STAT 3502 A **MIDTERM TEST I with solutions** **October 20, 2017**

Name:	**Student Number:**

Instructions: The test consists of 4 multiple-choice questions and 3 long-answer questions. The total of marks is 100. Marks for individual questions are given in [ ]. This is a 60-minute **closed-book** test. Non-programmable calculators are allowed. For long-answer questions, show all appropriate steps of your work; otherwise only partial marks may be awarded. In writing your answers, justify your claims in terms of results and theorems given in the lectures and the text. Formula sheet and a blank page for rough calculations are attached to the test paper.

Multiple-Choice Questions. **Please circle one answer only.**

[10] 1. A box in a supply room contains 15 compact fluorescent lightbulbs, of which 5 are rated 13-watt, 6 are rated 18-watt, and 4 are rated 23-watt. Suppose that three of these bulbs are randomly selected. What is the probability that one bulb of each type is selected?

(a) 0.1172
(b) 0.1719
$\boxed{\textbf{(c)}}$ 0.2637
(d) 0.2904

Solution: Using the classical definition of probability and the Multiplication Rule, the desired probability is equal to

$$\mathbf{P}(\text{one bulb of each type is selected}) = \frac{\binom{5}{1}\binom{6}{1}\binom{4}{1}}{\binom{5+6+4}{3}} = \frac{120}{455} = 0.2637.$$

Hence the correct answer is (c).

[10] 2. A box contains three coins with a head on each side, four coins with a tail on each side, and two fair coins. If one of these nine coins is selected at random and tossed once, what is the probability that a head will be obtained?

(a) 1/9
$\boxed{\textbf{(b)}}$ 4/9
(c) 5/9
(d) 8/9

Solution: Let A_1 denote the event that the selected coin has a head on each side, let A_2 denote the event that it has a tail on each side, let A_3 denote the event that it is fair, and let B denote the event that a head is obtained. Then

$$\mathbf{P}(A_1) = 3/9, \quad \mathbf{P}(A_2) = 4/9, \quad \mathbf{P}(A_3) = 2/9, \quad \mathbf{P}(B|A_1) = 1, \quad \mathbf{P}(B|A_2) = 0, \quad \mathbf{P}(B|A_3) = 1/2.$$

Therefore, by the Law of Total Probability,

$$\mathbf{P}(B) = \sum_{i=1}^{3} \mathbf{P}(A_i)\mathbf{P}(B|A_i) = 4/9.$$

Hence the correct answer is (b).

[10] 3. The number of requests for assistance received by a towing service is a Poisson process with rate $\alpha = 4$ per hour. If the operators of the towing service take a 30-min break, what is the probability that they do not miss any calls for assistance?

(a) 0.0015
(b) 0.0992
$\boxed{\textbf{(c)}}$ 0.1353
(d) 0.2154

Solution: If we denote by $N(t)$ the number of requests for assistance during time interval $[0, t]$, then by assumption $\{N(t), t \geq 0\}$ is a Poisson process with rate 4 (time unit = 1 hour). Since for any $t > 0$, $N(t)$ is a Poisson random variable with mean $\mu = 4t$, it follows that

$$\mathbf{P}(\text{no calls will be missed during a 30-min break}) = \mathbf{P}(N(1/2) = 0) = \frac{e^{-2}2^0}{0!} = 0.1353.$$

Hence the correct answer is (c).

[10] 4. The probability that a randomly selected box of a certain type of cereal has a particular prize is 0.2. Suppose you purchase box after box until you have obtained two of theses prizes. How many boxes do you expect to purchase?

(a) 12
$\boxed{\textbf{(b)}}$ 10
(c) 8
(d) 3

Solution: Let X be the number of boxes that do NOT contain a prize until you find 2 prizes. Then $X \sim NegBin(2, 0.2)$, and the number of boxes that you expect to purchase is

$$\mathbf{E}(2 + X) = 2 + \mathbf{E}(X) = 2 + \frac{2(1 - 0.2)}{0.2} = 2 + 8 = 10.$$

Hence the correct answer is (b).

Long-Answer Questions.

[15] 5. Let an experiment consists of throwing a balanced die twice. Let A be the event that in the second throw the die lands 1, 2, or 5; B the event that in the second throw it lands 4, 5, or 6; and C the event that the sum of the two outcomes is 9. Are the events A, B, and C independent? Why or why not?

Solution: Although

$$\mathbf{P}(A \cap B \cap C) = \frac{1}{36} = \mathbf{P}(A)\mathbf{P}(B)\mathbf{P}(C),$$

we have that

$$\mathbf{P}(A \cap B) = \frac{1}{6} \neq \frac{1}{4} = \mathbf{P}(A)\mathbf{P}(B), \quad \mathbf{P}(A \cap C) = \frac{1}{36} \neq \frac{1}{18} = \mathbf{P}(A)\mathbf{P}(C),$$

$$\mathbf{P}(B \cap C) = \frac{1}{12} \neq \frac{1}{18} = \mathbf{P}(B)\mathbf{P}(C).$$

Hence the events A, B, and C are not (mutually) independent. The same conclusion could be drawn without doing calculations by noting that our chance of getting a total of 9 depends on the outcome of the second throw.

[15] 6. A chemical supply company currently has in stock 100 lb of a certain chemical, which it sells to customers in 5-lb batches. Let X be the number of batches ordered by a randomly chosen customer, and suppose that X has the following probability mass function (pmf):

x	1	2	3	4
$p(x)$	0.2	0.4	0.3	0.1

Compute the expected number of pounds left after the next customer's order is shipped and the variance of the number of pounds left. (**Hint:** The number of pounds left is a linear function of X.)

Solution: Each lot weighs 5 lbs, so the number Y of pounds left is

$$Y = 100 - 5X.$$

First, we have

$$\mathbf{E}(X) = \sum_{x=1}^{4} xp(x) = 2.3, \quad \mathbf{E}(X) = \sum_{x=1}^{4} x^2 p(x) = 6.1,$$
$$\mathbf{Var}(X) = \mathbf{E}(X^2) - (\mathbf{E}(X))^2 = 6.1 - (2.3)^2 = 0.81.$$

Next,

$$\begin{aligned} \mathbf{E}(Y) &= \mathbf{E}(100 - 5X) = 100 - 5\mathbf{E}(X) = 88.5, \\ \mathbf{Var}(Y) &= \mathbf{Var}(100 - 5X) = 25\mathbf{Var}(X) = 20.25. \end{aligned}$$

7. Consider randomly selecting a single individual and having that person test drive 3 different vehicles. Define the events A_1, A_2, and A_3 by

$$A_1 = \{\text{the individual likes vehicle \# 1}\}, \quad A_2 = \{\text{the individual likes vehicle \# 2}\},$$
$$A_3 = \{\text{the individual likes vehicle \# 3}\},$$

Suppose that $\mathbf{P}(A_1) = 0.55$, $\mathbf{P}(A_2) = 0.65$, $\mathbf{P}(A_3) = 0.70$, $\mathbf{P}(A_1 \cup A_2) = 0.80$, $\mathbf{P}(A_2 \cap A_3) = 0.40$.

[15] a. What is the probability that the individual likes both vehicle #1 and vehicle #2?

[15] b. Determine and interpret $\mathbf{P}(A_2|A_3)$?

Solution: (a) The probability of interest is $\mathbf{P}(A_1 \cap A_2)$. It now follows from the Addition Law that

$$\mathbf{P}(A_1 \cap A_2) = \mathbf{P}(A_1) + \mathbf{P}(A_2) - \mathbf{P}(A_2 \cup A_2) = 0.55 + 0.65 - 0.80 = 0.40.$$

(b) We have

$$\mathbf{P}(A_2|A_3) = \frac{\mathbf{P}(A_2 \cap A_3)}{\mathbf{P}(A_3)} = \frac{0.40}{0.70} = 4/7 = 0.5714.$$

If a person likes vehicle #3, there is a 57.14% chance he/she will also like vehicle #2.