Solution to Midterm Examination (version A)

MAT 1322D, Fall 2011

1. (6 marks) Consider the following improper integrals:

(i) $\int_0^\infty \frac{1}{\sqrt{1+x^2}}\,dx$.

(ii) $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$;

If the integral converges, use the definition of the improper integral to find its value. If the improper integral diverges, use the comparison test to justify your conclusion.

Solution. (i) When $x > 1$, $\frac{1}{\sqrt{x^2+1}} > \frac{1}{\sqrt{x^2+x^2}} = \frac{1}{\sqrt{2}x}$. Since $\int_0^\infty \frac{1}{\sqrt{2}x}\,dx = \frac{1}{\sqrt{2}}\int_0^\infty \frac{1}{x}\,dx$ diverges, this integral diverges.

(ii) $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{b\to 1}\int_0^b \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{b\to 1}(\arcsin b - \arcsin 0) = \arcsin 1 = \frac{\pi}{2}$. This improper integral is convergent and its value is $\pi/2$.

2. (4 marks) Find the volume of the solid obtained by revolving the region in x-y plane bounded by the graphs of $y = x^2$ and $y = 3x$ about the y-axis.

Solution. The volume is $V = \pi\int_0^9 ((\sqrt{y})^2 - (y/3)^2)dy = \pi\left[\frac{y^2}{2} - \frac{y^3}{27}\right]_{y=0}^9 = \frac{27}{2}\pi$.

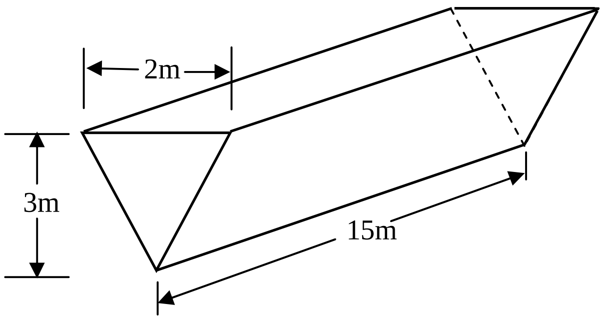
3. (3 marks) Suppose the density of a rod is $\delta(x) = e^{2x}$, $0 \le x \le 1$. Find the total mass (1 mark) and the center of mass of the rod (2 marks).

Solution. $m = \int_0^1 e^{2x}dx = \frac{1}{2}\left[e^{2x}\right]_{x=0}^1 = \frac{1}{2}e^2 - \frac{1}{2} \approx 3.1945$. The moment is

$$M = \int_0^1 xe^{2x}dx = \frac{1}{4}\left[(2x-1)e^{2x}\right]_{x=0}^1 = \frac{1}{4}(e^2+1) \approx 2.0973.$$

The center of mass is $\bar{x} = \frac{M}{m} = \frac{e^2+1}{2(e^2-1)} \approx 0.6565$.

4. (4 marks) Find the work (in Joules) needed to pump the water of a container of the shape show in the following figure to a point 4 meters above the top of the container. (The density of water is 1000 kg/m^3, and the acceleration of gravity g = 9.8 m/sec^2).



Solution. Consider a layer of water of distance x above the bottom with thickness dx. The volume of this layer is $15 \times (2/3)x\,dx = 10x\,dx$. The weight of this layer (in Newtons) is $dw = 10\delta g x dx$. To pump this layer to a point 4 meters above the top of the container, the work needed is $dW = 10\delta g x(7 - x)dx$. The total work needed is

$$W = 10\delta g \int_0^3 x(7-x)dx \approx 2.2 \times 10^6 \text{ Joule.}$$

5. (3 marks) Consider the initial-value problem $y' = \dfrac{1+ty}{y^2}$, $y(0) = 1$. Find an approximation $y(0.1)$ using step-size $h = 0.025$. (Use at least 4 digits after the decimal point in your calculation).

Solution. Use the iteration formula $y_{i+1} = y_i + 0.025\dfrac{1+t_i y_i}{y_i^2}$. We have

$t_1 = 0.025$, $y(0.025) \approx y_1 = 1.0250$,
$t_2 = 0.05$, $y(0.05) \approx y_2 = 1.49405$,
$t_3 = 0.075$, $y(0.075) \approx y_3 \approx 1.73298$,
$t_4 = 0.1$, $y(0.1) \approx y_4 \approx 1.096747$.