version 11

THE UNIVERSITY OF CALGARY
DEPARTMENT OF MATHEMATICS AND STATISTICS
MIDTERM TEST 2 - SOULTIONS
MATH 267 L01 (Fall 2017)

November 16, 2017 Time: 2 hours

Fill out the following information as it appears on your Unicard.

	Example	Your info
FIRST NAME	Alex	
LAST NAME	Richardson	
I.D. NUMBER	19995790	

NOTE: No calculators. No other aids. Closed-book.

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Question	Score
MC	/35
B1	/5
B2	/5
B3	/5
TOTAL	/50

PART (A) Multiple Choice Questions (35 marks)

Instructions: On the scantron, fill in one answer per question. Questions 1,3,5,7,9,11,13 are worth 3 marks each. Questions 2,4,6,8,10,12,14 are worth 2 marks each.

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1. (3 marks) What is the value of

$$\sum_{n=0}^{\infty} \frac{3^{2n+1}}{10^{n+2}}?$$

   (a) $\frac{3}{10}$. [***CORRECT***]
   (b) 1.
   (c) $\frac{3}{100}$.
   (d) $\frac{9}{10}$.
   (e) ∞.

   This is similar to **LAB 5 - Q2**.

$$\sum_{n=0}^{\infty} \frac{3^{2n+1}}{10^{n+2}} = \frac{3}{10^2}\sum_{n=0}^{\infty} \frac{3^{2n}}{10^{n}} = \frac{3}{10^2}\sum_{n=0}^{\infty} \left(\frac{9}{10}\right)^n = \frac{3}{10^2}\frac{1}{1-\frac{9}{10}} = \frac{3}{10^2}\frac{1}{\frac{1}{10}} = \frac{3}{10}$$

2. (2 marks) Let $\{a_n\}_{n=1}^{\infty}$ be the sequence defined by the relationship

$$a_{n+1} = 1 - a_n$$

   where $a_1 = 0$. In this case, $\{a_n\}_{n=1}^{\infty}$ is:

   (a) unbounded and it diverges,
   (b) bounded and it diverges, [***CORRECT***]
   (c) increasing and decreasing,
   (d) monotone and it diverges,
   (e) monotone and it converges.

   This is from **LECTURE 4-3**. The sequence is:

$$0, 1, 0, 1, 0, 1, 0, \ldots$$

3. (3 marks) For which values of p does

$$\sum_{k=0}^{\infty} e^{pk}$$

converge? (Choose the most accurate answer.)

(a) Only when $p > 1$.
(b) Only when $p < 1$.
(c) Only when $p < 0$. [***CORRECT***]
(d) Only when $p > 0$.
(e) Only when $p = 0$.

You saw this on **Midterm 1** (about integrals) and is similar to **LECTURE 6-1, page 4**. This question can be solved using the integral test, the root test or the ratio test.

4. (2 marks) For the series

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which of the following is the best upper bound?

(a) $S \leq \frac{1}{2}$.
(b) $S \leq 1$.
(c) $S \leq 2$. [***CORRECT***]
(d) $S \leq 3$.
(e) $S = \infty$.

This is from **LECTURE 6-1, page 9** and is simpler than **LAB 6, Q1.i**. Use the integral test error bound. Note that the term $a_1 = 1$ so the full sum is definitely at least 1 (i.e. answers a and b are impossible.)

5. (3 marks) The following two inequalities are true for all $n \geq 1$:

$$\frac{1}{n^3 + \sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{n}} \tag{1}$$

$$\frac{1}{n^3 + \sqrt[3]{n}} \leq \frac{1}{n^3} \tag{2}$$

Together with the basic comparison test, what can you conclude?

(a) (1) implies that $\sum \frac{1}{n^3+\sqrt[3]{n}}$ converges.
(b) (1) implies that $\sum \frac{1}{n^3+\sqrt[3]{n}}$ diverges.
(c) (2) implies that $\sum \frac{1}{n^3+\sqrt[3]{n}}$ converges. [*****CORRECT*****]
(d) (2) implies that $\sum \frac{1}{n^3+\sqrt[3]{n}}$ diverges.
(e) Nothing can be concluded since $\lim_{n\to\infty} \frac{1}{n^3 + \sqrt[3]{n}}$ diverges.

This is from **LECTURE 6-2, page 4**, **Lyryx**, and **LAB 6 - Q1a**.

6. (2 marks) What does the **limit comparison test** tell you about

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{n^2 + 2^n + 2}{n^3 + 3^n + 3}?$$

(a) It converges by comparing a_n with $b_n = \left(\frac{2}{3}\right)^n$. [*****CORRECT*****]
(b) It diverges by comparing a_n with $b_n = \left(\frac{2}{3}\right)^n$.
(c) It converges by comparing a_n with $b_n = \frac{n^2}{n^3}$.
(d) It diverges by comparing a_n with $b_n = \frac{n^2}{n^3}$.
(e) It diverges by comparing a_n with $b_n = \frac{2}{3}$.

This is from **LECTURE 6-2, page 7**, and **LAB 6 - Q4**.

7. (3 marks) Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

   (a) It converges absolutely.
   (b) It converges conditionally. [*****CORRECT*****]
   (c) It diverges.

   This is from **LECTURE 6-3, page 4 and page 9**; it is the alternating harmonic series. Use the alternating series test.

8. (2 marks) Does the following series converge?

$$\sum_{n=2}^{\infty} \frac{\cos(n)}{n^2 - 1}$$

   (a) It converges absolutely. [*****CORRECT*****]
   (b) It converges conditionally.
   (c) It diverges.

   This is from **LECTURE 7-1, page 11**, **LAB 7, Q2** and is similar to **SAMPLE BASIC Q3**. Take absolute values of the terms and use that $|\cos n| \leq 1$.

9. (3 marks) Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{(2n-1)!}{n!\,(n+2)!}$$

(a) It converges absolutely.
(b) It converges conditionally.
(c) It diverges. [*****CORRECT*****]

This is from **LAB 7, Q3**. Use the ratio test (or the root test if you really want).

10. (2 marks) Does the following series converge?

$$\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^{n^2}$$

(a) It converges absolutely. [*****CORRECT*****]
(b) It converges conditionally.
(c) It diverges.

This is from **LAB 7, Q3** and is similar to **LECTURE 7-2, page 6**. Use the root test.

11. (3 marks) What is the exact value of:

$$\sum_{k=0}^{\infty} \frac{k+1}{3^k}?$$

> **Hint**. Use $x = \frac{1}{3}$.

(a) $\frac{3}{2}$.
(b) $\frac{9}{4}$. [***CORRECT***]
(c) 4.
(d) 9.
(e) ∞.

This is from **LECTURE 8-2, page 6** (except with a 3 instead of a 2). Term-by-term differentiate the geometric series.

12. (2 marks) Suppose that $\sum_{k=0}^{\infty} a_k(x-1)^k$ is a power series that:

- converges at $x = 3$, and
- diverges at $x = -1.1$.

Which of these statements is necessarily true about that power series?

(a) It converges at $x = \pi$.
(b) It diverges at $x = \pi$. [***CORRECT***]
(c) It has an infinite radius of convergence.
(d) It converges at $x = -1$.
(e) It diverges at $x = -1$.

The centre is 1, and the series must converge there (**LECTURE 7-3**). The radius of convergence is less than 2.1 since the series diverges at -1.1. Since $\pi \approx 3.14$ is more than 2.1 units away from the the centre (1), the series must diverge at $x = \pi$.

13. (3 marks) Find the degree 3 Taylor polynomial for $f(x) = \tan(x)$.

    (a) $x + \frac{x^3}{3}$. [***CORRECT***]
    (b) $x + \frac{x^3}{3!}$.
    (c) $x - \frac{x^3}{3!}$.
    (d) $x + 2x^3$.
    (e) $\frac{x - \frac{x^3}{3!}}{1 - \frac{x^2}{2}}$.

$$f(x) = \tan x,\ f'(x) = \sec^2 x,\ f^{(2)}(x) = 2\sec x \sec x \tan x,\ f^{(3)}(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$$

$$f(0) = 0,\ f'(0) = 1,\ f^{(2)}(0) = 0,\ f^{(3)}(0) = 2$$

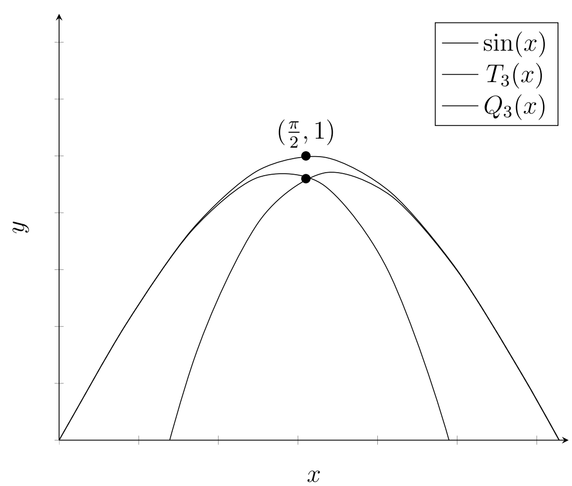
14. (2 marks) Let $T_3(x)$ be the degree 3 Taylor polynomial for $\sin(x)$ centred at 0. Let $Q_3(x)$ be the degree 3 Taylor polynomial for $\sin(x)$ centred at π.

    Which of these polynomials gives a better estimate for $\sin(x)$ at $x = \frac{\pi}{2}$?

    (a) $T_3(x)$ gives a better estimate.
    (b) $Q_3(x)$ gives a better estimate.
    (c) $T_3(x)$ and $Q_3(x)$ give the same estimate. [***CORRECT***]
    (d) $T_3(x)$ and $Q_3(x)$ give equally good estimates, but $T_3(\frac{\pi}{2}) \neq Q_3(\frac{\pi}{2})$.
    (e) Trick question: One of these polynomials is not defined.

    You are given on the formula sheet that $T_3(x) = x - \frac{x^3}{6}$. Computing $Q_3(x)$ by hand gives $Q_3(x) = -(x - \pi) + \frac{(x-\pi)^3}{6}$. Using both to estimate $\sin(\frac{\pi}{2})$ gives:

$$Q_3(\frac{\pi}{2}) = -(\frac{\pi}{2} - \pi) + \frac{(\frac{\pi}{2} - \pi)^3}{6} = \frac{\pi}{2} - \frac{(\frac{\pi}{2})^3}{6} = T_3(\frac{\pi}{2})$$



PART (B): Long Answer Questions (15 marks).

Instructions: Answer Questions 1,2,3 in the space provided. Be sure to show all of your work, as partially correct answers may be worth partial credit. Each question is worth 5 marks.

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1. (5 marks) Suppose that $\sum_{n=0}^{\infty} a_n$ converges, and $0 < a_n < \frac{\pi}{2}$ for each a_n. Show that

$$\sum_{n=0}^{\infty} \tan(a_n)$$

converges.

> **Hint**. This is very similar to the $\sin(a_n)$ version in lecture. Set up and use the Limit comparison test.

Solution. Modify the argument for the $\sin(a_n)$ version in **LECTURE 6-2**. This was **SAMPLE LONG ANSWER Q3**.

Since $\sum_{n=0}^{\infty} a_n$ converges, we must have $\lim_{n\to\infty} a_n = 0$ by the nth term test.

We set up and use the limit comparison test:

$$\lim_{n\to\infty} \frac{\tan a_n}{a_n} = \lim_{n\to\infty} \frac{\sin a_n}{a_n \cos a_n} = 1$$

since $\lim_{n\to\infty} \cos a_n = 1$, as $a_n \to 0$. Similarly, $\lim_{n\to\infty} \frac{\sin a_n}{a_n} = 1$, as $a_n \to 0$, which follows from $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Since $0 < 1 < \infty$, and $\sum_{n=0}^{\infty} a_n$ converges, the limit comparison test says that $\sum_{n=0}^{\infty} \tan a_n$ also converges. ■

2. (5 marks) A piece of paper has thickness 0.05mm (millimeters). If you fold it once it will have thickness 0.1mm, and if you fold it again it will have thickness 0.2mm, etc..

This was **SAMPLE LONG ANSWER Q4**.

(a.) (3 marks) Give a formula for how thick the paper will be after k folds ($k = 0, 1, 2, \ldots$).

Solution. Call T_k the thickness after k folds. It doubles after every fold, so the recursive formula is

$$T_{k+1} = 2T_k$$

Unwinding this (or observing it directly) we get the answer: ■

ANSWER: After k folds the paper will have thickness: $\underline{2^k \times 0.05 \text{ mm}}$

(b.) (2 marks) Show that after 57 folds the paper will have thickness greater than the thickness of the Earth ($\approx 1.3 \times 10^{10}$mm). You may use that $\ln(0.05) \approx -3, \ln(2) \approx \frac{1}{2}, \ln(10) \approx 2.5$ and $\ln(1.3) \approx \frac{1}{4}$.

Solution - Set up. We need to find a k such that

$$2^k \times 0.05 > 1.3 \times 10^{10} \tag{3}$$

Using that $0.05 = \frac{1}{20}$, this is the same as finding:

$$2^k > 20 \times 1.3 \times 10^{10}$$

■

Solution 1. To find such a k in (3) we take logs and get

$$\begin{aligned} \ln(2^k \times 0.05) &> \ln(1.3 \times 10^{10}) \\ k\ln(2) + \ln(0.05) &> \ln(1.3) + 10\ln(10) \\ k &> \frac{\ln(1.3) + 10\ln(10) - \ln(0.05)}{\ln(2)} \\ &\approx \frac{\frac{1}{4} + 10(2.5) - (-3)}{\frac{1}{2}} \\ &= \frac{2}{4} + 2(25) + 2(3) \\ &= 56.5 \end{aligned}$$

So taking $k = 57 (> 56.5)$ many folds will give the desired thickness. ■

Solution 2. To find such a k in (2) we avoid taking logs and instead use powers of 2 to estimate the quantity on the right. Note that:

$$\begin{aligned} 2^5 = 32 &> 20 \\ 2^1 = 2 &> 1.3 \\ 2^4 = 16 &> 10 \\ \text{so } 2^{40} = (2^4)^{10} &> 10^{10} \end{aligned}$$

Multiplying these together gives:

$$2^{46} = 2^{5+1+40} > 20 \times 1.3 \times 10^{10}$$

So in fact, 46 folds is enough to get the desired thickness (and folding 11 more times will only make it *even* thicker). ∎

3. (5 marks) Show that the following series converges:

$$\sum_{n=1}^{\infty} \frac{((1+\frac{1}{n})^n)^n}{1+3+9+\ldots+3^{n-1}}.$$

Hint. Figure out a closed form for the denominator. Then use an appropriate test.

Solution. From the formula sheet we get that:

$$1+3+9+\ldots+3^{n-1} = \frac{3^n-1}{3-1} = \frac{3^n-1}{2}.$$

So the series becomes

$$2\sum_{n=1}^{\infty} \frac{((1+\frac{1}{n})^n)^n}{3^n-1}$$

and this looks like the root test will be helpful. Note that the scalar factor of 2 does not affect convergence. Also note that all terms are positive, so we will drop the absolute value.

$$\sqrt[n]{|a_n|} = \left(\frac{((1+\frac{1}{n})^n)^n}{3^n-1}\right)^{\frac{1}{n}} = \frac{(1+\frac{1}{n})^n}{\sqrt[n]{3^n-1}}$$

The numerator is a special limit which appears on the formula sheet: it goes to e. The denominator is also a special limit which appears on the formula sheet: it goes to 3. Thus

$$\sqrt[n]{|a_n|} = \frac{e}{3} < 1$$

Since this is less than 1 (the formula sheet reminds us that $e \approx 2.71828\ldots < 3$), the root test tells us that the original series converges. ∎