version 11

THE UNIVERSITY OF CALGARY
DEPARTMENT OF MATHEMATICS AND STATISTICS
MIDTERM TEST 1 - SOLUTIONS
MATH 267 L01 (Fall 2017)

October 12, 2017 Time: 2 hours

These solutions are based off of version 11 of midterm 1. The others are similar.

I.D. NUMBER	SURNAME	OTHER NAMES

NOTE: No calculators. No other aids. Closed-book.

FOR TA USE ONLY	
Question	Score
MC	/35
B1	/5
B2	/5
B3	/5
TOTAL	/50

PART (A) Multiple Choice Questions (35 marks)

Instructions: On the scantron, fill in one answer per question. Questions 1,3,5,7,9,11,13 are worth 3 marks each. Questions 2,4,6,8,10,12,14 are worth 2 marks each.

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1. (3 marks) Evaluate
$$\int 2x\cos(x^2)\ dx.$$
   (a) $x^2\sin(\frac{x^3}{3})+C$,
   (b) $x^2\sin(x^2)+C$,
   (c) $\sin(x^2)+C$, [***CORRECT***. This is a basic u-sub. Very similar to an example in Lecture 1-1 and 1-2.]
   (d) $\cos(x^2)+C$,
   (e) 0.

2. (2 marks) Which u-substitution will turn
$$\int \frac{dx}{\sqrt{x}\ln(x)}$$
into
$$\int \frac{du}{\ln(u)}?$$
   (a) $u=\sqrt{x}$, [***CORRECT***. Remember $\ln(u^2)=2\ln(u)$.]
   (b) $u=x$,
   (c) $u=\ln(x)$,
   (d) $\dfrac{1}{\sqrt{x}\ln(x)}$,
   (e) $u=e^x$.

3. (3 marks) Evaluate

$$\int \cos^2(x)\sin^3(x)\ dx.$$

(a) $-\frac{\sin^3(x)}{3}\frac{\cos^4(x)}{4}+C$,

(b) $-\frac{\sin^3(x)}{3}+\frac{\cos^4(x)}{4}+C$,

(c) $-\frac{\cos^3(x)}{3}\frac{\cos^5(x)}{5}+C$,

(d) $-\frac{\cos^3(x)}{3}+\frac{\cos^5(x)}{5}+C$, [***CORRECT***. Use pythagoras on a $\sin^2 x$, then it is a u-sub. This was an example in Lecture 2-1.]

(e) $-\frac{\cos^3(x)}{3}$.

4. (2 marks) Evaluate

$$\int_0^\pi \sin^2(\theta)\ d\theta.$$

(a) 0,

(b) 1,

(c) $\frac{\pi}{2}$, [***CORRECT***. Use the half-angle formula for sine that reduces the power. The double-angle formula is not needed.]

(d) π,

(e) 2π.

5. (3 marks) Evaluate
$$\int \frac{dx}{\sqrt{4-9x^2}}.$$
   (a) $\frac{1}{3}\arcsin(\frac{1}{2}x)+C$,
   (b) $\frac{2}{3}\arcsin(\frac{1}{2}x)+C$,
   (c) $\frac{2}{3}\arcsin(\frac{3}{2}x)+C$,
   (d) $\frac{1}{3}\arcsin(\frac{3}{2}x)+C$, [***CORRECT***. Use the trig sub $x=\frac{2}{3}\sin\theta$. This was an example in class.]
   (e) $\frac{1}{3}\arcsin(x)+C$.

6. (2 marks) By using an appropriate trig substitution, evaluate
$$\int \frac{dx}{x(x^2+1)}.$$
   (a) $\arctan(x)+C$,
   (b) $\ln|\frac{x}{\sqrt{x^2-1}}|+C$,
   (c) $\ln|\frac{1}{\sqrt{1+x^2}}|+C$,
   (d) $\ln|\frac{x}{\sqrt{1+x^2}}|+C$, [***CORRECT***. Use the trig sub $x=\tan\theta$. This is a simplified version of a Lyryx problem from assignment 1. This was also example 2 in lecture 3-1, where we solved it using partial fractions.]
   (e) $x\arctan(x)+C$.

7. (3 marks) Evaluate
$$\int x \ln x \; dx.$$
   (a) $\frac{x^2}{2}(\ln x - 1) + C$,
   (b) $\frac{x^2}{4}(\ln x - 1) + C$,
   (c) $\frac{x^2}{2}(\ln x - \frac{1}{2}) + C$, [***CORRECT***. By parts: $u = \ln(x), dv = xdx$. This was the second example in lecture 2-2.]
   (d) $\frac{x^2}{4}(\ln x - \frac{1}{2}) + C$,
   (e) $\frac{x^2}{2}(x \ln x - x) + C$.

8. (2 marks) Evaluate
$$\int (\ln x)(\ln x) \; dx.$$
For this question, you may use the fact that
$$\int \ln x \; dx = x(\ln x) - x + C.$$
   (a) $\frac{(\ln x)^3}{3} + C$,
   (b) $\frac{(x(\ln x) - x)^3}{3} + C$,
   (c) $(x(\ln x) - x)^2 + C$,
   (d) $x(\ln x)^2 - 2x(\ln x) + C$,
   (e) $x(\ln x)^2 - 2x(\ln x) + 2x + C$. [***CORRECT***. By parts: $u = \ln(x), dv = \ln(x)dx$. You end up with the integral $\int \ln x - 1 dx$, and you can use the above formula again.]

9. (3 marks) Finley has written down

$$\frac{x^2+2}{(x-2)(x+3)^3(x^2+4)^2} = \frac{A}{x-2} + \frac{B_1}{x+3} + \frac{B_2}{(x+3)^2} + \frac{B_3}{(x+3)^3} + \frac{M_1x+C_1}{(x^2+4)^2}$$

which she thinks is the correct rational decomposition. Which term is missing from the full decomposition?

(a) Nothing is missing. All of the needed terms are there. (*Good work Finley!*)

(b) $\dfrac{C_2}{x^2+4}$

(c) $\dfrac{M_2x}{x^2+4}$

(d) $\dfrac{M_2x+C_2}{x^2+4}$ [***CORRECT***. Similar to problem 1 in Lab 1, and an example in lecture 3-2]

(e) $\dfrac{(M_2x+C_2)^2}{x^2+4}$

10. (2 marks) Evaluate

$$\int \frac{50x^{49}}{x^{100}+6x^{50}+10}\,dx.$$

Hint. The numerator should suggest a useful first step.

(a) $\arctan(x+3)+C$,

(b) $\arctan(x^{49}+3)+C$,

(c) $\arctan(x^{50}+3)+C$, [***CORRECT***. First use $u=x^{50}$ and then complete the square.]

(d) $\arctan(x^{100}+3)+C$

(e) $\dfrac{-1}{x^{50}} + \dfrac{50}{6}\ln x + \dfrac{x^{50}}{10} + C$.

11. (3 marks) Does the integral $\int_2^\infty \frac{dx}{x+1}$ converge?

    (a) Yes, it converges to $\frac{1}{9}$.
    (b) Yes, it converges to $\frac{1}{3}$.
    (c) Yes, it converges to $\ln 2$.
    (d) Yes, it converges to $\ln 3$.
    (e) No, it does not converge. [***CORRECT***. The partial integral grows like $\ln(b+1)$ as $b \to \infty$. Similar to an example in Lecture 3-3.]

12. (2 marks) For which values of p does $\int_0^\infty e^{px}\, dx$ converge? (Choose the most accurate answer.)

    (a) Only when $p > 1$.
    (b) Only when $p < 1$.
    (c) Only when $p < 0$. [***CORRECT***. The partial integral grows like $\frac{e^{px}}{p}$. Similar to problem 4.c in Lab 3.]
    (d) Only when $p > 0$.
    (e) Only when $p = 0$.

13. (3 marks) What is the length of the curve $y = \sqrt{x}$ for $1 \leq x \leq 9$?

 (a) $\frac{1}{2}\int_1^9 \frac{1}{\sqrt{x}}\sqrt{4x+1}\, dx$, [***CORRECT***. Use formula for arc length and factor out a $\sqrt{\frac{1}{4x}}$. Very similar to a computation in Lecture 4-1.]

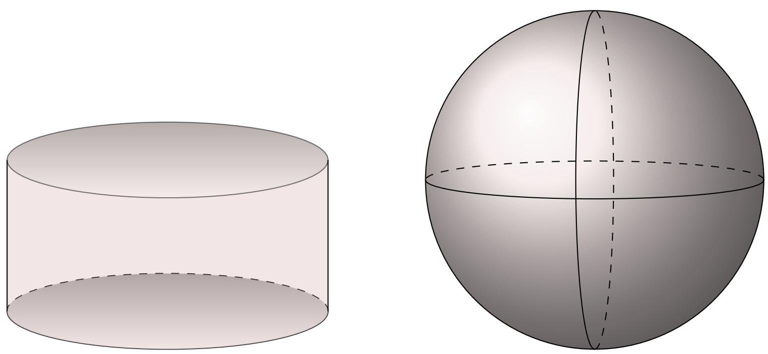
 (b) $\frac{1}{4}\int_1^9 \frac{1}{\sqrt{x}}\sqrt{4x+1}\, dx$,

 (c) $\frac{1}{4}\int_1^9 \sqrt{x^2+1}\, dx$,

 (d) $\frac{1}{4}\int_1^9 x+1\, dx$,

 (e) $\int_1^9 \frac{1}{\sqrt{x}}\sqrt{4x+1}\, dx$.

14. (2 marks) The surface area of a sphere with radius r is $4\pi r^2$. Suppose you have a cylinder with the **same radius** r **as the sphere**, and the height is $h = r$ (see picture). Which of these statements is most correct, when we consider the surface area of the cylinder to also include the top and bottom discs?



 (a) The surface area of the cylinder is always **larger than** the surface area of the sphere.

 (b) The surface area of the cylinder is always **smaller than** the surface area of the sphere.

 (c) The surface area of the cylinder is always **the same as** the surface area of the sphere. [***CORRECT***. The formula for the surface area of a cylinder *without* top and bottom is $2\pi rh$ (Example 1 in Lecture 4-1); this can be derived from the formula for surface area if needed. Each disc has area πr^2. Since $h = r$, the total surface area is $4\pi r^2$, the same as the sphere. This question is similar to the one in Lecture 4-1 about a cylinder vs a cone.]

 (d) For some values of r the surface area of the cylinder is larger than the surface area of the sphere, and for some values the surface area of the sphere is larger than the surface area of the cylinder.

PART (B): Long Answer Questions (15 marks).

Instructions: Answer Questions 1,2,3 in the space provided. Be sure to show all of your work, as partially correct answers may be worth partial credit. If required, please put your answer in the answer box. Each question is worth 5 marks.

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1. (5 marks) Show that

$$\int \tan^{102}\theta \; d\theta = \frac{\tan^{101}\theta}{101} - \int \tan^{100}\theta \; d\theta.$$

Solution. We covered the reduction formula for $\sec^{n+2}\theta$ in lecture and problem 4 in lab 3. This is a simpler reduction formula that doesn't need integration by parts.

This uses the Pythagoras identity $\sec^2\theta = \tan^2\theta + 1$, which can be deduced from $1 = \sin^2\theta + \cos^2\theta$ by dividing by $\cos^2\theta$.

Let $n = 100$.

$$\begin{aligned}
\int \tan^{n+2}\theta \; d\theta &= \int \tan^{n}\theta \tan^2\theta \; d\theta \\
&= \int \tan^{n}\theta(\sec^2\theta - 1) \; d\theta \\
&= \int \tan^{n}\theta \sec^2\theta \; d\theta - \int \tan^{n}\theta \; d\theta \qquad u\text{-sub, } u = \tan\theta, du = \sec^2\theta \; d\theta. \\
&= \frac{\tan^{n+1}\theta}{n+1} - \int \tan^{n}\theta \; d\theta
\end{aligned}$$

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2. (5 marks) For which p values does $\int_0^1 \frac{dx}{x^p}$ converge? Justify your answer. You may use any facts or results from class.

Solution. This is similar to an example in lecture 3-3.

$\boxed{p = 1}$. In this case, the integral is

$$\begin{aligned} \int_0^1 \frac{dx}{x^p} &= \lim_{b \to 0^+} \int_b^1 \frac{dx}{x} \\ &= \lim_{b \to 0^+} \ln(1) - \ln(b) \\ &= \infty \end{aligned}$$

So it doesn't converge.

$\boxed{p \neq 1}$. In this case, the integral is

$$\begin{aligned} \int_0^1 \frac{dx}{x^p} &= \lim_{b \to 0^+} \int_b^1 x^{-p}\, dx \\ &= \lim_{b \to 0^+} \frac{x^{-p+1}}{-p+1} \Big|_{x=b}^{x=1} \\ &= \lim_{b \to 0^+} \frac{1}{-p+1} - \frac{b^{-p+1}}{-p+1} \end{aligned}$$

The $\frac{1}{-p+1}$ is always finite, so we only need to investigate $\lim_{b \to 0^+} \frac{b^{-p+1}}{-p+1}$.

When $-p+1 < 0$ then this limit is infinite, so the integral diverges. When $-p+1 > 0$, then this limit is 0 (hence finite) so the integral converges.

In other words, then integral converges when $1 \leq p$ and diverges when $p > 1$.

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ANSWER:

3. (5 marks) Evaluate $\int \csc\theta \, d\theta$, using whatever method you prefer. You may use any facts or results from class.

> **Strategy 1**. One strategy is very similar to the one we used in class to evaluate $\int \sec\theta \, d\theta$.
> **Strategy 2**. Another strategy is to mimic the formula for $\int \sec\theta \, d\theta$, then guess a formula for $\int \csc\theta \, d\theta$, then check it.
> **Strategy 3**. Yet another strategy is to transform $\csc\theta$ to $\sec\theta$ by reflections and translations, then use the formula for $\int \sec\theta \, d\theta$.

Strategy 1. This is very similar to the integral for secant which we discussed in lecture 3-1. This is problem 2 in lab 3.

$$\begin{aligned}
\int \csc(\theta) \, d\theta &= \int \frac{d\theta}{\sin(\theta)} \\
&= \int \frac{\sin(\theta)d\theta}{\sin^2(\theta)} \text{ multiply by } \frac{\sin(\theta)}{\sin(\theta)} \\
&= \int \frac{\sin(\theta)d\theta}{1-\cos^2(\theta)} \text{ Pythagoras} \\
&= \int \frac{-du}{1-u^2} \text{ Use } u = \cos(\theta), du = -\sin(\theta)d\theta
\end{aligned}$$

The partial fractions decomposition is

$$\frac{-1}{(1-u)(1+u)} = \frac{A}{1-u} + \frac{B}{1+u}$$

which is equivalent to

$$-1 = A(1+u) + B(1-u) = (A-B)u + (A+B).$$

We can solve this using back-substitution if we want. In this case, it is so similar to the system we received in the secant integral that we can just adapt that solution (in essence we are making an educated guess).

$$A = B = \frac{-1}{2}.$$

So

$$\begin{aligned}
\int \csc(\theta) \, d\theta &= \ldots \\
&= \int \frac{-du}{1-u^2} \\
&= \int \frac{-1}{2}\frac{1}{1-u} + \frac{-1}{2}\frac{1}{1+u} du \\
&= \frac{-1}{2}\int \frac{du}{1-u} + \frac{-1}{2}\int \frac{du}{1+u} \\
&= \frac{-1}{2}\ln|1-u| - \frac{1}{2}\ln|1+u| + C \\
&= \frac{-1}{2}\ln|1-\cos(\theta)| - \frac{1}{2}\ln|1+\cos(\theta)| + C
\end{aligned}$$

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Strategy 2. From the formula sheet:

$$\boxed{\int \sec(\theta)\ d\theta = \ln|\sec\theta + \tan\theta| + C}$$

Let's guess

$$\int \csc(\theta)\ d\theta =^{?} \ln|\csc\theta + \cot\theta| + C.$$

Taking a derivative gives:

$$\frac{-\csc\theta\cot\theta - \csc^2\theta}{\csc\theta + \cot\theta} = -\csc\theta$$

We're off by a minus sign, so the correct integral is

$$\int \csc(\theta)\ d\theta = -\ln|\csc\theta + \cot\theta| + C.$$

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Strategy 3. Notice that $\sin(\theta) = \cos(\theta - \frac{\pi}{2})$ (this can be deduced from the pictures of the graphs). So

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{\cos(\theta - \frac{\pi}{2})} = \sec(\theta - \frac{\pi}{2}).$$

So the formula for secant on the formula sheet tells us:

$$\int \csc\theta\ d\theta = \int \sec(\theta - \frac{\pi}{2})\ d\theta = \ln|\sec(\theta - \frac{\pi}{2}) + \tan(\theta - \frac{\pi}{2})| + C$$

I would accept this as a final answer.

If you want, you can use the fact that $\sec(\theta - \frac{\pi}{2}) = \csc\theta$ and reduce this to

$$\ln|\csc\theta + \tan(\theta - \frac{\pi}{2})| + C$$

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Bonus Strategy. There is a clever way to find the integral of secant by multiplying the integrand by $\frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}$. A similar strategy will work here by multiplying by $\frac{\csc\theta + \cot\theta}{\csc\theta + \cot\theta}$, namely:

$$\int \csc\theta\ d\theta = \int \frac{\csc^2\theta + \csc\theta\cot\theta}{\csc\theta + \cot\theta} d\theta$$

Let $u = \csc\theta + \cot\theta$ and $du = -\csc\theta\cot\theta - \csc^2\theta\ d\theta$, so you get

$$\int \frac{-1}{u}\ d\theta = -\ln u + C = -\ln(\csc\theta + \cot\theta) + C.$$

Notice how similar this is to strategy 2. ■