4 \cdot 5 + 1}{6 \cdot 7} c_5; \quad \dots; \quad c_{2n+1} = -\frac{(2n-2)(2n-1)+1}{2n(2n+1)} c_{2n-1}; \quad \dots$$

So, the first terms of the series are

$$y = c_0 \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{13}{240}x^6 + \dots \right) + c_1 \left(x - \frac{1}{6}x^3 + \frac{7}{120}x^5 - \frac{147}{5040}x^7 + \dots \right).$$

To find the convergence radius, we use the Ratio Test.

$$\frac{c_{k+2}}{c_k} = -\frac{k(k-1)+1}{(k+1)(k+2)} = -\frac{1 \cdot (1-1/k) + 1/k^2}{(1+1/k)(1+2/k)} \rightarrow 1 \quad \text{as } k \rightarrow \infty.$$

Hence, the convergence radius $R = 1$.

