

THE UNIVERSITY OF CALGARY
 DEPARTMENT OF MATHEMATICS AND STATISTICS
MIDTERM TEST 2
MATH 267 L01 (Fall 2017)

November 16, 2017

Time: 2 hours

Fill out the following information as it appears on your Unicard.



	Example	Your info
FIRST NAME	Alex	
LAST NAME	Richardson	
I.D. NUMBER	19995790	

NOTE: No calculators. No other aids. Closed-book.

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Question	Score
MC	/35
B1	/5
B2	/5
B3	/5
TOTAL	/50

PART (A) Multiple Choice Questions (35 marks)

Instructions: On the scantron, fill in one answer per question. Questions 1,3,5,7,9,11,13 are worth 3 marks each. Questions 2,4,6,8,10,12,14 are worth 2 marks each.

1. (3 marks) What is the value of

$$\sum_{n=0}^{\infty} \frac{3^{2n+1}}{10^{n+2}}?$$

- (a) $\frac{3}{10}$.
- (b) 1.
- (c) $\frac{3}{100}$.
- (d) $\frac{9}{10}$.
- (e) ∞ .

2. (2 marks) Let $\{a_n\}_{n=1}^{\infty}$ be the sequence defined by the relationship

$$a_{n+1} = 1 - a_n$$

where $a_1 = 0$. In this case, $\{a_n\}_{n=1}^{\infty}$ is:

- (a) unbounded and it diverges,
- (b) bounded and it diverges,
- (c) increasing and decreasing,
- (d) monotone and it diverges,
- (e) monotone and it converges.

3. (3 marks) For which values of p does

$$\sum_{k=0}^{\infty} e^{pk}$$

converge? (Choose the most accurate answer.)

- (a) Only when $p > 1$.
- (b) Only when $p < 1$.
- (c) Only when $p < 0$.
- (d) Only when $p > 0$.
- (e) Only when $p = 0$.

4. (2 marks) For the series

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which of the following is the best upper bound?

- (a) $S \leq \frac{1}{2}$.
- (b) $S \leq 1$.
- (c) $S \leq 2$.
- (d) $S \leq 3$.
- (e) $S = \infty$.

5. (3 marks) The following two inequalities are true for all $n \geq 1$:

$$\frac{1}{n^3 + \sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{n}} \quad (1)$$

$$\frac{1}{n^3 + \sqrt[3]{n}} \leq \frac{1}{n^3} \quad (2)$$

Together with the basic comparison test, what can you conclude?

- (a) (1) implies that $\sum \frac{1}{n^3 + \sqrt[3]{n}}$ converges.
- (b) (1) implies that $\sum \frac{1}{n^3 + \sqrt[3]{n}}$ diverges.
- (c) (2) implies that $\sum \frac{1}{n^3 + \sqrt[3]{n}}$ converges.
- (d) (2) implies that $\sum \frac{1}{n^3 + \sqrt[3]{n}}$ diverges.
- (e) Nothing can be concluded since $\lim_{n \rightarrow \infty} \frac{1}{n^3 + \sqrt[3]{n}}$ diverges.

6. (2 marks) What does the **limit comparison test** tell you about

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{n^2 + 2^n + 2}{n^3 + 3^n + 3}?$$

- (a) It converges by comparing a_n with $b_n = \left(\frac{2}{3}\right)^n$.
- (b) It diverges by comparing a_n with $b_n = \left(\frac{2}{3}\right)^n$.
- (c) It converges by comparing a_n with $b_n = \frac{n^2}{n^3}$.
- (d) It diverges by comparing a_n with $b_n = \frac{n^2}{n^3}$.
- (e) It diverges by comparing a_n with $b_n = \frac{2}{3}$.

7. (3 marks) Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

- (a) It converges absolutely.
- (b) It converges conditionally.
- (c) It diverges.

8. (2 marks) Does the following series converge?

$$\sum_{n=2}^{\infty} \frac{\cos(n)}{n^2 - 1}$$

- (a) It converges absolutely.
- (b) It converges conditionally.
- (c) It diverges.

9. (3 marks) Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{(2n-1)!}{n!(n+2)!}$$

- (a) It converges absolutely.
- (b) It converges conditionally.
- (c) It diverges.

10. (2 marks) Does the following series converge?

$$\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^{n^2}$$

- (a) It converges absolutely.
- (b) It converges conditionally.
- (c) It diverges.

11. (3 marks) What is the exact value of:

$$\sum_{k=0}^{\infty} \frac{k+1}{3^k}?$$

Hint. Use $x = \frac{1}{3}$.

- (a) $\frac{3}{2}$.
- (b) $\frac{9}{4}$.
- (c) 4.
- (d) 9.
- (e) ∞ .

12. (2 marks) Suppose that $\sum_{k=0}^{\infty} a_k(x-1)^k$ is a power series that:

- converges at $x = 3$, and
- diverges at $x = -1.1$.

Which of these statements is necessarily true about that power series?

- (a) It converges at $x = \pi$.
- (b) It diverges at $x = \pi$.
- (c) It has an infinite radius of convergence.
- (d) It converges at $x = -1$.
- (e) It diverges at $x = -1$.

13. (3 marks) Find the degree 3 Taylor polynomial for $f(x) = \tan(x)$.

(a) $x + \frac{x^3}{3}$.

(b) $x + \frac{x^3}{3!}$.

(c) $x - \frac{x^3}{3!}$.

(d) $x + 2x^3$.

(e) $\frac{x - \frac{x^3}{3!}}{1 - \frac{x^2}{2}}$.

14. (2 marks) Let $T_3(x)$ be the degree 3 Taylor polynomial for $\sin(x)$ centred at 0. Let $Q_3(x)$ be the degree 3 Taylor polynomial for $\sin(x)$ centred at π .

Which of these polynomials gives a better estimate for $\sin(x)$ at $x = \frac{\pi}{2}$?

(a) $T_3(x)$ gives a better estimate.

(b) $Q_3(x)$ gives a better estimate.

(c) $T_3(x)$ and $Q_3(x)$ give the same estimate.

(d) $T_3(x)$ and $Q_3(x)$ give equally good estimates, but $T_3(\frac{\pi}{2}) \neq Q_3(\frac{\pi}{2})$.

(e) Trick question: One of these polynomials is not defined.

PART (B): Long Answer Questions (15 marks).

Instructions: Answer Questions 1,2,3 in the space provided. Be sure to show all of your work, as partially correct answers may be worth partial credit. Each question is worth 5 marks.

1. (5 marks) Suppose that $\sum_{n=0}^{\infty} a_n$ converges, and $0 < a_n < \frac{\pi}{2}$ for each a_n . Show that

$$\sum_{n=0}^{\infty} \tan(a_n)$$

converges.

Hint. This is very similar to the $\sin(a_n)$ version in lecture. Set up and use the Limit comparison test.

2. (5 marks) A piece of paper has thickness 0.05mm (millimeters). If you fold it once it will have thickness 0.1mm, and if you fold it again it will have thickness 0.2mm, etc..
- (a.) (3 marks) Give a formula for how thick the paper will be after k folds ($k = 0, 1, 2, \dots$).

ANSWER: After k folds the paper will have thickness: _____ mm

- (b.) (2 marks) Show that after 57 folds the paper will have thickness greater than the thickness of the Earth ($\approx 1.3 \times 10^{10}$ mm). You may use that $\ln(0.05) \approx -3$, $\ln(2) \approx \frac{1}{2}$, $\ln(10) \approx 2.5$ and $\ln(1.3) \approx \frac{1}{4}$.

3. (5 marks) Show that the following series converges:

$$\sum_{n=1}^{\infty} \frac{\left(1 + \frac{1}{n}\right)^n}{1 + 3 + 9 + \dots + 3^{n-1}}.$$

Hint. Figure out a closed form for the denominator. Then use an appropriate test.

End of Midterm Test 2

SCRAP WORK - NOT FOR MARKING

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MIDTERM TEST 2 - Formula sheet

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan \theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

Trigonometric Facts:

$$\tan \theta = \frac{\sin \theta}{\cos \theta}; \cot \theta = \frac{1}{\tan \theta}; \sec \theta = \frac{1}{\cos \theta}; \csc \theta = \frac{1}{\sin \theta}$$

Trigonometric Identities:

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}; \cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$$

Derivative formulas:

$$\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}$$

$$\frac{d}{dx}(\sin x) = \cos x; \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x; \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \tan x \sec x; \frac{d}{dx}(\csc x) = -\cot x \csc x$$

Some special values: $\pi \approx 3.14159\dots$, $e \approx 2.71828\dots$

Some special limits: $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x$, $\lim_{n \rightarrow \infty} \frac{\alpha^n}{n!} = 0$ for all α ,

$$\lim_{n \rightarrow \infty} (\alpha^n + b)^{\frac{1}{n}} = \alpha \text{ for all } \alpha > 1, \text{ for all } b$$

Some series

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad -1 < r < 1$$

$$\sum_{n=0}^N r^n = \frac{r^{N+1} - 1}{r - 1}, \quad r \neq 1$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, \quad -\infty < x < \infty$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n, \quad -1 < x \leq 1$$

Taylor Remainder: $|R_n(x)| \leq \max_{c \in [a,x]} \frac{|f^{(n+1)}(c)| |x-a|^{n+1}}{(n+1)!}$