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The University of British Columbia

MATH 215/255, Sections 101–105

Final Exam – December 2014

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No notes nor calculators.

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Problem	Points	Score
1	8	
2	12	
3	10	
4	13	
5	8	
6	12	
7	10	
8	12	
9	15	
Total:	100	

1. Solve the initial value problem

$$(3xy + y^2) + (x^2 + xy)y' = 0, \quad y(1) = 2.$$

(3 points) (a) Verify that $\mu(x) = x$ is an integrating factor, that is, $x(3xy + y^2)dx + x(x^2 + xy)dy = 0$ is exact.

Answer. Let $M = x(3xy + y^2) = 3x^2y + xy^2$ and $N = x(x^2 + xy) = x^3 + x^2y$, then

$$M_y = 3x^2 + 2xy, \quad \text{and} \quad N_x = 3x^2 + 2xy.$$

Then $M_y = N_x$. So $x(3xy + y^2)dx + x(x^2 + xy)dy = 0$ is exact.

(5 points) (b) Solve the initial value problem.

Answer. Since $Mdx + Ndy = 0$ is exact. Then there exists some $\phi(x, y)$ such that

$$\phi_x = M = 3x^2y + xy^2, \quad \text{and} \quad \phi_y = N = x^3 + x^2y.$$

Since $\phi_x = 3x^2y + xy^2$, then

$$\phi(x, y) = \int (3x^2y + xy^2)\ dx + g(y) = x^3y + \frac{1}{2}x^2y^2 + g(y).$$

Since $\phi_y = x^3 + x^2y$, then

$$x^3 + x^2y = \phi_y = x^3 + x^2y + g'(y),$$

which implies that $g'(y) = 0$, so $g(y) = \text{constant}$. Hence

$$x^3y + \frac{1}{2}x^2y^2 = C.$$

Since $y(1) = 2$, then $2 + 2 = C$, that is, $C = 4$. Therefore, the solution is:

$$x^3y + \frac{1}{2}x^2y^2 = 4.$$

2. A second order chemical reaction can be modeled by the equation

$$\frac{dy}{dx} = \alpha(y-p)(y-q),$$

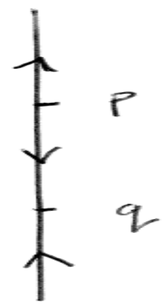
where α, p and q are constants

(4 points) (a) Assume that $\alpha > 0$ and $p > q > 0$. Find equilibrium points and classify stabilities of these equilibrium points.

Answer. Let's solve $\alpha(y-p)(y-q) = 0$, since $\alpha > 0$ and $p > q > 0$, then $y = p$ or $y = q$. So we have two equilibrium points:

$$y = p, \quad \text{and} \quad y = q.$$

The phase diagram is:



So $y = p$ is unstable, $y = q$ is stable.

(4 points) (b) Assume that $\alpha = 1$, $p = 0$, $q = 1$ and $y(0) = -1$, solve the initial value problem and determine the limiting value of $y(x)$ as $x \to \infty$.

Answer. Since $\alpha = 1$, $p = 0$ and $q = 1$, then our differential equation becomes

$$\frac{dy}{dx} = y(y-1).$$

So $\frac{dy}{y(y-1)} = dx$, that is, $\left(\frac{1}{y-1} - \frac{1}{y}\right) dy = dx$. Then

$$\ln\left|\frac{y-1}{y}\right| = x + C.$$

Since $y(0) = -1 < 0$, then $y(x) < 0$ for all $x \geq 0$, then

$$\ln\left(\frac{y-1}{y}\right) = x + C.$$

Then

$$y(x) = \frac{1}{1 - e^{x+C}}.$$

Since $y(0) = -1$, then $C = \ln 2$. So solution is $y(x) = \frac{1}{1-2e^x}$, which implies that $\lim_{x\to\infty} y(x) = 0$.

(4 points) (c) Assume that $\alpha = 1$, $p = q = 0$ and $y(0) = 1$. Use Euler's method to approximate $y(2)$ with the step size $h = 1$.

Answer. Since $\alpha = 1$, $p = q = 0$ and $y(0) = 1$, then our differential equation becomes

$$\frac{dy}{dx} = y^2, \quad \text{and} \quad y(0) = 1.$$

Let $f(y) = y^2$, $x_0 = 1$ and $y_0 = y(x_0) = y(0) = 1$, then

$$\begin{aligned} x_1 &= x_0 + h \\ &= 1 \\ y_1 &= y_0 + f(y_0)h \\ &= 1 + 1^2 \\ &= 2 \\ x_2 &= x_0 + 2h \\ &= 2 \\ y_2 &= y_1 + f(y_1)h \\ &= 2 + 2^2 \cdot 1 \\ &= 6. \end{aligned}$$

(5 points) 3. (a) Solve $y'' + 4y' + 13y = 0, \quad y(0) = 2, \quad y'(0) = 5.$

Answer. The characteristic equation is $r^2 + 4r + 13 = (r+2)^2 + 9 = 0$ with roots

$$r = -2 \pm 3i. \qquad \text{(1pt)}$$

The general solution is

$$y(t) = c_1 e^{-2t} \cos 3t + c_2 e^{-2t} \sin 3t. \qquad \text{(1pt)}$$

Hence

$$y'(t) = (-2c_1 + 3c_2)e^{-2t} \cos 3t + (-3c_1 - 2c_2)e^{-2t} \sin 3t.$$

The initial conditions give

$$c_1 + 0 = 2, \quad (-2c_1 + 3c_2) + 0 = 5. \qquad \text{(2pt)}$$

Thus $c_1 = 2$, $c_2 = 3$, and

$$y(t) = 2e^{-2t} \cos 3t + 3e^{-2t} \sin 3t. \qquad \text{(1pt)}$$

(5 points) (b) Find a particular solution of

$$y'' + 3y' = e^{-3t}$$

using **either** the method of undetermined coefficients, **or** the method of variation of parameters.

Answer. #1. (method of undetermined coefficients.) The characteristic equation is $r^2 + 3r = 0$, with roots $-3, 0$. Hence

$$y_c(t) = c_1 + c_2 e^{-3t}. \qquad \text{(2pt)}$$

For $f = e^{-3t}$, the initial guess is $Y(t) = ae^{-3t}$. However ae^{-3t} is part of y_c and hence we set

$$Y(t) = ate^{-3t}. \qquad \text{(1pt)}$$

We have

$$Y' = (a - 3at)e^{-3t}, \quad Y'' = (-3a - 3a + 9at)e^{-3t},$$

$$Y'' + 3Y' = -3ae^{-3t}. \qquad \text{(1pt)}$$

Thus $a = -\frac{1}{3}$ and

$$Y(t) = -\frac{1}{3}te^{-3t}. \qquad \text{(1pt)}$$

Answer #2. (method of variation of parameters.) The characteristic equation is $r^2 + 3r = 0$, with roots $-3, 0$. Hence two independent solutions are

$$y_1(t) = 1, \quad y_2(t) = e^{-3t}. \qquad \text{(2pt)}$$

A particular solution is of the form $Y(t) = y_1(t)u_1(t) + y_2(t)u_2(t)$ with

$$u_1' + e^{-3t}\, u_2' = 0, \quad 0\, u_1' - 3e^{-3t}\, u_2' = e^{-3t}.$$

Thus

$$u_2' = -\frac{1}{3}, \quad u_1' = \frac{1}{3}e^{-3t}. \qquad \text{(2pt)}$$

We can solve $u_1 = -\frac{1}{9}e^{-3t}$ and $u_2 = -\frac{1}{3}t$, and hence

$$Y = 1(-\frac{1}{9}e^{-3t}) + e^{-3t}(-\frac{1}{3}t) = -\frac{1}{9}e^{-3t} - \frac{1}{3}te^{-3t}. \qquad \text{(1pt)}$$

4. Consider a vibrating system described by the initial value problem

$$u'' + cu' + 4u = \cos 2t, \qquad u(0) = 0, \quad u'(0) = 2.$$

where $c > 0$ is the damping coefficient.

(5 points) (a) Find the steady periodic part of the solution (the part of the solution which remains as $t \to \infty$) of this problem, and find its amplitude. Do not find the transient part.

Answer. The steady state is the particular solution of the form

$$U(t) = a\cos 2t + b\sin 2t.$$

Thus $U' = 2b\cos 2t - 2a\sin 2t$ and $U'' = -4a\cos 2t - 4b\sin 2t$. Substitution into the equation yields

$$c(2b\cos 2t - 2a\sin 2t) = \cos 2t.$$

Thus $a = 0$, $b = \frac{1}{2c}$, and

$$U(t) = \frac{1}{2c}\sin 2t.$$

Its amplitude is

$$\frac{1}{2c}.$$

(2 points) (b) Let $A(c)$ denote the maximum amplitude of the steady state solutions of the systems

$$u'' + cu' + 4u = \cos\omega t, \qquad u(0) = 0, \quad u'(0) = 2$$

among all possible $\omega > 0$. What happens to $A(c)$ as $c \to 0_+$? Explain why.
Hint. You do not need to solve $A(c)$ explicitly.

Answer. $A(c)$ goes to infinity, because $A(c) \geq \frac{1}{2c}$ and $\lim_{c \to 0_+} \frac{1}{2c} = \infty$.

(6 points) (c) Find a particular solution of

$$y'' + y = \frac{1}{\cos t}, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

Hint: $\int \tan t\, dt = -\ln|\cos t|$, $\int \cot t\, dt = \ln|\sin t|$

Answer. The homogeneous equation has two independent solutions

$$y_1(t) = \sin t, \quad y_2(t) = \cos t.$$

Using the method of variation of parameters, a particular solution is given by $Y(t) = y_1(t)u_1(t) + y_2(t)u_2(t)$, in which

$$\begin{aligned} y_1 u_1' + y_2 u_2' &= 0, \\ y_1' u_1' + y_2' u_2' &= f. \end{aligned}$$

That is,

$$\begin{aligned} \sin t\, u_1' + \cos t\, u_2' &= 0, \\ \cos t\, u_1' - \sin t\, u_2' &= \frac{1}{\cos t}. \end{aligned}$$

Thus

$$u_1' = 1, \quad u_2' = -\tan t$$

$$u_1(t) = \int 1 dt = t,$$

$$u_2(t) = -\int \tan t\, dt = \ln|\cos t| = \ln(\cos t) \quad (-\frac{\pi}{2} < t < \frac{\pi}{2})$$

Hence a particular solution is

$$Y(t) = t\sin t + \cos t\ \ln(\cos t)$$

(8 points) 5. Use the Laplace transform to solve the system $x''(t) + 2x'(t) + 2x(t) = 2$ with initial conditions $x(0) = 0$, $x'(0) = 0$.

Answer. Writing $X(s)$ for the Laplace transform of $x(t)$, we obtain

$$(s^2 + 2s + 2)X(s) = \frac{2}{s}$$

and therefore

$$X(s) = \frac{2}{s(s^2 + 2s + 2)}.$$

By partial fraction decomposition, we have

$$X(s) = \frac{-s - 2}{s^2 + 2s + 2} + \frac{1}{s}.$$

By completion of the square, this can be put in a convenient form to later apply the first shifting formula:

$$X(s) = \frac{-(s + 1)}{(s + 1)^2 + 1} - \frac{1}{(s + 1)^2 + 1} + \frac{1}{s}.$$

Taking inverse Laplace transforms and applying the said shifting formula, we obtain

$$\begin{aligned} x(t) &= -e^{-t}\mathcal{L}^{-1}\left\{\frac{s}{s^2 + 1}\right\}(t) - e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\}(t) + 1 \\ &= -e^{-t}(\cos t + \sin t) + 1. \end{aligned}$$

(6 points) 6. (a) Solve the system $x''(t) + 4x(t) = \delta(t-2)$ with initial conditions $x(0) = 0$ and $x'(0) = 0$.

Answer. Writing $X(s) = \mathcal{L}\{x(t)\}$ and taking Laplace transforms, we obtain

$$\begin{aligned} (s^2+4)X(s) &= e^{-2s} \\ \Rightarrow \qquad X(s) &= \frac{e^{-2s}}{s^2+4}. \end{aligned}$$

By the second shifting formula, we have therefore

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\Big\{\frac{1}{s^2+4}\Big\}(t-2)\cdot u(t-2) \\ &= \tfrac{1}{2}\sin(2(t-2))\cdot u(t-2). \end{aligned}$$

(6 points) (b) Find the Laplace transform of

$$f(t) = \begin{cases} 1 & \text{if } t < 1 \\ t & \text{if } 1 \le t < 2 \\ 0 & \text{if } t \ge 2. \end{cases}$$

Answer. We can rewrite the piecewise definition of f using Heavyside functions:

$$\begin{aligned} f(t) &= 1\cdot(1-u(t-1)) + t\cdot\big(u(t-1)-u(t-2)\big) + 0\cdot u(t-2) \\ &= 1 + (t-1)u(t-1) - tu(t-2) \\ &= 1 + (t-1)u(t-1) - (t-2)u(t-2) - 2u(t-2). \end{aligned}$$

This last expression makes it easier to apply the second shifting formula:

$$\begin{aligned} F(s) &= \frac{1}{s} + e^{-s}\mathcal{L}\{t\}(s) - e^{-2s}\mathcal{L}\{t\}(s) - \frac{2e^{-2s}}{s} \\ &= \frac{1}{s} + \frac{e^{-s}}{s} - e^{-2s}\Big(\frac{1}{s^2}+\frac{2}{s}\Big). \end{aligned}$$

(7 points) 7. (a) Find general solution of

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad A = \begin{bmatrix} p & 4 \\ -1 & p \end{bmatrix},$$

assuming that p is real.

Answer. We first find the eigenvalues of A by solving $\det(A - \lambda I) = (p - \lambda)^2 + 4 = 0$. This gives two complex conjugated eigenvalues $\lambda_{1,2} = p \pm 2i$. The eigenvector that corresponds to $\lambda_1 = p + 2i$ is

$$\begin{bmatrix} -2i & 4 \\ -1 & -2i \end{bmatrix} \mathbf{v_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v_1} = \begin{bmatrix} 2 \\ i \end{bmatrix}.$$

The general solution is then given by

$$\mathbf{x} = C_1 \mathrm{Re}\Big(\begin{bmatrix} 2 \\ i \end{bmatrix} e^{(p+2i)t} \Big) + C_2 \mathrm{Im}\Big(\begin{bmatrix} 2 \\ i \end{bmatrix} e^{(p+2i)t} \Big),$$

which can be simplified to

$$\mathbf{x} = C_1 e^{pt} \begin{bmatrix} 2\cos(2t) \\ -\sin(2t) \end{bmatrix} + C_2 e^{pt} \begin{bmatrix} 2\sin(2t) \\ \cos(2t) \end{bmatrix}.$$

(3 points) (b) Describe the behaviour of the system (do not draw phase portrait) for all possible real values of p.

Answer. Since the eigenvalues are $\lambda_{1,2} = p \pm 2i$ and $\mathrm{Re}(\lambda_{1,2}) = p$ is real, there are only three cases. If $p > 0$, then the system behaves like a spiral source. If $p < 0$, then the system behaves like a spiral sink. If $p = 0$, then the system behaves like a center.

(12 points) 8. Solve

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} - \begin{bmatrix} 0 \\ 3t+2 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}.$$

Answer. We start by finding the complimentary solution. The eigenvalues of matrix A can be found from $-\lambda(2-\lambda) - 3 = 0$, which gives $\lambda_1 = 3$ and $\lambda_2 = -1$. The corresponding eigenvectors are

$$\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} \mathbf{v_1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v_1} = \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \mathbf{v_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The complimentary solution is then

$$\mathbf{x}_c = C_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{3t} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}.$$

Method 1. By undetermined coefficients, we seek for the particular solution in the form $\mathbf{x}_p = \mathbf{a}t + \mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors with unknown components. Substituting $\mathbf{x}_p$ into the equation gives

$$\mathbf{a} = A\mathbf{a}t + A\mathbf{b} - \begin{bmatrix} 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 3 \end{bmatrix} t,$$

which implies that

$$\mathbf{a} = A\mathbf{b} - \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \qquad A\mathbf{a} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

Computing the the inverse of A gives

$$A^{-1} = \frac{1}{3} \begin{bmatrix} -2 & 1 \\ 3 & 0 \end{bmatrix},$$

which can be used to find

$$\mathbf{a} = A^{-1} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{b} = A^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

So, the particular solution is

$$\mathbf{x}_p = \begin{bmatrix} t \\ 1 \end{bmatrix}.$$

Method 2. To use variation of parameters, we first need to find fundamental matrix solution. Using the obtained complimentary solution, one has

$$X = \begin{bmatrix} e^{3t} & e^{-t} \\ 3e^{3t} & -e^{-t} \end{bmatrix}, \qquad X^{-1} = \frac{1}{4} \begin{bmatrix} e^{-3t} & e^{-3t} \\ 3e^{t} & -e^{t} \end{bmatrix}.$$

Particular solution can be found using

$$\mathbf{x}_p = X \int X^{-1} \begin{bmatrix} 0 \\ -3t-2 \end{bmatrix} dt = \frac{1}{4} \begin{bmatrix} e^{3t} & e^{-t} \\ 3e^{3t} & -e^{-t} \end{bmatrix} \begin{bmatrix} \int (-3t-2)e^{-3t}\, dt \\ \int (3t+2)e^{t}\, dt \end{bmatrix}.$$

The integrals can be calculated as

$$\begin{bmatrix} \int(-3t-2)e^{-3t}\,dt \\ \int(3t+2)e^{t}\,dt \end{bmatrix} = \begin{bmatrix} (t+1)e^{-3t} \\ (3t-1)e^{t} \end{bmatrix}.$$

Finally, the particular solution is

$$\mathbf{x}_p = \frac{1}{4}\begin{bmatrix} e^{3t} & e^{-t} \\ 3e^{3t} & -e^{-t} \end{bmatrix}\begin{bmatrix} (t+1)e^{-3t} \\ (3t-1)e^{t} \end{bmatrix} = \begin{bmatrix} t \\ 1 \end{bmatrix}.$$

The general solution is

$$\mathbf{x} = C_1\begin{bmatrix} 1 \\ 3 \end{bmatrix}e^{3t} + C_2\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t} + \begin{bmatrix} t \\ 1 \end{bmatrix}.$$

Using the initial condition gives

$$\mathbf{x}(0) = C_1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + C_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

which implies that $C_1 = C_2 = 0$. Finally, the solution is

$$\mathbf{x} = \begin{bmatrix} t \\ 1 \end{bmatrix}.$$

9. Consider the nonlinear system

$$\frac{dx}{dt} = x^2 - \frac{y^2}{2} + 1, \qquad \frac{dy}{dt} = -4x - 2y. \tag{1}$$

(5 points) (a) Find the equilibria (critical points) for the system (1).

Answer. The equilibria are solutions of the equations $x^2 - \frac{y^2}{2} + 1 = 0$ and $-4x - 2y = 0$. From the second equation, we obtain $y = -2x$. Substituting this into the first equation, we have $-x^2 + 1 = 0$. Hence the equilibria are $(1, -2)$ and $(-1, 2)$.

(6 points) (b) Find the Jacobian (partial derivative) matrix for the system (1), compute the linearized system at each equilibrium, and compute the eigenvalues for each of the coefficient matrices.

Answer. The Jacobian matrix is

$$\begin{bmatrix} 2x & -y \\ -4 & -2 \end{bmatrix}.$$

The linearized system at $(1, -2)$ is $\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$.
The characteristic equation is $\lambda^2 + 4 = 0$ and the eigenvalues are $\pm 2i$.
The linearized system at $(-1, 2)$ is $\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$.
The characteristic equation is $\lambda^2 + 4\lambda - 4 = 0$, and the eigenvalues are

$$\frac{-4 \pm \sqrt{32}}{2} = -2 \pm 2\sqrt{2}.$$

(4 points) (c) Identify the equilibrium of (1) at which the linearized system (not system (1)) has a center. Classify the other equilibrium and indicate whether it is (asymptotically) stable or unstable in system (1).

Answer. The eigenvalues of the coefficient matrix for the linearized system at equilibrium $(1, -2)$ are are purely imaginary. Therefore the linearized system at $(1, -2)$ has a center.

The coefficient matrix for the linearized system at equilibrium $(-1, 2)$ has a positive eigenvalue and a negative eigenvalue. Hence, the equilibrium is a saddle, which is unstable, in system (1).

Table of Laplace transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}, \quad s > 0$
2. e^{-at}	$\frac{1}{s+a}, \quad s > -a$
3. t^n, n positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
4. $\sin(at)$	$\frac{a}{s^2+a^2}, \quad s > 0$
5. $\cos(at)$	$\frac{s}{s^2+a^2}, \quad s > 0$
6. $\sinh(at)$	$\frac{a}{s^2-a^2}, \quad s > \lvert a \rvert$
7. $\cosh(at)$	$\frac{s}{s^2-a^2}, \quad s > \lvert a \rvert$
8. $u(t-a)$	$\frac{e^{-as}}{s}, \quad s > 0$
9. $u(t-a)f(t-a)$	$e^{-as}F(s)$
10. $e^{-at}f(t)$	$F(s+a)$
11. $\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$
12. $\int_0^t f(\tau)d\tau$	$\frac{F(s)}{s}$
13. $\delta(t-a)$	e^{-as}
14. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - ... - f^{(n-1)}(0)$

Variation of parameters

If $y_1(x)$ and $y_2(x)$ are two solutions of $Ly = 0$, then the particular solution of $Ly = f(x)$ is

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x),$$

$$\begin{aligned} y_1u_1' + y_2u_2' &= 0, \\ y_1'u_1' + y_2'u_2' &= f(x). \end{aligned}$$