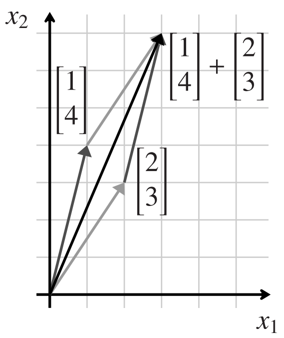
CHAPTER 1 Euclidean Vector Spaces

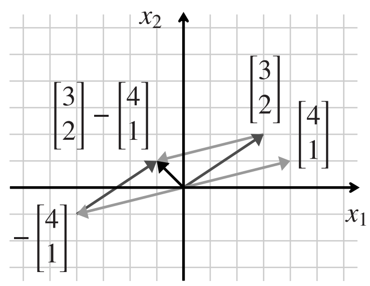
1.1 Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

Practice Problems

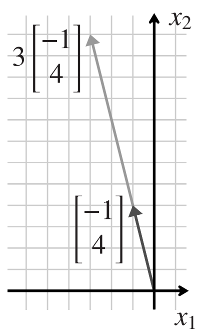
A1 (a) $\begin{bmatrix}1\\4\end{bmatrix}+\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}1+2\\4+3\end{bmatrix}=\begin{bmatrix}3\\7\end{bmatrix}$



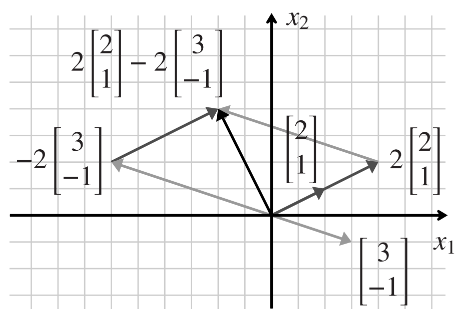
(b) $\begin{bmatrix}3\\2\end{bmatrix}-\begin{bmatrix}4\\1\end{bmatrix}=\begin{bmatrix}3-4\\2-1\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}$



(c) $3\begin{bmatrix}-1\\4\end{bmatrix}=\begin{bmatrix}3(-1)\\3(4)\end{bmatrix}=\begin{bmatrix}-3\\12\end{bmatrix}$



(d) $2\begin{bmatrix}2\\1\end{bmatrix}-2\begin{bmatrix}3\\-1\end{bmatrix}=\begin{bmatrix}4\\2\end{bmatrix}-\begin{bmatrix}6\\-2\end{bmatrix}=\begin{bmatrix}-2\\4\end{bmatrix}$



A2 (a) $\begin{bmatrix}4\\-2\end{bmatrix}+\begin{bmatrix}-1\\3\end{bmatrix}=\begin{bmatrix}4+(-1)\\-2+3\end{bmatrix}=\begin{bmatrix}3\\1\end{bmatrix}$

(b) $\begin{bmatrix}-3\\-4\end{bmatrix}-\begin{bmatrix}-2\\5\end{bmatrix}=\begin{bmatrix}-3-(-2)\\-4-5\end{bmatrix}=\begin{bmatrix}-1\\-9\end{bmatrix}$

(c) $-2\begin{bmatrix}3\\-2\end{bmatrix}=\begin{bmatrix}(-2)3\\(-2)(-2)\end{bmatrix}=\begin{bmatrix}-6\\4\end{bmatrix}$

(d) $\frac{1}{2}\begin{bmatrix}2\\6\end{bmatrix}+\frac{1}{3}\begin{bmatrix}4\\3\end{bmatrix}=\begin{bmatrix}1\\3\end{bmatrix}+\begin{bmatrix}4/3\\1\end{bmatrix}=\begin{bmatrix}7/3\\4\end{bmatrix}$

(e) $\frac{2}{3}\begin{bmatrix}3\\1\end{bmatrix}-2\begin{bmatrix}1/4\\1/3\end{bmatrix}=\begin{bmatrix}2\\2/3\end{bmatrix}-\begin{bmatrix}1/2\\2/3\end{bmatrix}=\begin{bmatrix}3/2\\0\end{bmatrix}$

(f) $\sqrt{2}\begin{bmatrix}\sqrt{2}\\\sqrt{3}\end{bmatrix}+3\begin{bmatrix}1\\\sqrt{6}\end{bmatrix}=\begin{bmatrix}2\\\sqrt{6}\end{bmatrix}+\begin{bmatrix}3\\3\sqrt{6}\end{bmatrix}=\begin{bmatrix}5\\4\sqrt{6}\end{bmatrix}$

A3 (a) $\begin{bmatrix}2\\3\\4\end{bmatrix} - \begin{bmatrix}5\\1\\-2\end{bmatrix} = \begin{bmatrix}2-5\\3-1\\4-(-2)\end{bmatrix} = \begin{bmatrix}-3\\2\\6\end{bmatrix}$

(b) $\begin{bmatrix}2\\1\\-6\end{bmatrix} + \begin{bmatrix}-3\\1\\-4\end{bmatrix} = \begin{bmatrix}2+(-3)\\1+1\\-6+(-4)\end{bmatrix} = \begin{bmatrix}-1\\2\\-10\end{bmatrix}$

(c) $-6\begin{bmatrix}4\\-5\\-6\end{bmatrix} = \begin{bmatrix}(-6)4\\(-6)(-5)\\(-6)(-6)\end{bmatrix} = \begin{bmatrix}-24\\30\\36\end{bmatrix}$

(d) $-2\begin{bmatrix}-5\\1\\1\end{bmatrix} + 3\begin{bmatrix}-1\\0\\-1\end{bmatrix} = \begin{bmatrix}10\\-2\\-2\end{bmatrix} + \begin{bmatrix}-3\\0\\-3\end{bmatrix} = \begin{bmatrix}7\\-2\\-5\end{bmatrix}$

(e) $2\begin{bmatrix}2/3\\-1/3\\2\end{bmatrix} + \frac{1}{3}\begin{bmatrix}3\\-2\\1\end{bmatrix} = \begin{bmatrix}4/3\\-2/3\\4\end{bmatrix} + \begin{bmatrix}1\\-2/3\\1/3\end{bmatrix} = \begin{bmatrix}7/3\\-4/3\\13/3\end{bmatrix}$

(f) $\sqrt{2}\begin{bmatrix}1\\1\\1\end{bmatrix} + \pi\begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}\sqrt{2}\\\sqrt{2}\\\sqrt{2}\end{bmatrix} + \begin{bmatrix}-\pi\\0\\\pi\end{bmatrix} = \begin{bmatrix}\sqrt{2}-\pi\\\sqrt{2}\\\sqrt{2}+\pi\end{bmatrix}$

A4 (a) $2\vec{v} - 3\vec{w} = \begin{bmatrix}2\\4\\-4\end{bmatrix} - \begin{bmatrix}6\\-3\\9\end{bmatrix} = \begin{bmatrix}-4\\7\\-13\end{bmatrix}$

(b) $-3(\vec{v} + 2\vec{w}) + 5\vec{v} = -3\left(\begin{bmatrix}1\\2\\-2\end{bmatrix} + \begin{bmatrix}4\\-2\\6\end{bmatrix}\right) + \begin{bmatrix}5\\10\\-10\end{bmatrix} = -3\begin{bmatrix}5\\0\\4\end{bmatrix} + \begin{bmatrix}5\\10\\-10\end{bmatrix} = \begin{bmatrix}-15\\0\\-12\end{bmatrix} + \begin{bmatrix}5\\10\\-10\end{bmatrix} = \begin{bmatrix}-10\\10\\-22\end{bmatrix}$

(c) We have $\vec{w} - 2\vec{u} = 3\vec{v}$, so $2\vec{u} = \vec{w} - 3\vec{v}$ or $\vec{u} = \frac{1}{2}(\vec{w} - 3\vec{v})$. This gives

$$\vec{u} = \frac{1}{2}\left(\begin{bmatrix}2\\-1\\3\end{bmatrix} - \begin{bmatrix}3\\6\\-6\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}-1\\-7\\9\end{bmatrix} = \begin{bmatrix}-1/2\\-7/2\\9/2\end{bmatrix}$$

(d) We have $\vec{u} - 3\vec{v} = 2\vec{u}$, so $\vec{u} = -3\vec{v} = \begin{bmatrix}-3\\-6\\6\end{bmatrix}$.

A5 (a) $\frac{1}{2}\vec{v} + \frac{1}{2}\vec{w} = \begin{bmatrix}3/2\\1/2\\1/2\end{bmatrix} + \begin{bmatrix}5/2\\-1/2\\-1\end{bmatrix} = \begin{bmatrix}4\\0\\-1/2\end{bmatrix}$

(b) $2(\vec{v} + \vec{w}) - (2\vec{v} - 3\vec{w}) = 2\begin{bmatrix}8\\0\\-1\end{bmatrix} - \left(\begin{bmatrix}6\\2\\2\end{bmatrix} - \begin{bmatrix}15\\-3\\-6\end{bmatrix}\right) = \begin{bmatrix}16\\0\\-2\end{bmatrix} - \begin{bmatrix}-9\\5\\8\end{bmatrix} = \begin{bmatrix}25\\-5\\-10\end{bmatrix}$

(c) We have $\vec{w} - \vec{u} = 2\vec{v}$, so $\vec{u} = \vec{w} - 2\vec{v}$. This gives $\vec{u} = \begin{bmatrix}5\\-1\\-2\end{bmatrix} - \begin{bmatrix}6\\2\\2\end{bmatrix} = \begin{bmatrix}-1\\-3\\-4\end{bmatrix}$.

(d) We have $\frac{1}{2}\vec{u} + \frac{1}{3}\vec{v} = \vec{w}$, so $\frac{1}{2}\vec{u} = \vec{w} - \frac{1}{3}\vec{v}$, or $\vec{u} = 2\vec{w} - \frac{2}{3}\vec{v} = \begin{bmatrix} 10 \\ -2 \\ -4 \end{bmatrix} - \begin{bmatrix} 2 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 8 \\ -8/3 \\ -14/3 \end{bmatrix}$.

A6

$$\vec{PQ} = \vec{OQ} - \vec{OP} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$$

$$\vec{PR} = \vec{OR} - \vec{OP} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{PS} = \vec{OS} - \vec{OP} = \begin{bmatrix} -5 \\ 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 \\ -2 \\ 4 \end{bmatrix}$$

$$\vec{QR} = \vec{OR} - \vec{OQ} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}$$

$$\vec{SR} = \vec{OR} - \vec{OS} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} - \begin{bmatrix} -5 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -5 \end{bmatrix}$$

Thus,

$$\vec{PQ} + \vec{QR} = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ -2 \\ 4 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \\ -5 \end{bmatrix} = \vec{PS} + \vec{SR}$$

A7 (a) The equation of the line is $\vec{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + t\begin{bmatrix} -5 \\ 1 \end{bmatrix}, t \in \mathbb{R}$

(b) The equation of the line is $\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + t\begin{bmatrix} -4 \\ -6 \end{bmatrix}, t \in \mathbb{R}$

(c) The equation of the line is $\vec{x} = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} + t\begin{bmatrix} 4 \\ -2 \\ -11 \end{bmatrix}, t \in \mathbb{R}$

(d) The equation of the line is $\vec{x} = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}, t \in \mathbb{R}$

A8 Note that alternative correct answers are possible.

(a) The direction vector $\vec{d}$ of the line is given by the directed line segment joining the two points: $\vec{d} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$. This, along with one of the points, may be used to obtain an equation for the line

$$\vec{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + t\begin{bmatrix} 3 \\ -5 \end{bmatrix}, \quad t \in \mathbb{R}$$

(b) The direction vector $\vec{d}$ of the line is given by the directed line segment joining the two points: $\vec{d} = \begin{bmatrix} -2 \\ -1 \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -2 \end{bmatrix}$. This, along with one of the points, may be used to obtain an equation for the line

$$\vec{x} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + t \begin{bmatrix} -6 \\ -2 \end{bmatrix}, \quad t \in \mathbb{R}$$

(c) The direction vector $\vec{d}$ of the line is given by the directed line segment joining the two points: $\vec{d} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 5 \end{bmatrix}$. This, along with one of the points, may be used to obtain an equation for the line

$$\vec{x} = \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix} + t \begin{bmatrix} -3 \\ -2 \\ 5 \end{bmatrix}, \quad t \in \mathbb{R}$$

(d) The direction vector $\vec{d}$ of the line is given by the directed line segment joining the two points: $\vec{d} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 1 \end{bmatrix}$. This, along with one of the points, may be used to obtain an equation for the line

$$\vec{x} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 6 \\ 1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

(e) The direction vector $\vec{d}$ of the line is given by the directed line segment joining the two points: $\vec{d} = \begin{bmatrix} -1 \\ 1 \\ 1/3 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/4 \\ 1 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 3/4 \\ -2/3 \end{bmatrix}$. This, along with one of the points, may be used to obtain an equation for the line

$$\vec{x} = \begin{bmatrix} 1/2 \\ 1/4 \\ 1 \end{bmatrix} + t \begin{bmatrix} -3/2 \\ 3/4 \\ -2/3 \end{bmatrix}, \quad t \in \mathbb{R}$$

A9 (a) We have

$$\begin{aligned} x_2 &= 3x_1 + 2 \\ x_2 + 1 &= 3x_1 + 3 \\ x_2 + 1 &= 3(x_1 + 1) \end{aligned}$$

Let $t = x_1 + 1$. Then, from the equation above we get $x_2 + 1 = 3t$. Solving the equations for x_1 and x_2 we find that the parametric equations are $x_1 = -1 + t$, $x_2 = -1 + 3t$, $t \in \mathbb{R}$ and the corresponding vector equation is $\vec{x} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $t \in \mathbb{R}$.

(b) We have

$$\begin{aligned} 2x_1 + 3x_2 &= 5 \\ 2x_1 - 2 &= -3x_2 + 3 \\ 2(x_1 - 1) &= -3(x_2 - 1) \\ \frac{1}{3}(x_1 - 1) &= -\frac{1}{2}(x_2 - 1) \end{aligned}$$

Let $t = -\frac{x_2 - 1}{2}$. Then, $\frac{1}{3}(x_1 - 1) = t$. Solving the equations for x_1 and x_2 we find that the parametric equations are $x_1 = 1 + 3t$, $x_2 = 1 - 2t$, $t \in \mathbb{R}$ and the corresponding vector equation is $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, $t \in \mathbb{R}$

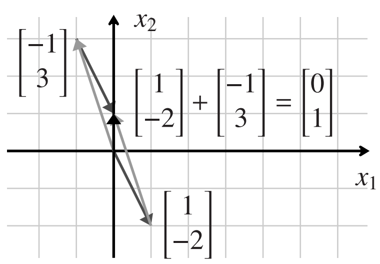
A10 (a) Let P, Q, and R be three points in $\mathbb{R}^n$, with corresponding vectors $\vec{p}$, $\vec{q}$, and $\vec{r}$. If P, Q, and R are collinear, then the directed line segments $\vec{PQ}$ and $\vec{PR}$ should define the same line. That is, the direction vector of one should be a non-zero scalar multiple of the direction vector of the other. Therefore, $\vec{PQ} = t\vec{PR}$, for some $t \in \mathbb{R}$.

(b) We have $\vec{PQ} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\vec{PR} = \begin{bmatrix} -5 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \end{bmatrix} = -2\vec{PQ}$, so they are collinear.

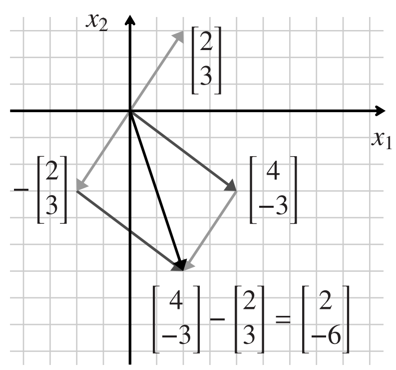
(c) We have $\vec{ST} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$ and $\vec{SU} = \begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ -2 \end{bmatrix}$. Therefore, the points S, T, and U are not collinear because $\vec{SU} \neq t\vec{ST}$ for any real number t.

Homework Problems

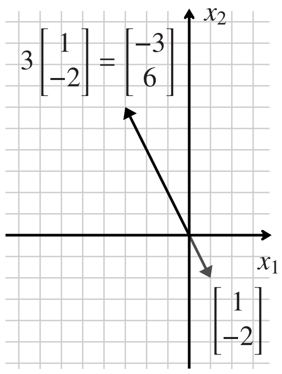
B1 (a)



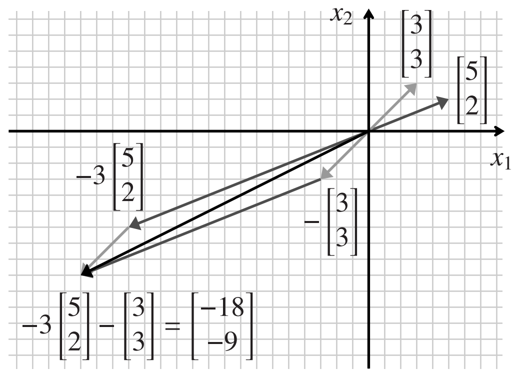
(b)



(c)



(d)



B2 (a) $\begin{bmatrix}0\\0\end{bmatrix}$ (b) $\begin{bmatrix}0\\0\end{bmatrix}$ (c) $\begin{bmatrix}-4\\-2\end{bmatrix}$ (d) $\begin{bmatrix}0\\27/5\end{bmatrix}$ (e) $\begin{bmatrix}2\sqrt{3}\\\sqrt{2}-3\sqrt{3}/2\end{bmatrix}$

B3 (a) $\begin{bmatrix}3\\-3\\-6\end{bmatrix}$ (b) $\begin{bmatrix}6\\0\\0\end{bmatrix}$ (c) $\begin{bmatrix}8\\-20\\-4\end{bmatrix}$ (d) $\begin{bmatrix}0\\0\\0\end{bmatrix}$ (e) $\begin{bmatrix}7/3\\-4/3\\13/3\end{bmatrix}$ (f) $\begin{bmatrix}0\\0\\0\end{bmatrix}$

B4 (a) $\begin{bmatrix}21\\4\\-5\end{bmatrix}$ (b) $\begin{bmatrix}-1\\-12\\1\end{bmatrix}$ (c) $\begin{bmatrix}-7\\-13/2\\2\end{bmatrix}$ (d) $\begin{bmatrix}9\\-1/2\\-2\end{bmatrix}$

B5 (a) $\begin{bmatrix}-2\\0\\4\end{bmatrix}$ (b) $\begin{bmatrix}0\\1/4\\-3/4\end{bmatrix}$ (c) $\vec{u}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ (d) $\vec{u}=\begin{bmatrix}4\\-7/2\\5/2\end{bmatrix}$

B6 (a) $\vec{PQ}=\begin{bmatrix}3\\-1\\-2\end{bmatrix}$, $\vec{PR}=\begin{bmatrix}-2\\0\\1\end{bmatrix}$, $\vec{PS}=\begin{bmatrix}7\\2\\-6\end{bmatrix}$, $\vec{QR}=\begin{bmatrix}-5\\1\\3\end{bmatrix}$, $\vec{SR}=\begin{bmatrix}-9\\-2\\7\end{bmatrix}$

(b) $\vec{PQ}=\begin{bmatrix}-1\\9\\-4\end{bmatrix}$, $\vec{PR}=\begin{bmatrix}0\\3\\4\end{bmatrix}$, $\vec{PS}=\begin{bmatrix}-5\\6\\-2\end{bmatrix}$, $\vec{QR}=\begin{bmatrix}1\\-6\\8\end{bmatrix}$, $\vec{SR}=\begin{bmatrix}5\\-3\\6\end{bmatrix}$

B7

(a) $\vec{x}=\begin{bmatrix}-3\\4\end{bmatrix}+t\begin{bmatrix}4\\-3\end{bmatrix}, t\in\mathbb{R}.$ (b) $\vec{x}=t\begin{bmatrix}1\\3\\1\end{bmatrix}, t\in\mathbb{R}.$

(c) $\vec{x}=\begin{bmatrix}2\\3\\-1\end{bmatrix}+t\begin{bmatrix}2\\-4\\8\end{bmatrix}, t\in\mathbb{R}.$ (d) $\vec{x}=\begin{bmatrix}3\\1\\2\end{bmatrix}+t\begin{bmatrix}2\\-3\\2\end{bmatrix}, t\in\mathbb{R}.$

B8

(a) $\vec{x}=\begin{bmatrix}3\\1\end{bmatrix}+t\begin{bmatrix}-2\\1\end{bmatrix}, \quad t\in\mathbb{R}$ (b) $\vec{x}=t\begin{bmatrix}-1\\2\\-1\end{bmatrix}, \quad t\in\mathbb{R}$

(c) $\vec{x}=\begin{bmatrix}2\\-6\\3\end{bmatrix}+t\begin{bmatrix}-3\\11\\-1\end{bmatrix}, \quad t\in\mathbb{R}$ (d) $\vec{x}=\begin{bmatrix}1\\-1\\1/2\end{bmatrix}+t\begin{bmatrix}-1/2\\4/3\\1/2\end{bmatrix}, \quad t\in\mathbb{R}$

B9 Other correct answers are possible.

(a) $x_1 = -\frac{1}{2}t + \frac{3}{2}$, $x_2 = t$; $\vec{x} = \begin{bmatrix} 3/2 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$

(b) $x_1 = t$, $x_2 = -\frac{1}{2}t + \frac{3}{2}$; $\vec{x} = \begin{bmatrix} 0 \\ 3/2 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$

B10 (a) Since $-2\vec{PQ} = \begin{bmatrix} 2 \\ -2 \\ -4 \end{bmatrix} = \vec{PR}$, the point P, Q, and R must be collinear.

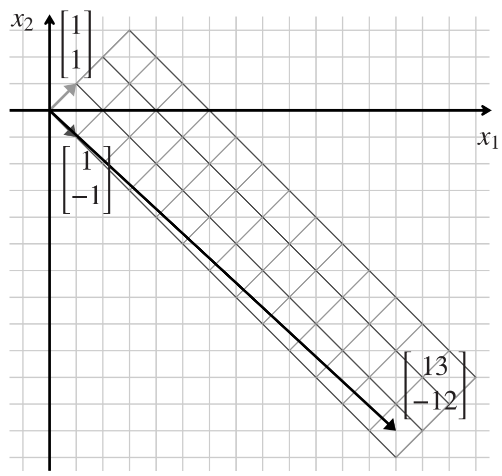
(b) Since $-\vec{ST} = \begin{bmatrix} -5 \\ -1 \\ -1 \end{bmatrix} = \vec{SU}$, the point S, T, and U must be collinear.

Computer Problems

C1 (a) $\begin{bmatrix} -2322 \\ -1761 \\ 1667 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Conceptual Problems

D1 (a) We need to find t_1 and t_2 such that $\begin{bmatrix} 13 \\ -12 \end{bmatrix} = t_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + t_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} t_1 + t_2 \\ t_1 - t_2 \end{bmatrix}$.
That is, we need to solve the two equations in two unknowns $t_1 + t_2 = 13$ and $t_1 - t_2 = -12$. Using substitution and/or elimination we find that $t_1 = \frac{1}{2}$ and $t_2 = \frac{25}{2}$.



(b) We use the same approach as in part (a). We need to find t_1 and t_2 such that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + t_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} t_1 + t_2 \\ t_1 - t_2 \end{bmatrix}$$

Solving $t_1 + t_2 = x_1$ and $t_1 - t_2 = x_2$ by substitution and/or elimination gives $t_1 = \frac{1}{2}(x_1 + x_2)$ and $t_2 = \frac{1}{2}(x_1 - x_2)$.

(c) We have $x_1 = \sqrt{2}$ and $x_2 = \pi$, so we get $t_1 = \frac{1}{2}(\sqrt{2} + \pi)$ and $t_2 = \frac{1}{2}(\sqrt{2} - \pi)$.

D2 (a) $\vec{PQ} + \vec{QR} + \vec{RP}$ can be described informally as "start at P and move to Q, then move from Q to R, then from R to P; the net result is a zero change in position."

(b) We have $\vec{PQ} = \vec{q} - \vec{p}$, $\vec{QR} = \vec{r} - \vec{q}$, and $\vec{RP} = \vec{p} - \vec{r}$. Thus,

$$\vec{PQ} + \vec{QR} + \vec{RP} = \vec{q} - \vec{p} + \vec{r} - \vec{q} + \vec{p} - \vec{r} = \vec{0}$$

D3 Assume that $\vec{x} = \vec{p} + t\vec{d}$, $t \in \mathbb{R}$, is a line in $\mathbb{R}^2$ passing through the origin. Then, there exists a real number t_1 such that $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{p} + t_1\vec{d}$. Hence, $\vec{p} = -t_1\vec{d}$ and so $\vec{p}$ is a scalar multiple of $\vec{d}$. On the other hand, assume that $\vec{p}$ is a scalar multiple of $\vec{d}$. Then, there exists a real number t_1 such that $\vec{p} = t_1\vec{d}$. Hence, if we take $t = -t_1$, we get that the line with vector equation $\vec{x} = \vec{p} + t\vec{d}$ passes through the point $\vec{p} + (-t_1)\vec{d} = t_1\vec{d} - t_1\vec{d} = \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as required.

D4 Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. Then,

$$t(\vec{x} + \vec{y}) = t\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix} = \begin{bmatrix} t(x_1 + y_1) \\ t(x_2 + y_2) \\ t(x_3 + y_3) \end{bmatrix} = \begin{bmatrix} tx_1 + ty_1 \\ tx_2 + ty_2 \\ tx_3 + ty_3 \end{bmatrix} = t\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + t\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = t\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + t\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = t\vec{x} + t\vec{y}$$

1.2 Vectors in $\mathbb{R}^n$

Practice Problems

A1 (a) $\begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 4 \\ 6 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ 0 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ -2 \\ 5 \\ 1 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 1 \\ 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 3 \\ -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} -3 \\ 3 \\ 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 6 \\ -2 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ -7 \\ 10 \\ -5 \end{bmatrix}$

(c) $2\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -2 \\ 1 \\ 2 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} 4 \\ -4 \\ 2 \\ 4 \\ 2 \end{bmatrix} - \begin{bmatrix} 6 \\ 0 \\ 3 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 3 \end{bmatrix}$

A2 (a) Since the condition of the set contains the square of a variable in it, we suspect that it is not a subspace. To prove it is not a subspace we just need to find one example where the set is not closed under addition.

Let $\vec{x} = \begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\vec{y} = \begin{bmatrix}2\\1\\3\end{bmatrix}$. Observe that $\vec{x}$ and $\vec{y}$ are in the set since $x_1^2 - x_2^2 = 1^2 - 1^2 = 0 = x_3$ and $y_1^2 - y_2^2 = 2^2 - 1^2 = 3 = y_3$, but $\vec{x}+\vec{y} = \begin{bmatrix}3\\2\\3\end{bmatrix}$ is not in the set since $3^2 - 2^2 = 5 \neq 3$.

(b) Since the condition of the set only contains linear variables, we suspect that this is a subspace. To prove it is a subspace we need to show that it satisfies the definition of a subspace.
Call the set S. First, observe that S is a subset of $\mathbb{R}^3$ and is non-empty since the zero vector satisfies the conditions of the set. Pick any vectors $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ and $\vec{y} = \begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$ in S. Then they must satisfy the condition of S, so $x_1 = x_3$ and $y_1 = y_3$. We now need to show that $\vec{x}+\vec{y} = \begin{bmatrix}x_1+y_1\\x_2+y_2\\x_3+y_3\end{bmatrix}$ satisfies the conditions of the set. In particular, we need to show that the first entry of $\vec{x}+\vec{y}$ equals its third entry. Since $x_1 = x_3$ and $y_1 = y_3$ we get $x_1 + y_1 = x_3 + y_3$ as required. Thus, S is closed under addition. Similarly, to show S is closed under scalar multiplication, we let t be any real number and show that $t\vec{x} = \begin{bmatrix}tx_1\\tx_2\\tx_3\end{bmatrix}$ satisfies the conditions of the set. Using $x_1 = x_3$ we get $tx_1 = tx_3$ as required. Thus, S is a subspace of $\mathbb{R}^3$.

(c) Since the condition of the set only contains linear variables, we suspect that this is a subspace. Call the set S. First, observe that S is a subset of $\mathbb{R}^2$ and is non-empty since the zero vector satisfies the conditions of the set. Pick any vectors $\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$ and $\vec{y} = \begin{bmatrix}y_1\\y_2\end{bmatrix}$ in S. Then they must satisfy the condition of S, so $x_1 + x_2 = 0$ and $y_1 + y_2 = 0$. Then $\vec{x}+\vec{y} = \begin{bmatrix}x_1+y_1\\x_2+y_2\end{bmatrix}$ satisfies the conditions of the set since $(x_1 + y_1) + (x_2 + y_2) = x_1 + x_2 + y_1 + y_2 = 0 + 0 = 0$. Thus, S is closed under addition. Similarly, for any real number t we have that $t\vec{x} = \begin{bmatrix}tx_1\\tx_2\end{bmatrix}$ and $tx_1 + tx_2 = t(x_1 + x_2) = t(0) = 0$, so S is also closed under scalar multiplication. Thus, S is a subspace of $\mathbb{R}^2$.

(d) The condition of the set involves multiplication of entries, so we suspect that it is not a subspace. Observe that if $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}$, then $\vec{x}$ is in the set since $x_1x_2 = 1(1) = 1 = x_3$, but $2\vec{x} = \begin{bmatrix}2\\2\\2\end{bmatrix}$ is not in the set since $2(2) = 4 \neq 2$. Therefore, the set is not a subspace.

(e) At first glance this might not seem like a subspace since we are adding the vector $\begin{bmatrix}2\\3\\4\end{bmatrix}$. However, the key observation to make is that $\begin{bmatrix}2\\3\\4\end{bmatrix}$ is equal to $1\begin{bmatrix}1\\1\\1\end{bmatrix} + 1\begin{bmatrix}1\\2\\3\end{bmatrix}$. In particular, this is the equation of a plane in $\mathbb{R}^3$ which passes through the origin. So, this should be a subspace of $\mathbb{R}^3$. We could use the definition of a subspace to prove this, but the point of proving theorems is to make problems easier. Therefore, we instead observe that this is a vector equation of the set Span $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}\right\}$ and hence is a subspace by Theorem 1.2.2.

(f) The set is a subspace of $\mathbb{R}^4$ by Theorem 1.2.2.

A3 (a) Since the condition of the set only contains linear variables, we suspect that this is a subspace. Call the set S. By definition S is a subset of $\mathbb{R}^4$ and is non-empty since the zero vector satisfies the conditions of the set. Pick any vectors $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$ and $\vec{y} = \begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}$ in S, then $x_1 + x_2 + x_3 + x_4 = 0$ and $y_1 + y_2 + y_3 + y_4 = 0$. We have $\vec{x} + \vec{y} = \begin{bmatrix}x_1 + y_1\\x_2 + y_2\\x_3 + y_3\\x_4 + y_4\end{bmatrix}$ satisfies the conditions of the set since $(x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) + (x_4 + y_4) = x_1 + x_2 + x_3 + x_4 + y_1 + y_2 + y_3 + y_4 = 0 + 0 = 0$. Thus, S is closed under addition. Similarly, for any real number t we have that $t\vec{x} = \begin{bmatrix}tx_1\\tx_2\\tx_3\\tx_4\end{bmatrix}$ and $tx_1 + tx_2 + tx_3 + tx_4 = t(x_1 + x_2 + x_3 + x_4) = t(0) = 0$, so S is also closed under scalar multiplication. Thus, S is a subspace of $\mathbb{R}^4$.

(b) The set clearly does not contain the zero vector and hence cannot be a subspace.

(c) The conditions of the set only contain linear variables, but we notice that the first equation $x_1 + 2x_3 = 5$ excludes $x_1 = x_3 = 0$. Hence the zero vector is not in the set so it is not a subspace.

(d) The conditions of the set involve a multiplication of variables, so we suspect that it is not a subspace. Using the knowledge gained from problem A2(d) we take $\vec{x} = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$. Then, $\vec{x}$ is in the set since $x_1 = 1 = 1(1) = x_3x_4$ and $x_2 - x_4 = 1 - 1 = 0$. But, $2\vec{x} = \begin{bmatrix}2\\2\\2\\2\end{bmatrix}$ is not in the set since $2 \neq 2(2)$.

(e) Since the conditions of the set only contains linear variables, we suspect that this is a subspace. Call the set S. By definition S is a subset of $\mathbb{R}^4$ and is non-empty since the zero vector satisfies the conditions of the set. Pick any vectors $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$ and $\vec{y} = \begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}$ in S, then $2x_1 = 3x_4$, $x_2 - 5x_3 = 0$, $2y_1 = 3y_4$, and $y_2 - 5y_3 = 0$. We have $\vec{x} + \vec{y} = \begin{bmatrix}x_1 + y_1\\x_2 + y_2\\x_3 + y_3\\x_4 + y_4\end{bmatrix}$ satisfies the conditions of the set since $2(x_1 + y_1) = 2x_1 + 2y_1 = 3x_4 + 3y_4 = 3(x_4 + y_4)$ and $(x_2 + y_2) - 5(x_3 - y_3) = x_2 - 5x_3 + y_2 - 5y_3 = 0 + 0 = 0$. Thus, S is closed under addition. Similarly, for any real number t we have that $t\vec{x} = \begin{bmatrix} tx_1 \\ tx_2 \\ tx_3 \\ tx_4 \end{bmatrix}$, $2(tx_1) = t(2x_1) = t(3x_4) = 3(tx_4)$, and $(tx_2) - t(5x_3) = t(x_2 - 5x_3) = t(0) = 0$. Therefore, S is also closed under scalar multiplication. Thus, S is a subspace of $\mathbb{R}^4$.

(f) Since $x_3 = 2$ the zero vector cannot be in the set, so it is not a subspace.

A4 Alternative correct answers are possible.

(a) $1\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(b) $0\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(c) $1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(d) $1\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

A5 Alternative correct answers are possible.

(a) We observe that neither vector is a scalar multiple of the other. Hence, this is a linearly independent set of two vectors in $\mathbb{R}^4$. Hence, it is a plane in $\mathbb{R}^4$ with basis $\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}\right\}$.

(b) This is a subset of the standard basis for $\mathbb{R}^4$ and hence is a linearly independent set of three vectors in $\mathbb{R}^4$. Hence, it is a hyperplane in $\mathbb{R}^4$ with basis $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$.

(c) Observe that the second and third vectors are just scalar multiples of the first vector. Hence, by Theorem 1.2.3, we can write

$$\operatorname{Span}\left\{\begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -2 \\ 0 \end{bmatrix}\right\} = \operatorname{Span}\left\{\begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix}\right\}$$

Therefore, it is a line in $\mathbb{R}^4$ with basis $\left\{\begin{bmatrix} 3 \\ 1 \\ -1 \\ 0 \end{bmatrix}\right\}$.

(d) Observe that the third vector is the sum of the first two vectors. hence, by Theorem 1.2.3 we can write

$$\operatorname{Span}\left\{\begin{bmatrix}1\\1\\0\\2\end{bmatrix},\begin{bmatrix}1\\0\\0\\-1\end{bmatrix},\begin{bmatrix}2\\1\\0\\1\end{bmatrix}\right\} = \operatorname{Span}\left\{\begin{bmatrix}1\\1\\0\\2\end{bmatrix},\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}\right\}$$

Since $\left\{\begin{bmatrix}1\\1\\0\\2\end{bmatrix},\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}\right\}$ is linearly independent, we get that it spans a plane in $\mathbb{R}^4$ with basis $\left\{\begin{bmatrix}1\\1\\0\\2\end{bmatrix},\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}\right\}$.

A6 If $\vec{x} = \vec{p} + t\vec{d}$ is a subspace of $\mathbb{R}^n$, then it contains the zero vector. Hence, there exists t_1 such that $\vec{0} = \vec{p} + t_1\vec{d}$. Thus, $\vec{p} = -t_1\vec{d}$ and so $\vec{p}$ is a scalar multiple of $\vec{d}$. On the other hand, if $\vec{p}$ is a scalar multiple of $\vec{d}$, say $\vec{p} = t_1\vec{d}$, then we have $\vec{x} = \vec{p} + t\vec{d} = t_1\vec{d} + t\vec{d} = (t_1 + t)\vec{d}$. Hence, the set is $\operatorname{Span}\{\vec{d}\}$ and thus is a subspace.

A7 Assume there is a non-empty subset $\mathcal{B}_1 = \{\vec{v}_1, \dots, \vec{v}_\ell\}$ of $\mathcal{B}$ that is linearly dependent. Then there exists c_i not all zero such that

$$\vec{0} = c_1\vec{v}_1 + \cdots + c_\ell\vec{v}_\ell = c_1\vec{v}_1 + \cdots + c_\ell\vec{v}_\ell + 0\vec{v}_{\ell+1} + \cdots + 0\vec{v}_n$$

which contradicts the fact that $\mathcal{B}$ is linearly independent. Hence, $\mathcal{B}_1$ must be linearly independent.

Homework Problems

B1 (a) $\begin{bmatrix}-2\\3\\-2\\5\end{bmatrix}$ (b) $\begin{bmatrix}10\\2\\13\\10\\-1\end{bmatrix}$ (c) $\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}$

B2 (a) It is not a subspace of $\mathbb{R}^2$.

(b) It is not a subspace of $\mathbb{R}^3$.

(c) It is a subspace of $\mathbb{R}^3$.

(d) It is not a subspace of $\mathbb{R}^3$.

(e) It is a subspace of $\mathbb{R}^3$.

(f) It is a subspace of $\mathbb{R}^3$.

B3 (a) It is not a subspace of $\mathbb{R}^4$.

(b) It is not a subspace of $\mathbb{R}^4$.

(c) It is a subspace of $\mathbb{R}^4$.

(d) It is a subspace of $\mathbb{R}^4$.

(e) It is not a subspace of $\mathbb{R}^4$.

(f) It is a subspace of $\mathbb{R}^4$.

B4 Alternative correct answers are possible.

(a) $(-2)\begin{bmatrix}1\\1\\1\end{bmatrix}+\begin{bmatrix}2\\2\\2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

(b) $2\begin{bmatrix}3\\-2\\4\end{bmatrix}+0\begin{bmatrix}1\\2\\-1\end{bmatrix}+\begin{bmatrix}-6\\4\\-8\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

(c) $1\begin{bmatrix}2\\1\\2\end{bmatrix}-\begin{bmatrix}5\\4\\5\end{bmatrix}+3\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

(d) $4\begin{bmatrix}3\\1\end{bmatrix}-7\begin{bmatrix}1\\2\end{bmatrix}+5\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

B5 Alternative correct answers are possible.

(a) The line in $\mathbb{R}^4$ with basis $\left\{\begin{bmatrix}2\\-1\\1\\2\end{bmatrix}\right\}$.

(b) The hyperplane in $\mathbb{R}^4$ with basis $\left\{\begin{bmatrix}2\\0\\0\\1\end{bmatrix},\begin{bmatrix}0\\2\\0\\1\end{bmatrix},\begin{bmatrix}0\\0\\2\\1\end{bmatrix}\right\}$.

(c) The plane in $\mathbb{R}^4$ with basis $\left\{\begin{bmatrix}1\\2\\2\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\1\end{bmatrix}\right\}$.

(d) The plane in $\mathbb{R}^4$ with basis $\left\{\begin{bmatrix}3\\1\\2\\1\end{bmatrix},\begin{bmatrix}1\\1\\2\\1\end{bmatrix}\right\}$.

Computer Problems

C1 (a) $\begin{bmatrix}-53.72\\17.45\\13.16\\2.88\end{bmatrix}$ (b) $\begin{bmatrix}-16.66902697\\11.82240799\\3.147235505\\-1.434240231\end{bmatrix}$

Conceptual Problems

D1 Let $\vec{x}=\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}$ and let $s,t\in\mathbb{R}$. Then

$$(s+t)\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}=\begin{bmatrix}(s+t)x_1\\\vdots\\(s+t)x_n\end{bmatrix}=\begin{bmatrix}sx_1+tx_1\\\vdots\\sx_n+tx_n\end{bmatrix}=\begin{bmatrix}sx_1\\\vdots\\sx_n\end{bmatrix}+\begin{bmatrix}tx_1\\\vdots\\tx_n\end{bmatrix}=s\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}+t\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}$$

D2 Let $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, and let $t \in \mathbb{R}$.

$$t(\vec{x}+\vec{y}) = t\begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} = \begin{bmatrix} t(x_1 + y_1) \\ \vdots \\ t(x_n + y_n) \end{bmatrix} = \begin{bmatrix} tx_1 + ty_1 \\ \vdots \\ tx_n + ty_n \end{bmatrix}$$
$$= \begin{bmatrix} tx_1 \\ \vdots \\ tx_n \end{bmatrix} + \begin{bmatrix} ty_1 \\ \vdots \\ ty_n \end{bmatrix} = t\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + t\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = t\vec{x} + t\vec{y}$$

D3 (a) By definition $U \cap V$ is a subset of $\mathbb{R}^n$, and $\vec{0} \in U$ and $\vec{0} \in V$ since they are both subspaces. Thus, $\vec{0} \in U \cap V$. Let $\vec{x}, \vec{y} \in U \cap V$. Then $\vec{x}, \vec{y} \in U$ and $\vec{x}, \vec{y} \in V$. Since U is a subspace it is closed under addition and scalar multiplication, so $\vec{x} + \vec{y} \in U$ and $t\vec{x} \in U$ for all $t \in \mathbb{R}$. Similarly, V is a subspace, so $\vec{x} + \vec{y} \in V$ and $t\vec{x} \in V$ for all $t \in \mathbb{R}$. Hence, $\vec{x} + \vec{y} \in U \cap V$ and $t\vec{x} \in U \cap V$, so $U \cap V$ is closed under addition and scalar multiplication and thus is a subspace of $\mathbb{R}^n$.

(b) Consider the subspaces $U = \left\{ \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \mid x_1 \in \mathbb{R} \right\}$ and $V = \left\{ \begin{bmatrix} 0 \\ x_2 \end{bmatrix} \mid x_2 \in \mathbb{R} \right\}$ of $\mathbb{R}^2$. Then $\vec{x} = \begin{bmatrix} x \\ 0 \end{bmatrix} \in U$ and $\vec{y} = \begin{bmatrix} 0 \\ y \end{bmatrix} \in V$, but $\vec{x} + \vec{y}$ is not in U and not in V, so it is not in $U \cup V$. Thus, $U \cup V$ is not a subspace.

(c) Since U and V are subspaces of $\mathbb{R}^n$, $\vec{u}, \vec{v} \in \mathbb{R}^n$ for any $\vec{u} \in U$ and $\vec{v} \in V$, so $\vec{u} + \vec{v} \in \mathbb{R}^n$ since $\mathbb{R}^n$ is closed under addition. Hence, $U + V$ is a subset of $\mathbb{R}^n$. Also, since U and V are subspace of $\mathbb{R}^n$, we have $\vec{0} \in U$ and $\vec{0} \in V$, thus $\vec{0} = \vec{0} + \vec{0} \in U + V$. Pick any vectors $\vec{x}, \vec{y} \in U + V$. Then, there exists vectors $\vec{u}_1, \vec{u}_2 \in U$ and $\vec{v}_1, \vec{v}_2 \in V$ such that $\vec{x} = \vec{u}_1 + \vec{v}_1$ and $\vec{y} = \vec{u}_2 + \vec{v}_2$. We have $\vec{x} + \vec{y} = (\vec{u}_1 + \vec{v}_1) + (\vec{u}_2 + \vec{v}_2) = (\vec{u}_1 + \vec{u}_2) + (\vec{v}_1 + \vec{v}_2)$ with $\vec{u}_1 + \vec{u}_2 \in U$ and $\vec{v}_1 + \vec{v}_2 \in V$ since U and V are both closed under addition. Hence, $\vec{x} + \vec{y} \in U + V$. Similarly, for any $t \in \mathbb{R}$ we have $t\vec{x} = t(\vec{u}_1 + \vec{v}_1) = t\vec{u}_1 + t\vec{v}_1$ where $t\vec{u}_1 \in U$ and $t\vec{v}_1 \in V$, so $t\vec{x} \in U + V$. Therefore, $U + V$ is a subspace of $\mathbb{R}^n$.

D4 There are many possible solutions.

(a) Pick $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

(b) Pick $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

(c) Pick $\vec{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

(d) Pick $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

D5 By definition of spanning, every $\vec{x} \in \operatorname{Span}\mathcal{B}$ can be written as a linear combination of the vectors in $\mathcal{B}$. Now, assume that we have $\vec{x} = s_1\vec{v}_1 + \cdots + s_k\vec{v}_k$ and $\vec{x} = t_1\vec{v}_1 + \cdots + t_k\vec{v}_k$. Then, we have

$$\begin{aligned} s_1\vec{v}_1 + \cdots + s_k\vec{v}_k &= t_1\vec{v}_1 + \cdots + t_k\vec{v}_k \\ (s_1\vec{v}_1 + \cdots + s_k\vec{v}_k) - (t_1\vec{v}_1 + \cdots + t_k\vec{v}_k) &= \vec{0} \\ (s_1 - t_1)\vec{v}_1 + \cdots + (s_k - t_k)\vec{v}_k &= \vec{0} \end{aligned}$$

Since $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent, this implies that $s_i - t_i = 0$ for $1 \leq i \leq k$. That is, $s_i = t_i$. Therefore, there is a unique linear combination of the vectors in $\mathcal{B}$ which equals $\vec{x}$.

D6 Assume that $\{\vec{v}_1, \dots, \vec{v}_k, \vec{x}\}$ is linearly dependent. Then, there exist coefficients $t_1, \dots, t_k, t_{k+1}$ not all zero such that

$$t_1\vec{v}_1 + \cdots + t_k\vec{v}_k + t_{k+1}\vec{x} = \vec{0}$$

If $t_{k+1} = 0$, then we have $t_1\vec{v}_1 + \cdots + t_k\vec{v}_k = \vec{0}$ with some $t_i \neq 0$, which contradicts the fact that $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent. Thus, $t_{k+1} \neq 0$. Hence, we can solve the equation for $\vec{x}$ to get

$$\vec{x} = -\frac{t_1}{t_{k+1}}\vec{v}_1 - \cdots - \frac{t_k}{t_{k+1}}\vec{v}_k$$

This contradicts our assumption that $\vec{x} \notin \operatorname{Span}\mathcal{B}$. Therefore, $\mathcal{B}$ must be linearly independent.

D7 If $\vec{x} \in \operatorname{Span}\{\vec{v}_1, s\vec{v}_1 + t\vec{v}_2\}$, then

$$\vec{x} = c_1\vec{v}_1 + c_2(s\vec{v}_1 + t\vec{v}_2) = (c_1 + sc_2)\vec{v}_1 + c_2t\vec{v}_2 \in \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$$

Hence, $\operatorname{Span}\{\vec{v}_1, s\vec{v}_1 + t\vec{v}_2\} \subseteq \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$.
Since $t \neq 0$ we get that $\vec{v}_2 = \frac{-s}{t}\vec{v}_1 + \frac{1}{t}(s\vec{v}_1 + t\vec{v}_2)$. Hence, if $\vec{v} \in \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$, then

$$\begin{aligned} \vec{v} = b_1\vec{v}_1 + b_2\vec{v}_2 &= b_1\vec{v}_1 + b_2\left(\frac{-s}{t}\vec{v}_1 + \frac{1}{t}(s\vec{v}_1 + t\vec{v}_2)\right) \\ &= \left(b_1 - \frac{b_2 s}{t}\right)\vec{v}_1 + \frac{b_2}{t}(s\vec{v}_1 + t\vec{v}_2) \in \operatorname{Span}\{\vec{v}_1, s\vec{v}_1 + t\vec{v}_2\} \end{aligned}$$

Thus, $\operatorname{Span}\{\vec{v}_1, \vec{v}_2\} \subseteq \operatorname{Span}\{\vec{v}_1, s\vec{v}_1 + t\vec{v}_2\}$. Hence $\operatorname{Span}\{\vec{v}_1, \vec{v}_2\} = \operatorname{Span}\{\vec{v}_1, s\vec{v}_1 + t\vec{v}_2\}$.

D8
(a) TRUE. We can rearrange the equation to get $-t\vec{v}_1 + \vec{v}_2 = \vec{0}$ with at least one coefficient (the coefficient of $\vec{v}_1$) non-zero. Hence $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent by definition.

(b) FALSE. If $\vec{v}_2 = \vec{0}$ and $\vec{v}_1$ is any non-zero vector, then $\vec{v}_1$ is not a scalar multiple of $\vec{v}_2$ and $\{\vec{v}_1, \vec{v}_2\}$ is linearly dependent by Theorem 1.2.4.

(c) FALSE. If $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$. Then, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent, but $\vec{v}_1$ cannot be written as a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

(d) TRUE. If $\vec{v}_1 = s\vec{v}_2 + t\vec{v}_3$, then we have $\vec{v}_1 - s\vec{v}_2 - t\vec{v}_3 = \vec{0}$ with at least one coefficient (the coefficient of $\vec{v}_1$) non-zero. Hence, by definition, the set is linearly dependent.

(e) FALSE. The set $\{\vec{0}\} = \operatorname{Span}\{\vec{0}\}$ is a subspace by Theorem 1.2.2.

(f) TRUE. By Theorem 1.2.2.

1.3 Length and Dot Products

Practice Problems

A1 (a) $\left\|\begin{bmatrix}2\\-5\end{bmatrix}\right\| = \sqrt{2^2+(-5)^2} = \sqrt{29}$

(b) $\left\|\begin{bmatrix}2/\sqrt{29}\\-5/\sqrt{29}\end{bmatrix}\right\| = \sqrt{(2/\sqrt{29})^2+(-5/\sqrt{29})^2} = \sqrt{4/29+25/29} = 1$

(c) $\left\|\begin{bmatrix}1\\0\\-1\end{bmatrix}\right\| = \sqrt{1^2+0^2+(-1)^2} = \sqrt{2}$

(d) $\left\|\begin{bmatrix}2\\3\\-2\end{bmatrix}\right\| = \sqrt{2^2+3^2+(-2)^2} = \sqrt{17}$

(e) $\left\|\begin{bmatrix}1\\1/5\\-3\end{bmatrix}\right\| = \sqrt{1^2+(1/5)^2+(-3)^2} = \sqrt{251}/5$

(f) $\left\|\begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\-1/\sqrt{3}\end{bmatrix}\right\| = \sqrt{(1/\sqrt{3})^2+(1/\sqrt{3})^2+(-1/\sqrt{3})^2} = 1$

(g) $\left\|\begin{bmatrix}1\\-1\\0\\2\end{bmatrix}\right\| = \sqrt{1^2+(-1)^2+0^2+2^2} = \sqrt{6}$

(h) $\left\|\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\| = \left|\frac{1}{2}\right|\left\|\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\| = \frac{1}{2}\sqrt{1^2+1^2+1^2+1^2} = 1$

A2 (a) A unit vector in the direction of $\vec{x} = \begin{bmatrix}3\\-4\end{bmatrix}$ is $\frac{1}{\|\vec{x}\|}\vec{x} = \frac{1}{\sqrt{3^2+(-4)^2}}\begin{bmatrix}3\\-4\end{bmatrix} = \frac{1}{5}\begin{bmatrix}3\\-4\end{bmatrix} = \begin{bmatrix}3/5\\-4/5\end{bmatrix}$.

(b) A unit vector in the direction of $\vec{x} = \begin{bmatrix}1\\1\end{bmatrix}$ is $\frac{1}{\|\vec{x}\|}\vec{x} = \frac{1}{\sqrt{1^2+1^2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$.

(c) A unit vector in the direction of $\vec{x} = \begin{bmatrix}-1\\0\\2\end{bmatrix}$ is $\frac{1}{\|\vec{x}\|}\vec{x} = \frac{1}{\sqrt{(-1)^2+0^2+2^2}}\begin{bmatrix}-1\\0\\2\end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix}-1\\0\\2\end{bmatrix} = \begin{bmatrix}-1/\sqrt{5}\\0\\2/\sqrt{5}\end{bmatrix}$.

(d) A unit vector in the direction of $\vec{x} = \begin{bmatrix}0\\-3\\0\end{bmatrix}$ is $\frac{1}{\|\vec{x}\|}\vec{x} = \frac{1}{\sqrt{0^2+(-3)^2+0^2}}\begin{bmatrix}0\\-3\\0\end{bmatrix} = \frac{1}{3}\begin{bmatrix}0\\-3\\0\end{bmatrix} = \begin{bmatrix}0\\-1\\0\end{bmatrix}$.

(e) A unit vector in the direction of $\vec{x} = \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ is $\frac{1}{\|\vec{x}\|}\vec{x} = \frac{1}{\sqrt{(-2)^2 + (-2)^2 + 1^2 + 0^2}} \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -2/3 \\ 1/3 \\ 0 \end{bmatrix}$.

(f) A unit vector in the direction of $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ is $\frac{1}{\|\vec{x}\|}\vec{x} = \frac{1}{\sqrt{1^2 + 0^2 + 0^2 + (-1)^2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$.

A3 (a) The distance between P and Q is $\|\vec{PQ}\| = \left\|\begin{bmatrix} -4 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right\| = \left\|\begin{bmatrix} -6 \\ -2 \end{bmatrix}\right\| = \sqrt{(-6)^2 + (-2)^2} = 2\sqrt{10}$.

(b) The distance between P and Q is $\|\vec{PQ}\| = \left\|\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}\right\| = \left\|\begin{bmatrix} -4 \\ 0 \\ 3 \end{bmatrix}\right\| = \sqrt{(-4)^2 + 0^2 + 3^2} = 5$.

(c) The distance between P and Q is $\|\vec{PQ}\| = \left\|\begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix} - \begin{bmatrix} 4 \\ -6 \\ 1 \end{bmatrix}\right\| = \left\|\begin{bmatrix} -7 \\ 11 \\ 0 \end{bmatrix}\right\| = \sqrt{(-7)^2 + 11^2 + 0^2} = \sqrt{170}$.

(d) The distance between P and Q is $\|\vec{PQ}\| = \left\|\begin{bmatrix} 4 \\ 6 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 1 \\ 5 \end{bmatrix}\right\| = \left\|\begin{bmatrix} 2 \\ 5 \\ -3 \\ -4 \end{bmatrix}\right\| = \sqrt{2^2 + 5^2 + (-3)^2 + (-4)^2} = 3\sqrt{6}$.

A4 (a) We have $\|\vec{x}\| = \sqrt{4^2 + 3^2 + 1^2} = \sqrt{26}$, $\|\vec{y}\| = \sqrt{2^2 + 1^2 + 5^2} = \sqrt{30}$, $\|\vec{x} + \vec{y}\| = \left\|\begin{bmatrix} 6 \\ 4 \\ 6 \end{bmatrix}\right\| = \sqrt{6^2 + 4^2 + 6^2} = 2\sqrt{22}$, and $|\vec{x} \cdot \vec{y}| = 4(2) + 3(1) + 1(5) = 16$. The triangle inequality is satisfied since

$$2\sqrt{22} \approx 9.38 \leq \sqrt{26} + \sqrt{30} \approx 10.58$$

The Cauchy-Schwarz inequality is also satisfied since $16 \leq \sqrt{26(30)} \approx 27.93$.

(b) We have $\|\vec{x}\| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$, $\|\vec{y}\| = \sqrt{(-3)^2 + 2^2 + 4^2} = \sqrt{29}$, $\|\vec{x}+\vec{y}\| = \left\|\begin{bmatrix} -2 \\ 1 \\ 6 \end{bmatrix}\right\| = \sqrt{(-2)^2 + 1^2 + 6^2} = \sqrt{41}$, and $|\vec{x} \cdot \vec{y}| = 1(-3) + (-1)(2) + 2(4) = 3$. The triangle inequality is satisfied since

$$\sqrt{41} \approx 6.40 \leq \sqrt{6} + \sqrt{29} \approx 7.83$$

The Cauchy-Schwarz inequality is satisfied since $3 \leq \sqrt{6(29)} \approx 13.19$.

A5 (a) $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} = 1(2) + 3(-2) + 2(2) = 0$. Hence these vectors are orthogonal.

(b) $\begin{bmatrix} -3 \\ 1 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = (-3)(2) + 1(-1) + 7(1) = 0$. Hence these vectors are orthogonal.

(c) $\begin{bmatrix}2\\1\\1\end{bmatrix} \cdot \begin{bmatrix}-1\\4\\2\end{bmatrix} = 2(-1) + 1(4) + 1(2) = 4 \neq 0$. Therefore, these vectors are not orthogonal.

(d) $\begin{bmatrix}4\\1\\0\\-2\end{bmatrix} \cdot \begin{bmatrix}-1\\4\\3\\0\end{bmatrix} = 4(-1) + 1(4) + 0(3) + (-2)(0) = 0$. Hence these vectors are orthogonal.

(e) $\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \cdot \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = 0(x_1) + 0(x_2) + 0(x_3) + 0(x_4) = 0$. Hence these vectors are orthogonal.

(f) $\begin{bmatrix}1/3\\2/3\\-1/3\\3\end{bmatrix} \cdot \begin{bmatrix}3/2\\0\\-3/2\\1\end{bmatrix} = \frac{1}{3}\left(\frac{3}{2}\right) + \frac{2}{3}(0) + \left(-\frac{1}{3}\right)\left(-\frac{3}{2}\right) + 3(1) = 4$. Therefore, these vectors are not orthogonal.

A6 (a) The vectors are orthogonal when $0 = \begin{bmatrix}3\\-1\end{bmatrix} \cdot \begin{bmatrix}2\\k\end{bmatrix} = 3(2) + (-1)k = 6 - k$.
Thus, the vectors are orthogonal only when $k = 6$.

(b) The vectors are orthogonal when $0 = \begin{bmatrix}3\\-1\end{bmatrix} \cdot \begin{bmatrix}k\\k^2\end{bmatrix} = 3(k) + (-1)(k^2) = 3k - k^2 = k(3-k)$.
Thus, the vectors are orthogonal only when $k = 0$ or $k = 3$.

(c) The vectors are orthogonal when $0 = \begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}3\\-k\\k\end{bmatrix} = 1(3) + 2(-k) + 3(k) = 3 + k$.
Thus, the vectors are orthogonal only when $k = -3$.

(d) The vectors are orthogonal when $0 = \begin{bmatrix}1\\2\\3\\4\end{bmatrix} \cdot \begin{bmatrix}k\\k\\-k\\0\end{bmatrix} = 1(k) + 2(k) + 3(-k) + 4(0) = 0$.
Therefore, the vectors are always orthogonal.

A7 (a) The scalar equation of the plane is

$$\begin{aligned}
0 = \vec{n} \cdot (\vec{x} - \vec{p}) &= \begin{bmatrix}2\\4\\-1\end{bmatrix} \cdot \begin{bmatrix}x_1+1\\x_2-2\\x_3+3\end{bmatrix}\\
&= 2(x_1+1) + 4(x_2-2) + (-1)(x_3+3)\\
&= 2x_1 + 2 + 4x_2 - 8 - x_3 - 3\\
9 &= 2x_1 + 4x_2 - x_3
\end{aligned}$$

(b) The scalar equation of the plane is

$$\begin{aligned}0 = \vec{n} \cdot (\vec{x} - \vec{p}) &= \begin{bmatrix}3\\0\\5\end{bmatrix} \cdot \begin{bmatrix}x_1 - 2\\x_2 - 5\\x_3 - 4\end{bmatrix}\\
&= 3(x_1 - 2) + 0(x_2 - 5) + 5(x_3 - 4)\\
&= 3x_1 - 6 + 5x_3 - 20\\
26 &= 3x_1 + 5x_3\end{aligned}$$

(c) The scalar equation of the plane is

$$\begin{aligned}0 = \vec{n} \cdot (\vec{x} - \vec{p}) &= \begin{bmatrix}3\\-4\\1\end{bmatrix} \cdot \begin{bmatrix}x_1 - 1\\x_2 + 1\\x_3 - 1\end{bmatrix}\\
&= 3(x_1 - 1) + (-4)(x_2 + 1) + 1(x_3 - 1)\\
&= 3x_1 - 3 - 4x_2 - 4 + x_3 - 1\\
8 &= 3x_1 - 4x_2 + x_3\end{aligned}$$

(d) The scalar equation of the plane is

$$\begin{aligned}0 = \vec{n} \cdot (\vec{x} - \vec{p}) &= \begin{bmatrix}-4\\-2\\-2\end{bmatrix} \cdot \begin{bmatrix}x_1 - 2\\x_2 - 1\\x_3 - 1\end{bmatrix}\\
&= (-4)(x_1 - 2) + (-2)(x_2 - 1) + (-2)(x_3 - 1)\\
&= -4x_1 + 8 - 2x_2 + 2 - 2x_3 + 2\\
-12 &= -4x_1 - 2x_2 - 2x_3\end{aligned}$$

A8 (a) The scalar equation of the hyperplane is

$$\begin{aligned}0 = \vec{n} \cdot (\vec{x} - \vec{p}) &= \begin{bmatrix}3\\1\\4\end{bmatrix} \cdot \begin{bmatrix}x_1 - 1\\x_2 - 1\\x_3 + 1\end{bmatrix}\\
&= 3(x_1 - 1) + 1(x_2 - 1) + 4(x_3 + 1)\\
&= 3x_1 - 3 + x_2 - 1 + 4x_3 + 4\\
0 &= 3x_1 + x_2 + 4x_3\end{aligned}$$

(b) The scalar equation of the hyperplane is

$$\begin{aligned}0 = \vec{n} \cdot (\vec{x} - \vec{p}) &= \begin{bmatrix}0\\1\\3\\3\end{bmatrix} \cdot \begin{bmatrix}x_1 - 2\\x_2 + 2\\x_3\\x_4 - 1\end{bmatrix}\\
&= 0(x_1 - 2) + 1(x_2 + 2) + 3x_3 + 3(x_4 - 1)\\
&= x_2 + 2 + 3x_3 + 3x_4 - 3\\
1 &= x_2 + 3x_3 + 3x_4\end{aligned}$$

(c) The scalar equation of the hyperplane is

$$\begin{aligned} 0 = \vec{n} \cdot (\vec{x} - \vec{p}) &= \begin{bmatrix} 1 \\ -4 \\ 5 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \\ &= 1x_1 - 4x_2 + 5x_3 - 2x_4 \end{aligned}$$

(d) The scalar equation of the hyperplane is

$$\begin{aligned} 0 = \vec{n} \cdot (\vec{x} - \vec{p}) &= \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 - 1 \\ x_2 \\ x_3 - 1 \\ x_4 - 2 \\ x_5 - 1 \end{bmatrix} \\ &= 0(x_1 - 1) + 1x_2 + 2(x_3 - 1) + (-1)(x_4 - 2) + 1(x_5 - 1) \\ &= x_2 + 2x_3 - 2 - x_4 + 2 + x_5 - 1 \\ 1 &= x_2 + 2x_3 - x_4 + x_5 \end{aligned}$$

A9 The components of a normal vector for a hyperplane are the coefficients of the variables in the scalar equation of the hyperplane. Thus:

(a) $\vec{n} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (b) $\vec{n} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}$ (c) $\vec{n} = \begin{bmatrix} -4 \\ 3 \\ -5 \end{bmatrix}$ (d) $\vec{n} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -3 \end{bmatrix}$ (e) $\vec{n} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2 \\ -1 \end{bmatrix}$

A10 (a) A normal vector of any plane parallel to the given plane must be a scalar multiple of the normal vector for the given plane. Hence, a normal vector for the required plane is $\vec{n} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$. Therefore, an equation for the plane passing through $P(1, -3, -1)$ with normal vector $\vec{n}$ is

$$\begin{aligned} 0 = \vec{n} \cdot (\vec{x} - \vec{p}) &= \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} x_1 - 1 \\ x_2 + 3 \\ x_3 + 1 \end{bmatrix} \\ &= 2(x_1 - 1) + (-3)(x_2 + 3) + 5(x_3 + 1) \\ &= 2x_1 - 2 - 3x_2 - 9 + 5x_3 + 5 \\ 6 &= 2x_1 - 3x_2 + 5x_3 \end{aligned}$$

(b) A normal vector for the required plane is $\vec{n} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Hence, an equation for the plane passing through

$P(0,-2,4)$ with normal vector $\vec{n}$ is

$$\begin{aligned} 0 = \vec{n}\cdot(\vec{x}-\vec{p}) &= \begin{bmatrix}0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2+2\\x_3-4\end{bmatrix} \\ &= 0x_1 + 1(x_2+2) + 0(x_3-4) \\ -2 &= x_2 \end{aligned}$$

(c) A normal vector for the required plane is $\vec{n} = \begin{bmatrix}1\\-1\\3\end{bmatrix}$. Hence, an equation for the plane passing through $P(1,2,1)$ with normal vector $\vec{n}$ is

$$\begin{aligned} 0 = \vec{n}\cdot(\vec{x}-\vec{p}) &= \begin{bmatrix}1\\-1\\3\end{bmatrix}\cdot\begin{bmatrix}x_1-1\\x_2-2\\x_3-1\end{bmatrix} \\ &= 1(x_1-1) + (-1)(x_2-2) + 3(x_3-1) \\ &= x_1 - 1 - x_2 + 2 + 3x_3 - 3 \\ 2 &= x_1 - x_2 + 3x_3 \end{aligned}$$

A11 (a) FALSE. One possible counterexample is $\begin{bmatrix}1\\0\end{bmatrix}\cdot\begin{bmatrix}2\\2\end{bmatrix} = 2 = \begin{bmatrix}1\\0\end{bmatrix}\cdot\begin{bmatrix}2\\-97\end{bmatrix}$.

(b) Our counterexample in part (a) has $\vec{u} \neq \vec{0}$ so the result does not change.

Homework Problems

B1 (a) 0 (b) $\sqrt{10}$ (c) 7
(d) $\sqrt{30}/3$ (e) 1 (f) 4

B2 (a) $\begin{bmatrix}-3/5\\4/5\end{bmatrix}$ (b) $\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$ (c) $\begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}$

(d) $\begin{bmatrix}-1/\sqrt{10}\\3/\sqrt{10}\\0\end{bmatrix}$ (e) $\begin{bmatrix}2/3\\2/3\\0\\1/3\end{bmatrix}$ (f) $\begin{bmatrix}-1/\sqrt{12}\\-1/\sqrt{12}\\-1/\sqrt{12}\\3/\sqrt{12}\end{bmatrix}$

B3 (a) $\sqrt{37}$ (b) $\sqrt{14}$ (c) $\sqrt{89}$ (d) $\sqrt{68}$

B4 (a) We have $\|\vec{x}\| = \sqrt{2^2+(-6)^2+(-3)^2} = 7$, $\|\vec{y}\| = \sqrt{(-3)^2+4^2+5^2} = \sqrt{50}$, and

$$\|\vec{x}+\vec{y}\| = \left\|\begin{bmatrix}-1\\-2\\2\end{bmatrix}\right\| = \sqrt{(-1)^2+(-2)^2+2^2} = 3$$

Thus,

$$\|\vec{x}+\vec{y}\| = 3 \leq 7 + \sqrt{50} = \|\vec{x}\| + \|\vec{y}\|$$

Observe that $|\vec{x}\cdot\vec{y}| = |2(-3) + (-6)(4) + (-3)(5)| = 45$ and $\|\vec{x}\|\|\vec{y}\| = 7\sqrt{50} > 49$. Hence,

$$|\vec{x}\cdot\vec{y}| \leq \|\vec{x}\|\|\vec{y}\|$$

(b) We have $\|\vec{x}\| = \sqrt{4^2 + 1^2 + (-2)^2} = \sqrt{21}$, $\|\vec{y}\| = \sqrt{3^2 + 5^2 + 1^2} = \sqrt{35}$, and

$$\|\vec{x}+\vec{y}\| = \left\|\begin{bmatrix} 7 \\ 6 \\ -1 \end{bmatrix}\right\| = \sqrt{7^2 + 6^2 + (-1)^2} = \sqrt{86}$$

Thus,

$$\|\vec{x}+\vec{y}\| = \sqrt{86} \leq \sqrt{21} + \sqrt{35} = \|\vec{x}\| + \|\vec{y}\|$$

Observe that $|\vec{x}\cdot\vec{y}| = 4(3) + 1(5) + (-2)(1) = 15$ and $\|\vec{x}\|\|\vec{y}\| = \sqrt{21}\sqrt{35} = \sqrt{735} > 27$. Hence,

$$|\vec{x}\cdot\vec{y}| \leq \|\vec{x}\|\|\vec{y}\|$$

B5 (a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 2 \end{bmatrix} = 1(-2) + 2(2) = 2$, so they are not orthogonal.

(b) $\begin{bmatrix} 4 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 2 \end{bmatrix} = 4(-3) + 6(2) = 0$, so they are orthogonal.

(c) $\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 1 \\ -4 \end{bmatrix} = 1(-4) + 4(1) + 1(-4) = -4$, so they are not orthogonal.

(d) $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} = 1(3) + 3(-1) + 1(0) = 0$, so they are orthogonal.

(e) $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} = 1(-3) + 2(0) + 1(5) + 2(1) = 4$, so they are not orthogonal.

(f) $\begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} = 2(-2) + 1(1) + (-1)(-1) + (-2)(0) + 1(2) = 0$, so they are orthogonal.

B6 (a) $k = 0$ (b) $k = 0$ or $k = 2$
(c) $k = -4/5$ (d) $k = 0$ or $k = 2$

B7 (a) $-x_1 + 4x_2 + 7x_3 = -2$ (b) $4x_1 - 2x_2 + x_3 = 33$
(c) $x_1 + 2x_2 + x_3 = 0$ (d) $x_1 + x_2 + x_3 = 3$

B8 (a) $3x_1 - 2x_2 - 5x_3 + x_4 = 4$ (b) $2x_1 - 4x_2 + x_3 - 3x_4 = -19$
(c) $x_1 = 0$ (d) $x_2 - 2x_3 + x_4 + x_5 = 4$

B9 (a) $\begin{bmatrix}2\\3\end{bmatrix}$ (b) $\begin{bmatrix}-1\\-2\\5\end{bmatrix}$ (c) $\begin{bmatrix}1\\4\\0\\-1\end{bmatrix}$ (d) $\begin{bmatrix}1\\1\\1\\-2\end{bmatrix}$ (e) $\begin{bmatrix}1\\0\\0\\0\\-1\end{bmatrix}$ (f) $\begin{bmatrix}1\\1\\-2\\1\\-1\end{bmatrix}$

B10 (a) $5x_1 - x_2 - 2x_3 = 2$

(b) $2x_2 + 3x_3 = -11$

(c) $3x_1 - 2x_2 + 3x_3 = 7$

(d) $x_1 - 5x_2 + 3x_3 = 4$

Computer Problems

C1 (a) 7.597

(b) 4.89412336

(c) 0

(d) 47.0948

Conceptual Problems

D1 (a) Intuitively, if there is no point of intersection, the line is parallel to the plane, so the direction vector of the line is orthogonal to the normal to the plane: $\vec{d} \cdot \vec{n} = 0$. Moreover, the point P is not on the plane, so $\vec{p} \cdot \vec{n} \neq k$.

(b) Substitute $\vec{x} = \vec{p} + t\vec{d}$ into the equation of the plane to see whether for some t, $\vec{x}$ satisfies the equation of the plane.

$$\vec{n} \cdot (\vec{p} + t\vec{d}) = k$$

Isolate the term in t: $t(\vec{n} \cdot \vec{d}) = k - \vec{n} \cdot \vec{p}$.

There is one solution for t (and thus, one point of intersection of the line and the plane) exactly when $\vec{n} \cdot \vec{d} \neq 0$. If $\vec{n} \cdot \vec{d} = 0$, there is no solution for t unless we also have $\vec{n} \cdot \vec{p} = k$. In this case the equation is satisfied for all t and the line lies in the plane. Thus, to have no point of intersection, it is necessary and sufficient that $\vec{n} \cdot \vec{d} = 0$ and $\vec{n} \cdot \vec{p} \neq k$.

D2 Since $\vec{x} = \vec{x} - \vec{y} + \vec{y}$,

$$\|\vec{x}\| = \|\vec{x} - \vec{y} + \vec{y}\| = \|(\vec{x} - \vec{y}) + \vec{y}\| \leq \|\vec{x} - \vec{y}\| + \|\vec{y}\|$$

So, $\|\vec{x}\| - \|\vec{y}\| \leq \|\vec{x} - \vec{y}\|$. This is almost what we require, but the left-hand side might be negative. So, by a similar argument with $\vec{y}$, and using the fact that $\|\vec{y} - \vec{x}\| = \|\vec{x} - \vec{y}\|$, we obtain $\|\vec{y}\| - \|\vec{x}\| \leq \|\vec{x} - \vec{y}\|$. From this equation and the previous one, we can conclude that

$$\big|\|\vec{x}\| - \|\vec{y}\|\big| \leq \|\vec{x} - \vec{y}\|$$

D3 We have

$$\begin{aligned}\|\vec{v}_1 + \vec{v}_2\|^2 &= (\vec{v}_1 + \vec{v}_2) \cdot (\vec{v}_1 + \vec{v}_2) = \vec{v}_1 \cdot \vec{v}_1 + \vec{v}_1 \cdot \vec{v}_2 + \vec{v}_2 \cdot \vec{v}_1 + \vec{v}_2 \cdot \vec{v}_2 \\ &= \|\vec{v}_1\|^2 + 0 + 0 + \|\vec{v}_2\|^2 = \|\vec{v}_1\|^2 + \|\vec{v}_2\|^2\end{aligned}$$

D4 Let $\vec{x}$ be any point that is equidistant from P and Q. Then $\vec{x}$ satisfies $\|\vec{x}-\vec{p}\| = \|\vec{x}-\vec{q}\|$, or equivalently, $\|\vec{x}-\vec{p}\|^2 = \|\vec{x}-\vec{q}\|^2$. Hence,

$$\begin{aligned}
(\vec{x}-\vec{p})\cdot(\vec{x}-\vec{p}) &= (\vec{x}-\vec{q})\cdot(\vec{x}-\vec{q})\\
\vec{x}\cdot\vec{x}-\vec{x}\cdot\vec{p}-\vec{p}\cdot\vec{x}-\vec{p}\cdot\vec{p} &= \vec{x}\cdot\vec{x}-\vec{x}\cdot\vec{q}-\vec{q}\cdot\vec{x}+\vec{q}\cdot\vec{q}\\
-2\vec{p}\cdot\vec{x}+2\vec{q}\cdot\vec{x} &= \vec{q}\cdot\vec{q}-\vec{p}\cdot\vec{p}\\
2(\vec{q}-\vec{p})\cdot\vec{x} &= \|\vec{q}\|^2-\|\vec{p}\|^2
\end{aligned}$$

This is the equation of a plane with normal vector $2(\vec{q}-\vec{p})$.

D5 (a) A point $\vec{x}$ on the plane must satisfy $\left\|\vec{x}-\begin{bmatrix}2\\2\\5\end{bmatrix}\right\| = \left\|\vec{x}-\begin{bmatrix}-3\\4\\1\end{bmatrix}\right\|$. Square both sides and simplify.

$$\begin{aligned}
\left(\vec{x}-\begin{bmatrix}2\\2\\5\end{bmatrix}\right)\cdot\left(\vec{x}-\begin{bmatrix}2\\2\\5\end{bmatrix}\right) &= \left(\vec{x}-\begin{bmatrix}-3\\4\\1\end{bmatrix}\right)\cdot\left(\vec{x}-\begin{bmatrix}-3\\4\\1\end{bmatrix}\right)\\
\vec{x}\cdot\vec{x}-2\begin{bmatrix}2\\2\\5\end{bmatrix}\cdot\vec{x}+33 &= \vec{x}\cdot\vec{x}-2\begin{bmatrix}-3\\4\\1\end{bmatrix}\cdot\vec{x}+26\\
2\left(\begin{bmatrix}-3\\4\\1\end{bmatrix}-\begin{bmatrix}2\\2\\5\end{bmatrix}\right)\cdot\vec{x} &= 26-33\\
5x_1-2x_2+4x_3 &= 7/2
\end{aligned}$$

(b) A point equidistant from the points is $\frac{1}{2}\left(\begin{bmatrix}2\\2\\5\end{bmatrix}+\begin{bmatrix}-3\\4\\1\end{bmatrix}\right) = \begin{bmatrix}-1/2\\3\\3\end{bmatrix}$. The vector joining the two points, $\vec{n} = \begin{bmatrix}2\\2\\5\end{bmatrix}-\begin{bmatrix}-3\\4\\1\end{bmatrix} = \begin{bmatrix}5\\-2\\4\end{bmatrix}$ must be orthogonal to the plane. Thus, the equation of the plane is

$$5x_1-2x_2+4x_3 = \begin{bmatrix}5\\-2\\4\end{bmatrix}\cdot\begin{bmatrix}-1/2\\3\\3\end{bmatrix} = 7/2$$

D6 Let S denote the set of all vectors orthogonal to $\vec{u}$. By definition, a vector orthogonal to $\vec{u}$ must be in $\mathbb{R}^n$, so S is a subset of $\mathbb{R}^n$. Moreover, since $\vec{u}\cdot\vec{0}=0$, we have that S is non-empty. Let $\vec{x},\vec{y}\in S$. Then we have

$$(\vec{x}+\vec{y})\cdot\vec{u} = \vec{x}\cdot\vec{u}+\vec{y}\cdot\vec{u} = 0+0 = 0$$

Therefore, S is closed under addition. Similarly, for any $t\in\mathbb{R}$ we have

$$(t\vec{x})\cdot\vec{u} = t(\vec{x}\cdot\vec{u}) = t(0) = 0$$

so S is also closed under scalar multiplication. Therefore, S is a subspace of $\mathbb{R}^n$.

D7 By definition, a vector orthogonal to any vector in S must be in $\mathbb{R}^n$, so $S^\perp$ is a subset of $\mathbb{R}^n$. Moreover, since $\vec{v} \cdot \vec{0} = 0$ for any $\vec{v} \in S$, we have that $S^\perp$ is non-empty. Let $\vec{x}, \vec{y} \in S^\perp$. This implies that $\vec{x} \cdot \vec{v} = 0$ and $\vec{y} \cdot \vec{v} = 0$ for all $\vec{v} \in S$. Hence, for any $\vec{v} \in S$ we have

$$(\vec{x} + \vec{y}) \cdot \vec{v} = \vec{x} \cdot \vec{v} + \vec{y} \cdot \vec{v} = 0 + 0 = 0$$

Therefore,$\vec{x} + \vec{y} \in S^\perp$. Similarly, for any $t \in \mathbb{R}$ and any $\vec{v} \in S$ we have

$$(t\vec{x}) \cdot \vec{v} = t(\vec{x} \cdot \vec{v}) = t(0) = 0$$

so $(t\vec{x}) \in S^\perp$. Therefore, $S^\perp$ is a subspace of $\mathbb{R}^n$.

D8 Consider $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$. Then for $1 \leq i \leq k$ we have

$$\begin{aligned} 0 = \vec{0} \cdot \vec{v}_i &= (c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) \cdot \vec{v}_i \\ &= c_1(\vec{v}_1 \cdot \vec{v}_i) + \cdots + c_i(\vec{v}_i \cdot \vec{v}_i) + \cdots + c_k(\vec{v}_k \cdot \vec{v}_i) \\ &= 0 + \cdots + 0 + c_i\|\vec{v}_i\|^2 + 0 + \cdots + 0 \end{aligned}$$

The fact that $\|\vec{v}_i\| \neq 0$ implies that $c_i = 0$. Since this is valid for $1 \leq i \leq k$ we get $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent.

D9 (a) Let $\vec{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$. We have $\vec{n} \cdot \vec{e}_1 = \|\vec{n}\| \|\vec{e}_1\| \cos\alpha$. But, $\|\vec{n}\| = 1$ and $\|\vec{e}_1\| = 1$, so $\vec{n} \cdot \vec{e}_1 = \cos\alpha$. But, $\vec{n} \cdot \vec{e}_1 = n_1$, so $n_1 = \cos\alpha$. Similarly, $n_2 = \cos\beta$ and $n_3 = \cos\gamma$, so $\vec{n} = \begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix}$.

(b) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \|\vec{n}\|^2 = 1$, because $\vec{n}$ is a unit vector.

(c) In $\mathbb{R}^2$, the unit vector is $\vec{n} = \begin{bmatrix} \cos\alpha \\ \cos\beta \end{bmatrix}$, where α is the angle between $\vec{n}$ and $\vec{e}_1$ and β is the angle between $\vec{n}$ and $\vec{e}_2$. But in the plane $\alpha + \beta = \frac{\pi}{2}$, so $\cos\beta = \cos(\pi/2 - \alpha) = \sin\alpha$. Now let $\theta = \alpha$, and we have

$$1 = \|\vec{n}\|^2 = \cos^2\alpha + \cos^2\beta = \cos^2\theta + \sin^2\theta$$

1.4 Projections and Minimum Distance

Practice Problems

A1 (a) We have

$$\begin{aligned} \text{proj}_{\vec{v}}\,\vec{u} &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{-5}{1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \end{bmatrix} \\ \text{perp}_{\vec{v}}\,\vec{u} &= \vec{u} - \text{proj}_{\vec{v}}\,\vec{u} = \begin{bmatrix} 3 \\ -5 \end{bmatrix} - \begin{bmatrix} 0 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \end{aligned}$$

(b) We have

$$\operatorname{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{12/5}{1}\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 36/25 \\ 48/25 \end{bmatrix}$$
$$\operatorname{perp}_{\vec{v}} \vec{u} = \vec{u} - \operatorname{proj}_{\vec{v}} \vec{u} = \begin{bmatrix} -4 \\ 6 \end{bmatrix} - \begin{bmatrix} 36/25 \\ 48/25 \end{bmatrix} = \begin{bmatrix} -136/25 \\ 102/25 \end{bmatrix}$$

(c) We have

$$\operatorname{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{5}{1}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$
$$\operatorname{perp}_{\vec{v}} \vec{u} = \vec{u} - \operatorname{proj}_{\vec{v}} \vec{u} = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$$

(d) We have

$$\operatorname{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{-4/3}{1}\begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} -4/9 \\ 8/9 \\ -8/9 \end{bmatrix}$$
$$\operatorname{perp}_{\vec{v}} \vec{u} = \vec{u} - \operatorname{proj}_{\vec{v}} \vec{u} = \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} - \begin{bmatrix} -4/9 \\ 8/9 \\ -8/9 \end{bmatrix} = \begin{bmatrix} 40/9 \\ 1/9 \\ -19/9 \end{bmatrix}$$

(e) We have

$$\operatorname{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{0}{6}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\operatorname{perp}_{\vec{v}} \vec{u} = \vec{u} - \operatorname{proj}_{\vec{v}} \vec{u} = \begin{bmatrix} -1 \\ -1 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \\ -1 \end{bmatrix}$$

(f) We have

$$\operatorname{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{-1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \\ 0 \\ -1/2 \end{bmatrix}$$
$$\operatorname{perp}_{\vec{v}} \vec{u} = \vec{u} - \operatorname{proj}_{\vec{v}} \vec{u} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ -3 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 0 \\ 0 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 3 \\ 2 \\ -5/2 \end{bmatrix}$$

A2 (a) We have

$$\text{proj}_{\vec{v}}\,\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{0}{2}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$
$$\text{perp}_{\vec{v}}\,\vec{u} = \vec{u} - \text{proj}_{\vec{v}}\,\vec{u} = \begin{bmatrix}3\\-3\end{bmatrix} - \begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}3\\-3\end{bmatrix}$$

(b) We have

$$\text{proj}_{\vec{v}}\,\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{-1}{17}\begin{bmatrix}2\\3\\-2\end{bmatrix} = \begin{bmatrix}-2/17\\-3/17\\2/17\end{bmatrix}$$
$$\text{perp}_{\vec{v}}\,\vec{u} = \vec{u} - \text{proj}_{\vec{v}}\,\vec{u} = \begin{bmatrix}4\\-1\\3\end{bmatrix} - \begin{bmatrix}-2/17\\-3/17\\2/17\end{bmatrix} = \begin{bmatrix}70/17\\-14/17\\49/17\end{bmatrix}$$

(c) We have

$$\text{proj}_{\vec{v}}\,\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{-14}{6}\begin{bmatrix}-2\\1\\-1\end{bmatrix} = \begin{bmatrix}14/3\\-7/3\\7/3\end{bmatrix}$$
$$\text{perp}_{\vec{v}}\,\vec{u} = \vec{u} - \text{proj}_{\vec{v}}\,\vec{u} = \begin{bmatrix}5\\-1\\3\end{bmatrix} - \begin{bmatrix}14/3\\-7/3\\7/3\end{bmatrix} = \begin{bmatrix}1/3\\4/3\\2/3\end{bmatrix}$$

(d) We have

$$\text{proj}_{\vec{v}}\,\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{9}{6}\begin{bmatrix}1\\1\\-2\end{bmatrix} = \begin{bmatrix}3/2\\3/2\\-3\end{bmatrix}$$
$$\text{perp}_{\vec{v}}\,\vec{u} = \vec{u} - \text{proj}_{\vec{v}}\,\vec{u} = \begin{bmatrix}4\\1\\-2\end{bmatrix} - \begin{bmatrix}3/2\\3/2\\-3\end{bmatrix} = \begin{bmatrix}5/2\\-1/2\\1\end{bmatrix}$$

(e) We have

$$\text{proj}_{\vec{v}}\,\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{-5}{15}\begin{bmatrix}-1\\2\\1\\-3\end{bmatrix} = \begin{bmatrix}1/3\\-2/3\\-1/3\\1\end{bmatrix}$$
$$\text{perp}_{\vec{v}}\,\vec{u} = \vec{u} - \text{proj}_{\vec{v}}\,\vec{u} = \begin{bmatrix}2\\-1\\2\\1\end{bmatrix} - \begin{bmatrix}1/3\\-2/3\\-1/3\\1\end{bmatrix} = \begin{bmatrix}5/3\\-1/3\\7/3\\0\end{bmatrix}$$

A3 (a) A unit vector in the direction of $\vec{u}$ is

$$\hat{u} = \frac{1}{\|\vec{u}\|}\vec{u} = \begin{bmatrix} 2/7 \\ 6/7 \\ 3/7 \end{bmatrix}$$

(b) We have

$$\text{proj}_{\vec{u}} \vec{F} = \frac{\vec{F} \cdot \vec{u}}{\|\vec{u}\|^2}\vec{u} = \frac{110}{49}\begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 220/49 \\ 660/49 \\ 330/49 \end{bmatrix}$$

(c) We get

$$\text{perp}_{\vec{u}} \vec{F} = \vec{F} - \text{proj}_{\vec{u}} \vec{F} = \begin{bmatrix} 10 \\ 18 \\ -6 \end{bmatrix} - \begin{bmatrix} 220/49 \\ 660/49 \\ 330/49 \end{bmatrix} = \begin{bmatrix} 270/49 \\ 222/49 \\ -624/49 \end{bmatrix}$$

A4 (a) A unit vector in the direction of $\vec{u}$ is

$$\hat{u} = \frac{\vec{u}}{\|\vec{u}\|} = \begin{bmatrix} 3/\sqrt{14} \\ 1/\sqrt{14} \\ -2/\sqrt{14} \end{bmatrix}$$

(b) We have

$$\text{proj}_{\vec{u}} \vec{F} = \frac{\vec{F} \cdot \vec{u}}{\|\vec{u}\|^2}\vec{u} = \frac{16}{14}\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 24/7 \\ 8/7 \\ -16/7 \end{bmatrix}$$

(c) We get

$$\text{perp}_{\vec{u}} \vec{F} = \vec{F} - \text{proj}_{\vec{u}} \vec{F} = \begin{bmatrix} 3 \\ 11 \\ 2 \end{bmatrix} - \begin{bmatrix} 24/7 \\ 8/7 \\ -16/7 \end{bmatrix} = \begin{bmatrix} -3/7 \\ 69/7 \\ 30/7 \end{bmatrix}$$

A5 (a) We first pick a point P on the line, say $P(1,4)$. Then the point R on the line that is closest to $Q(0,0)$ satisfies $\vec{PR} = \text{proj}_{\vec{d}} \vec{PQ}$ where $\vec{PQ} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$ and $\vec{d} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ is a direction vector of the line. We get

$$\vec{PR} = \text{proj}_{\vec{d}} \vec{PQ} = \frac{\vec{PQ} \cdot \vec{d}}{\|\vec{d}\|^2}\vec{d} = \frac{-6}{8}\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -3/2 \end{bmatrix}$$

Therefore, we have

$$\vec{OR} = \vec{OP} + \vec{PR} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 3/2 \\ -3/2 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 5/2 \end{bmatrix}$$

Hence, the point on the line closest to Q is $R(5/2, 5/2)$. The distance from R to Q is

$$\|\text{perp}_{\vec{d}} \vec{PQ}\| = \left\| \begin{bmatrix} -1 \\ -4 \end{bmatrix} - \begin{bmatrix} 3/2 \\ -3/2 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -5/2 \\ -5/2 \end{bmatrix} \right\| = \frac{5}{\sqrt{2}}$$

(b) We first pick the point $P(3,7)$ on the line. Then the point R on the line that is closest to $Q(2,5)$ satisfies $\vec{PR} = \text{proj}_{\vec{d}} \vec{PQ}$ where $\vec{PQ} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ and $\vec{d} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$ is a direction vector of the line. We get

$$\vec{PR} = \text{proj}_{\vec{d}} \vec{PQ} = \frac{\vec{PQ} \cdot \vec{d}}{\|\vec{d}\|^2}\vec{d} = \frac{7}{17}\begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 7/17 \\ -28/17 \end{bmatrix}$$

Therefore, we have

$$\vec{OR} = \vec{OP} + \vec{PR} = \begin{bmatrix} 3 \\ 7 \end{bmatrix} + \begin{bmatrix} 7/17 \\ -28/17 \end{bmatrix} = \begin{bmatrix} 58/17 \\ 91/17 \end{bmatrix}$$

Hence, the point on the line closest to Q is $R(58/17, 91/17)$. The distance from R to Q is

$$\| \operatorname{perp}_{\vec{d}} \vec{PQ} \| = \left\| \begin{bmatrix} -1 \\ -2 \end{bmatrix} - \begin{bmatrix} 7/17 \\ -28/17 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -24/17 \\ -6/17 \end{bmatrix} \right\| = \frac{6}{\sqrt{17}}$$

(c) We first pick the point $P(2, 2, -1)$ on the line. Then the point R on the line that is closest to $Q(1, 0, 1)$ satisfies $\vec{PR} = \operatorname{proj}_{\vec{d}} \vec{PQ}$ where $\vec{PQ} = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$ and $\vec{d} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is a direction vector of the line. We get

$$\vec{PR} = \operatorname{proj}_{\vec{d}} \vec{PQ} = \frac{\vec{PQ} \cdot \vec{d}}{\|\vec{d}\|^2} \vec{d} = \frac{5}{6} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/6 \\ -5/3 \\ 5/6 \end{bmatrix}$$

Therefore, we have

$$\vec{OR} = \vec{OP} + \vec{PR} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 5/6 \\ -5/3 \\ 5/6 \end{bmatrix} = \begin{bmatrix} 17/6 \\ 1/3 \\ -1/6 \end{bmatrix}$$

Hence, the point on the line closest to Q is $R(17/6, 1/3, -1/6)$. The distance from R to Q is

$$\| \operatorname{perp}_{\vec{d}} \vec{PQ} \| = \left\| \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} - \begin{bmatrix} 5/6 \\ -5/3 \\ 5/6 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -11/6 \\ -1/3 \\ 7/6 \end{bmatrix} \right\| = \sqrt{\frac{29}{6}}$$

(d) We first pick the point $P(1, 1, -1)$ on the line. Then the point R on the line that is closest to $Q(2, 3, 2)$ satisfies $\vec{PR} = \operatorname{proj}_{\vec{d}} \vec{PQ}$ where $\vec{PQ} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\vec{d} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$ is a direction vector of the line. We get

$$\vec{PR} = \operatorname{proj}_{\vec{d}} \vec{PQ} = \frac{\vec{PQ} \cdot \vec{d}}{\|\vec{d}\|^2} \vec{d} = \frac{12}{18} \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 8/3 \\ 2/3 \end{bmatrix}$$

Therefore, we have

$$\vec{OR} = \vec{OP} + \vec{PR} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2/3 \\ 8/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 11/3 \\ -1/3 \end{bmatrix}$$

Hence, the point on the line closest to Q is $R(5/3, 11/3, -1/3)$. The distance from R to Q is

$$\| \operatorname{perp}_{\vec{d}} \vec{PQ} \| = \left\| \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 2/3 \\ 8/3 \\ 2/3 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 1/3 \\ -2/3 \\ 7/3 \end{bmatrix} \right\| = \sqrt{6}$$

A6 (a) We first pick any point P on the plane (that is, any point $P(x_1, x_2, x_3)$ such that $3x_1 - x_2 + 4x_3 = 5$). We pick $P(0, -5, 0)$. Then the distance from Q to the plane is the length of the projection of $\vec{PQ} = \begin{bmatrix} 2 \\ 8 \\ 1 \end{bmatrix}$ onto a normal vector of the plane, say $\vec{n} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$. Thus, the distance is

$$\| \operatorname{proj}_{\vec{n}} \vec{PQ} \| = \left| \frac{\vec{PQ} \cdot \vec{n}}{\|\vec{n}\|} \right| = \frac{2}{\sqrt{26}}$$

(b) We pick the point $P(0, 0, -1)$ on the plane and pick the normal vector for the plane $\vec{n} = \begin{bmatrix} 2 \\ -3 \\ -5 \end{bmatrix}$. Then the distance from Q to the plane is

$$\| \operatorname{proj}_{\vec{n}} \vec{PQ} \| = \left| \frac{\vec{PQ} \cdot \vec{n}}{\|\vec{n}\|} \right| = \frac{13}{\sqrt{38}}$$

(c) We pick the point $P(0, 0, -5)$ on the plane and pick the normal vector for the plane $\vec{n} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$. Then the distance from Q to the plane is

$$\| \operatorname{proj}_{\vec{n}} \vec{PQ} \| = \left| \frac{\vec{PQ} \cdot \vec{n}}{\|\vec{n}\|} \right| = \frac{4}{\sqrt{5}}$$

(d) We pick the point $P(2, 0, 0)$ on the plane and pick the normal vector for the plane $\vec{n} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$. Then the distance from Q to the plane is

$$\| \operatorname{proj}_{\vec{n}} \vec{PQ} \| = \left| \frac{\vec{PQ} \cdot \vec{n}}{\|\vec{n}\|} \right| = \sqrt{6}$$

A7 (a) Pick a point P on the hyperplane, say $P(0, 0, 0, 0)$. Then the point R on the hyperplane that is closest to $Q(1, 0, 0, 1)$ satisfies $\vec{OR} = \vec{OQ} + \operatorname{proj}_{\vec{n}} \vec{QP}$ where $\vec{n}$ is a normal vector of the hyperplane. We have $\vec{QP} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ and $\vec{n} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 1 \end{bmatrix}$, so

$$\vec{OR} = \vec{OQ} + \frac{\vec{QP} \cdot \vec{n}}{\|\vec{n}\|^2} \vec{n} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \frac{-3}{7} \begin{bmatrix} 2 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -6/7 \\ 3/7 \\ -3/7 \\ -3/7 \end{bmatrix} = \begin{bmatrix} 1/7 \\ 3/7 \\ -3/7 \\ 4/7 \end{bmatrix}$$

Hence, the point in the hyperplane closest to Q is $R(1/7, 3/7, -3/7, 4/7)$.

(b) We pick the point $P(1,0,0,0)$ on the hyperplane and pick the normal vector $\vec{n} = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 0 \end{bmatrix}$ for the hyperplane.

Then the point R in the hyperplane closest to Q satisfies

$$\vec{OR} = \vec{OQ} + \frac{\vec{QP}\cdot\vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} + \frac{1}{14}\begin{bmatrix} 1 \\ -2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1/14 \\ -2/14 \\ 3/14 \\ 0 \end{bmatrix} = \begin{bmatrix} 15/14 \\ 13/7 \\ 17/14 \\ 3 \end{bmatrix}$$

Hence, the point in the hyperplane closest to Q is $R(15/14, 13/7, 17/14, 3)$.

(c) We pick the point $P(0,0,0,0)$ on the hyperplane and pick the normal vector $\vec{n} = \begin{bmatrix} 3 \\ -1 \\ 4 \\ 1 \end{bmatrix}$ for the hyperplane.

Then the point R in the hyperplane closest to Q satisfies

$$\vec{OR} = \vec{OQ} + \frac{\vec{QP}\cdot\vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} 2 \\ 4 \\ 3 \\ 4 \end{bmatrix} + \frac{-18}{27}\begin{bmatrix} 3 \\ -1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ 2/3 \\ -8/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 14/3 \\ 1/3 \\ 10/3 \end{bmatrix}$$

Hence, the point in the hyperplane closest to Q is $R(0, 14/3, 1/3, 10/3)$.

(d) We pick the point $P(4,0,0,0)$ on the hyperplane and pick the normal vector $\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}$ for the hyperplane.

Then the point R in the hyperplane closest to Q satisfies

$$\vec{OR} = \vec{OQ} + \frac{\vec{QP}\cdot\vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} -1 \\ 3 \\ 2 \\ -2 \end{bmatrix} + \frac{-5}{7}\begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 2 \\ -2 \end{bmatrix} + \begin{bmatrix} -5/7 \\ -10/7 \\ -5/7 \\ 5/7 \end{bmatrix} = \begin{bmatrix} -12/7 \\ 11/7 \\ 9/7 \\ -9/7 \end{bmatrix}$$

Hence, the point in the hyperplane closest to Q is $R(-12/7, 11/7, 9/7, -9/7)$.

Homework Problems

B1 (a) $\text{proj}_{\vec{v}}\,\vec{u} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, $\text{perp}_{\vec{v}}\,\vec{u} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

(b) $\text{proj}_{\vec{v}}\,\vec{u} = \begin{bmatrix} -92/25 \\ 69/25 \end{bmatrix}$, $\text{perp}_{\vec{v}}\,\vec{u} = \begin{bmatrix} 42/25 \\ 56/25 \end{bmatrix}$

(c) $\text{proj}_{\vec{v}}\,\vec{u} = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}$, $\text{perp}_{\vec{v}}\,\vec{u} = \begin{bmatrix} 2 \\ -4 \\ 0 \end{bmatrix}$

(d) $\text{proj}_{\vec{v}}\,\vec{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\text{perp}_{\vec{v}}\,\vec{u} = \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}$

(e) $\text{proj}_{\vec{v}}\,\vec{u} = \begin{bmatrix} 8/3 \\ 8/3 \\ 0 \\ 8/3 \end{bmatrix}$, $\text{perp}_{\vec{v}}\,\vec{u} = \begin{bmatrix} 1/3 \\ 1/3 \\ -4 \\ -2/3 \end{bmatrix}$

(f) $\text{proj}_{\vec{v}}\,\vec{u} = \begin{bmatrix} 3/5 \\ 6/5 \\ -3/5 \\ 9/5 \end{bmatrix}$, $\text{perp}_{\vec{v}}\,\vec{u} = \begin{bmatrix} 7/5 \\ 9/5 \\ 13/5 \\ -4/5 \end{bmatrix}$

B2 (a) $\text{proj}_{\vec{v}}\,\vec{u} = \begin{bmatrix} -18/13 \\ -27/13 \end{bmatrix}$, $\text{perp}_{\vec{v}}\,\vec{u} = \begin{bmatrix} -21/13 \\ 14/13 \end{bmatrix}$

(b) $\text{proj}_{\vec{v}}\,\vec{u} = \begin{bmatrix} 11/10 \\ 33/10 \end{bmatrix}$, $\text{perp}_{\vec{v}}\,\vec{u} = \begin{bmatrix} 9/10 \\ -3/10 \end{bmatrix}$

(c) $\text{proj}_{\vec{v}}\,\vec{u} = \begin{bmatrix} 1/6 \\ -1/3 \\ 1/6 \end{bmatrix}$, $\text{perp}_{\vec{v}}\,\vec{u} = \begin{bmatrix} 29/6 \\ 4/3 \\ -13/6 \end{bmatrix}$

(d) $\text{proj}_{\vec{v}}\,\vec{u} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $\text{perp}_{\vec{v}}\,\vec{u} = \begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix}$

(e) $\text{proj}_{\vec{v}}\,\vec{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\text{perp}_{\vec{v}}\,\vec{u} = \begin{bmatrix} 6 \\ 2 \\ 6 \end{bmatrix}$

(f) $\text{proj}_{\vec{v}}\,\vec{u} = \begin{bmatrix} -1/3 \\ 0 \\ -1/6 \\ -1/6 \end{bmatrix}$, $\text{perp}_{\vec{v}}\,\vec{u} = \begin{bmatrix} -2/3 \\ 2 \\ -5/6 \\ 13/6 \end{bmatrix}$

B3 (a) $\hat{u} = \dfrac{\vec{u}}{\|\vec{u}\|} = \begin{bmatrix} -1/\sqrt{69} \\ 8/\sqrt{69} \\ 2/\sqrt{69} \end{bmatrix}$

(b) $\text{proj}_{\vec{v}}\,\vec{u} = \begin{bmatrix} 124/69 \\ -992/69 \\ -248/69 \end{bmatrix}$

(c) $\text{perp}_{\vec{v}}\,\vec{u} = \begin{bmatrix} 704/69 \\ -43/69 \\ 524/69 \end{bmatrix}$

B4 (a) $\hat{u} = \dfrac{\vec{u}}{\|\vec{u}\|} = \begin{bmatrix} 2/\sqrt{29} \\ -4/\sqrt{29} \\ 3/\sqrt{29} \end{bmatrix}$

(b) $\text{proj}_{\vec{v}}\,\vec{u} = \begin{bmatrix} -66/29 \\ 132/29 \\ -99/29 \end{bmatrix}$

(c) $\text{perp}_{\vec{v}}\,\vec{u} = \begin{bmatrix} 211/29 \\ 245/29 \\ 186/29 \end{bmatrix}$

B5 (a) $P(71/25, -128/25)$, $1/5$

(b) $P(14/9, 17/9, 8/9)$, $\sqrt{65}/3$

(c) $P(2/3, -4/3, -4/3)$, $\sqrt{14/3}$

(d) $P(106/21, 31/21, 2/21)$, $\sqrt{950/21}$

B6 (a) $11/\sqrt{38}$

(b) $13/\sqrt{21}$

(c) $5/\sqrt{3}$

(d) $5/\sqrt{6}$

B7 (a) $P(13/7, 1, 1/14, -17/14)$

(b) $P(16/9, 20/9, 7/18, 1/6)$

(c) $P(15/4, 3/4, 7/4, -23/4)$

(d) $P(14/3, 11/6, 7/3, 13/2)$

Computer Problems

C1 (a) $\begin{bmatrix} 0.08 \\ 0.16 \\ 0.53 \\ 0.31 \\ 0.46 \end{bmatrix}$

(b) $\begin{bmatrix} 0.92 \\ -1.17 \\ 0.49 \\ -1.34 \\ 0.58 \end{bmatrix}$

(c) $\begin{bmatrix} 1.59 \\ -1.61 \\ 1.62 \\ -1.64 \\ 1.65 \end{bmatrix}$

(d) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Conceptual Problems

D1 (a) We have

$$\begin{aligned} C(\vec{x}+\vec{y}) &= \mathrm{proj}_{\vec{u}}(\mathrm{proj}_{\vec{v}}(\vec{x}+\vec{y})) = \mathrm{proj}_{\vec{u}}(\mathrm{proj}_{\vec{v}}\,\vec{x}+\mathrm{proj}_{\vec{v}}\,\vec{y}) \\ &= \mathrm{proj}_{\vec{u}}(\mathrm{proj}_{\vec{v}}\,\vec{x}) + \mathrm{proj}_{\vec{u}}(\mathrm{proj}_{\vec{v}}\,\vec{y}) = C(\vec{x})+C(\vec{y}) \\ C(t\vec{x}) &= \mathrm{proj}_{\vec{u}}(\mathrm{proj}_{\vec{v}}(t\vec{x})) = \mathrm{proj}_{\vec{u}}(t\,\mathrm{proj}_{\vec{v}}\,\vec{x}) \\ &= t\,\mathrm{proj}_{\vec{u}}(\mathrm{proj}_{\vec{v}}\,\vec{x}) = tC(\vec{x}) \end{aligned}$$

(b) If $C(\vec{x}) = \vec{0}$ for all $\vec{x}$, then certainly

$$\vec{0} = C(\vec{v}) = \mathrm{proj}_{\vec{u}}(\mathrm{proj}_{\vec{v}}\,\vec{v}) = \mathrm{proj}_{\vec{u}}\,\vec{v} = \frac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u}$$

Hence, $\vec{v}\cdot\vec{u} = 0$, and the vectors $\vec{u}$ and $\vec{v}$ are orthogonal to each other.

D2

$$\mathrm{proj}_{-\vec{u}}\,\vec{x} = \frac{\vec{x}\cdot(-\vec{u})}{\|-\vec{u}\|^2}(-\vec{u}) = \frac{-(\vec{x}\cdot\vec{u})}{\|\vec{u}\|^2}(-\vec{u}) = \frac{\vec{x}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u} = \mathrm{proj}_{\vec{u}}\,\vec{x}$$

Geometrically, $\mathrm{proj}_{-\vec{u}}\,\vec{x}$ is a vector along the line through the origin with direction vector $-\vec{u}$, and this line is the same as the line with direction vector $\vec{u}$. We have that $\mathrm{proj}_{-\vec{u}}\,\vec{x}$ is the point on this line that is closest to $\vec{x}$ and this is the same as $\mathrm{proj}_{\vec{u}}\,\vec{x}$.

D3 (a)

$$\begin{aligned} \|\vec{x}+\vec{y}\|^2 &= (\vec{x}+\vec{y})\cdot(\vec{x}+\vec{y}) = \vec{x}\cdot\vec{x}+\vec{x}\cdot\vec{y}+\vec{y}\cdot\vec{x}+\vec{y}\cdot\vec{y} \\ &= \|\vec{x}\|^2 + 2\vec{x}\cdot\vec{y} + \|\vec{y}\|^2 \end{aligned}$$

Hence, $\|\vec{x}+\vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2$ if and only if $\vec{x}\cdot\vec{y} = 0$.

(b) Following the hint, we subtract and add $\mathrm{proj}_{\vec{d}}\,\vec{p}$:

$$\begin{aligned} \|\vec{p}-\vec{q}\|^2 &= \|\vec{p} - \mathrm{proj}_{\vec{d}}\,\vec{p} + \mathrm{proj}_{\vec{d}}\,\vec{p} - \vec{q}\|^2 \\ &= \left\| \mathrm{perp}_{\vec{d}}\,\vec{p} + \left(\frac{\vec{p}\cdot\vec{d}}{\|\vec{d}\|^2} - t\right)\vec{d} \right\|^2 \end{aligned}$$

Since, $\vec{d}\cdot\mathrm{perp}_{\vec{d}}\,\vec{p} = 0$, we can apply the result of (a) to get

$$\|\vec{p}-\vec{q}\|^2 = \|\,\mathrm{perp}_{\vec{d}}\,\vec{p}\|^2 + \|\,\mathrm{proj}_{\vec{d}}\,\vec{p} - \vec{q}\|^2$$

Since $\vec{p}$ and $\vec{d}$ are given, $\mathrm{perp}_{\vec{d}}\,\vec{p}$ is fixed, so to make this expression as small as possible choose $\vec{q} = \mathrm{proj}_{\vec{d}}\,\vec{p}$. Thus, the distance from the point $\vec{p}$ to a point on the line is minimized by the point $\vec{q} = \mathrm{proj}_{\vec{d}}\,\vec{p}$ on the line.

D4

$$\begin{aligned} \vec{OP} + \mathrm{perp}_{\vec{n}}\,\vec{PQ} &= \vec{OP} + \left(\vec{PQ} - \mathrm{proj}_{\vec{n}}\,\vec{PQ}\right) \\ &= \left(\vec{OP} + \vec{PQ}\right) + \mathrm{proj}_{\vec{n}}\left(-\vec{PQ}\right) = \vec{OQ} + \mathrm{proj}_{\vec{n}}\,\vec{QP} \end{aligned}$$

D5 (a)

$$\operatorname{perp}_{\vec{u}}\vec{x} = \vec{x} - \frac{\vec{x}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u} = \begin{bmatrix} 2/3 \\ 11/3 \\ 13/3 \end{bmatrix}$$

$$\operatorname{proj}_{\vec{u}}(\operatorname{perp}_{\vec{u}}\vec{x}) = \frac{\operatorname{perp}_{\vec{u}}\vec{x}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(b)

$$\begin{aligned}\operatorname{proj}_{\vec{u}}(\operatorname{perp}_{\vec{u}}\vec{x}) &= \left[\left(\vec{x} - \frac{\vec{x}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u}\right)\cdot\frac{\vec{u}}{\|\vec{u}\|^2}\right]\vec{u} \\ &= \left[\frac{\vec{x}\cdot\vec{u}}{\|\vec{u}\|^2} - \frac{(\vec{x}\cdot\vec{u})(\vec{u}\cdot\vec{u})}{\|\vec{u}\|^4}\right]\vec{u} \\ &= \left[\frac{\vec{x}\cdot\vec{u}}{\|\vec{u}\|^2} - \frac{\vec{x}\cdot\vec{u}}{\|\vec{u}\|^2}\right]\vec{u} \\ &= \vec{0}\end{aligned}$$

(c) $\operatorname{proj}_{\vec{u}}(\operatorname{perp}_{\vec{u}}(\vec{x})) = \vec{0}$ since $\operatorname{perp}_{\vec{u}}\vec{x}$ is orthogonal to $\vec{u}$ and $\operatorname{proj}_{\vec{u}}$ maps vectors orthogonal to $\vec{u}$ to the zero vector.

1.5 Cross-Products and Volumes

Practice Problems

A1 (a) $\begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix} \times \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} (-5)(5) - 1(2) \\ 2(-2) - 1(5) \\ 1(1) - (-5)(-2) \end{bmatrix} = \begin{bmatrix} -27 \\ -9 \\ -9 \end{bmatrix}$

(b) $\begin{bmatrix} 2 \\ -3 \\ -5 \end{bmatrix} \times \begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix} = \begin{bmatrix} (-3)(7) - (-5)(-2) \\ (-5)(4) - 2(7) \\ 2(-2) - (-3)(4) \end{bmatrix} = \begin{bmatrix} -31 \\ -34 \\ 8 \end{bmatrix}$

(c) $\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 0(5) - (-1)(4) \\ (-1)(0) - (-1)(5) \\ (-1)(4) - 0(0) \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -4 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \times \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2(0) - 0(-3) \\ 0(-1) - 1(0) \\ 1(-3) - 2(-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

(e) $\begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix} \times \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} (-2)(-3) - 6(1) \\ 6(-2) - 4(-3) \\ 4(1) - (-2)(-2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(f) $\begin{bmatrix}3\\1\\3\end{bmatrix} \times \begin{bmatrix}3\\1\\3\end{bmatrix} = \begin{bmatrix}1(3)-3(1)\\3(3)-3(3)\\3(1)-1(3)\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

A2 (a) $\vec{u} \times \vec{u} = \begin{bmatrix}4(2)-2(4)\\2(-1)-(-1)(2)\\(-1)(4)-4(-1)\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

(b) We have

$$\vec{u} \times \vec{v} = \begin{bmatrix}4(-1)-2(1)\\2(3)-(-1)(-1)\\(-1)(1)-4(3)\end{bmatrix} = \begin{bmatrix}-6\\5\\-13\end{bmatrix}$$

$$-\vec{v} \times \vec{u} = -\begin{bmatrix}1(2)-(-1)(4)\\(-1)(-1)-3(2)\\3(4)-1(-1)\end{bmatrix} = -\begin{bmatrix}6\\-5\\13\end{bmatrix} = \vec{u} \times \vec{v}$$

(c) We have

$$\vec{u} \times 3\vec{w} = \begin{bmatrix}-1\\4\\2\end{bmatrix} \times \begin{bmatrix}6\\-9\\-3\end{bmatrix} = \begin{bmatrix}4(-3)-2(-9)\\2(6)-(-1)(-3)\\(-1)(-9)-4(6)\end{bmatrix} = \begin{bmatrix}6\\9\\-15\end{bmatrix}$$

$$3(\vec{u} \times \vec{w}) = 3\begin{bmatrix}4(-1)-2(-3)\\2(2)-(-1)(-1)\\(-1)(-3)-4(2)\end{bmatrix} = 3\begin{bmatrix}2\\3\\-5\end{bmatrix} = \begin{bmatrix}6\\9\\-15\end{bmatrix}$$

(d) We have

$$\begin{aligned}\vec{u} \times (\vec{v} + \vec{w}) &= \begin{bmatrix}-1\\4\\2\end{bmatrix} \times \begin{bmatrix}5\\-2\\-2\end{bmatrix}\\ &= \begin{bmatrix}4(-2)-2(-2)\\2(5)-(-1)(-2)\\(-1)(-2)-4(5)\end{bmatrix} = \begin{bmatrix}-4\\8\\-18\end{bmatrix}\\ \vec{u} \times \vec{v} + \vec{u} \times \vec{w} &= \begin{bmatrix}-6\\5\\-13\end{bmatrix} + \begin{bmatrix}2\\3\\-5\end{bmatrix} = \begin{bmatrix}-4\\8\\-18\end{bmatrix}\end{aligned}$$

(e) We have

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{bmatrix}-1\\4\\2\end{bmatrix} \cdot \begin{bmatrix}1(-1)-(-1)(-3)\\(-1)(2)-3(-1)\\3(-3)-1(2)\end{bmatrix} = \begin{bmatrix}-1\\4\\2\end{bmatrix} \cdot \begin{bmatrix}-4\\1\\-11\end{bmatrix} = -14$$

$$\vec{w} \cdot (\vec{u} \times \vec{v}) = \begin{bmatrix}2\\-3\\-1\end{bmatrix} \cdot \begin{bmatrix}-6\\5\\-13\end{bmatrix} = -14$$

(f) From part (e) we have $\vec{u}\cdot(\vec{v}\times\vec{w}) = -14$. Then

$$\vec{v}\cdot(\vec{u}\times\vec{w}) = \begin{bmatrix}3\\1\\-1\end{bmatrix}\cdot\begin{bmatrix}2\\3\\-5\end{bmatrix} = 14 = -\vec{u}\cdot(\vec{v}\times\vec{w})$$

A3 (a) The area of the parallelogram is

$$\left\|\begin{bmatrix}1\\2\\1\end{bmatrix}\times\begin{bmatrix}2\\3\\-1\end{bmatrix}\right\| = \left\|\begin{bmatrix}-5\\3\\-1\end{bmatrix}\right\| = \sqrt{35}$$

(b) The area of the parallelogram is

$$\left\|\begin{bmatrix}1\\0\\1\end{bmatrix}\times\begin{bmatrix}1\\1\\4\end{bmatrix}\right\| = \left\|\begin{bmatrix}-1\\-3\\1\end{bmatrix}\right\| = \sqrt{11}$$

(c) The area of the parallelogram is

$$\left\|\begin{bmatrix}1\\2\\0\end{bmatrix}\times\begin{bmatrix}-2\\5\\0\end{bmatrix}\right\| = \left\|\begin{bmatrix}0\\0\\9\end{bmatrix}\right\| = 9$$

(d) As specified in the hint, we write the vectors as $\begin{bmatrix}-3\\1\\0\end{bmatrix}$ and $\begin{bmatrix}4\\3\\0\end{bmatrix}$. Hence, the area of the parallelogram is

$$\left\|\begin{bmatrix}-3\\1\\0\end{bmatrix}\times\begin{bmatrix}4\\3\\0\end{bmatrix}\right\| = \left\|\begin{bmatrix}0\\0\\-13\end{bmatrix}\right\| = 13$$

A4 (a) A normal vector of the plane is $\vec{n} = \begin{bmatrix}2\\3\\-1\end{bmatrix}\times\begin{bmatrix}4\\1\\0\end{bmatrix} = \begin{bmatrix}1\\-4\\-10\end{bmatrix}$. Thus, since the plane passes through the point $P(1,4,7)$, we get the scalar equation of the plane is

$$x_1 - 4x_2 - 10x_3 = 1(1) + (-4)(4) + (-10)(7) = -85$$

(b) A normal vector of the plane is $\vec{n} = \begin{bmatrix}1\\1\\0\end{bmatrix}\times\begin{bmatrix}-2\\1\\2\end{bmatrix} = \begin{bmatrix}2\\-2\\3\end{bmatrix}$. Thus, since the plane passes through the point $P(2,3,-1)$, we get the scalar equation of the plane is

$$2x_1 - 2x_2 + 3x_3 = 2(2) + (-2)(3) + 3(-1) = -5$$

(c) A normal vector of the plane is $\vec{n} = \begin{bmatrix}2\\-2\\1\end{bmatrix}\times\begin{bmatrix}0\\3\\1\end{bmatrix} = \begin{bmatrix}-5\\-2\\6\end{bmatrix}$. Thus, since the plane passes through the point $P(1,-1,3)$, we get the scalar equation of the plane is

$$-5x_1 - 2x_2 + 6x_3 = (-5)(1) + (-2)(-1) + 6(3) = 15$$

(d) A normal vector of the plane is $\vec{n} = \begin{bmatrix}1\\3\\2\end{bmatrix} \times \begin{bmatrix}-2\\4\\-3\end{bmatrix} = \begin{bmatrix}-17\\-1\\10\end{bmatrix}$. Thus, since the plane passes through the point $P(0,0,0)$, we get the scalar equation of the plane is

$$-17x_1 - x_2 + 10x_3 = 0$$

A5 (a) We have that the vectors $\vec{PQ} = \begin{bmatrix}2\\-4\\-3\end{bmatrix}$ and $\vec{PR} = \begin{bmatrix}0\\5\\-6\end{bmatrix}$ are vectors in the plane. Hence, a normal vector for the plane is $\vec{n} = \begin{bmatrix}2\\-4\\-3\end{bmatrix} \times \begin{bmatrix}0\\5\\-6\end{bmatrix} = \begin{bmatrix}39\\12\\10\end{bmatrix}$. Then, since $P(2,1,5)$ is a point on the plane we get a scalar equation of the plane is

$$39x_1 + 12x_2 + 10x_3 = 39(2) + 12(1) + 10(5) = 140$$

(b) We have that the vectors $\vec{PQ} = \begin{bmatrix}-5\\-1\\-2\end{bmatrix}$ and $\vec{PR} = \begin{bmatrix}-2\\3\\-5\end{bmatrix}$ are vectors in the plane. Hence, a normal vector for the plane is $\vec{n} = \begin{bmatrix}-5\\-1\\-2\end{bmatrix} \times \begin{bmatrix}-2\\3\\-5\end{bmatrix} = \begin{bmatrix}11\\-21\\-17\end{bmatrix}$. Then, since $P(3,1,4)$ is a point on the plane we get a scalar equation of the plane is

$$11x_1 - 21x_2 - 17x_3 = 11(3) - 21(1) - 17(4) = -56$$

(c) We have that the vectors $\vec{PQ} = \begin{bmatrix}4\\-3\\-3\end{bmatrix}$ and $\vec{PR} = \begin{bmatrix}3\\-7\\-3\end{bmatrix}$ are vectors in the plane. Hence, a normal vector for the plane is $\vec{n} = \begin{bmatrix}4\\-3\\-3\end{bmatrix} \times \begin{bmatrix}3\\-7\\-3\end{bmatrix} = \begin{bmatrix}-12\\3\\-19\end{bmatrix}$. Then, since $P(-1,4,2)$ is a point on the plane we get a scalar equation of the plane is

$$-12x_1 + 3x_2 - 19x_3 = -12(-1) + 3(4) - 19(2) = -14$$

(d) We have that the vectors $\vec{PQ} = \begin{bmatrix}-2\\0\\0\end{bmatrix}$ and $\vec{PR} = \begin{bmatrix}-1\\0\\-1\end{bmatrix}$ are vectors in the plane. Hence, a normal vector for the plane is $\vec{n} = \begin{bmatrix}-2\\0\\0\end{bmatrix} \times \begin{bmatrix}-1\\0\\-1\end{bmatrix} = \begin{bmatrix}0\\-2\\0\end{bmatrix}$. Then, since $R(0,0,0)$ is a point on the plane we get a scalar equation of the plane is $-2x_2 = 0$ or $x_2 = 0$.

A6 (a) The line of intersection must lie in both planes and hence it must be orthogonal to both normal vectors. Hence, a direction vector of the line is

$$\vec{d} = \begin{bmatrix}1\\3\\-1\end{bmatrix} \times \begin{bmatrix}2\\-5\\1\end{bmatrix} = \begin{bmatrix}-2\\-3\\-11\end{bmatrix}$$

To find a point on the line we set $x_3 = 0$ in the equations of both planes to get $x_1 + 3x_2 = 5$ and $2x_1 - 5x_2 = 7$. Solving the two equations in two unknowns gives the solution $x_1 = \frac{46}{11}$ and $x_2 = \frac{3}{11}$. Thus, an equation of the line is

$$\vec{x} = \begin{bmatrix} 46/11 \\ 3/11 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ -3 \\ -11 \end{bmatrix}, \quad t \in \mathbb{R}$$

(b) A direction vector of the line is

$$\vec{d} = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}$$

To find a point on the line we set $x_3 = 0$ to get $2x_1 = 7$ and $x_2 = 4$. Thus, an equation of the line is

$$\vec{x} = \begin{bmatrix} 7/2 \\ 4 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}, \quad t \in \mathbb{R}$$

A7 (a) The volume of the parallelepiped is

$$\left| \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) \right| = \left| \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right| = 1$$

(b) The volume of the parallelepiped is

$$\left| \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} \cdot \left(\begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix} \right) \right| = \left| \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 28 \\ 8 \\ -6 \end{bmatrix} \right| = 126$$

(c) The volume of the parallelepiped is

$$\left| \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \cdot \left(\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \times \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} \right) \right| = \left| \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -15 \\ 6 \end{bmatrix} \right| = |-5| = 5$$

(d) The volume of the parallelepiped is

$$\left| \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \right) \right| = \left| \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -7 \\ 0 \end{bmatrix} \right| = |-35| = 35$$

(e) The volume of the parallelepiped is

$$\left| \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \cdot \left(\begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix} \times \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right) \right| = \left| \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 4 \\ -12 \end{bmatrix} \right| = |-16| = 16$$

A8 $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$ means that $\vec{u}$ is orthogonal to $\vec{v} \times \vec{w}$. Therefore, $\vec{u}$ lies in the plane through the origin that contains $\vec{v}$ and $\vec{w}$. We can also see this by observing that $\vec{u} \cdot (\vec{v} \times \vec{w}) = 0$ means that the parallelepiped determined by $\vec{u}$, $\vec{v}$, and $\vec{w}$ has volume zero; this can happen only if the three vectors lie in a common plane.

A9 We have

$$\begin{aligned}(\vec{u} - \vec{v}) \times (\vec{u} + \vec{v}) &= \vec{u} \times (\vec{u} + \vec{v}) - \vec{v} \times (\vec{u} + \vec{v}) \\ &= \vec{u} \times \vec{u} + \vec{u} \times \vec{v} - \vec{v} \times \vec{u} - \vec{v} \times \vec{v} \\ &= \vec{0} + \vec{u} \times \vec{v} + \vec{u} \times \vec{v} - \vec{0} \\ &= 2(\vec{u} \times \vec{v})\end{aligned}$$

as required.

Homework Problems

B1 (a) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} -29 \\ 14 \\ 12 \end{bmatrix}$

(c) $\begin{bmatrix} 4 \\ 2 \\ -8 \end{bmatrix}$

(d) $\begin{bmatrix} 2 \\ 6 \\ -16 \end{bmatrix}$

(e) $\begin{bmatrix} -25/2 \\ -13/2 \\ -2 \end{bmatrix}$

(f) $\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$

B2 (a) $\vec{u} \times \vec{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(b) $\vec{u} \times \vec{v} = \begin{bmatrix} -6 \\ 8 \\ 16 \end{bmatrix} = -\vec{v} \times \vec{u}$

(c) $\vec{u} \times 2\vec{w} = \begin{bmatrix} 10 \\ -22 \\ -18 \end{bmatrix} = 2(\vec{u} \times \vec{w})$

(d) $\vec{u} \times (\vec{v} + \vec{w}) = \begin{bmatrix} -1 \\ -3 \\ 7 \end{bmatrix} = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

(e) $\vec{u} \cdot (\vec{v} \times \vec{w}) = 26 = \vec{w} \cdot (\vec{u} \times \vec{v})$

(f) $\vec{u} \cdot (\vec{v} \times \vec{w}) = 26 = -\vec{v} \cdot (\vec{u} \times \vec{w})$

B3 (a) $3\sqrt{10}$

(b) $\sqrt{542}$

(c) $\sqrt{222}$

(d) 19

B4 (a) $23x_1 + 3x_2 - 7x_3 = 121$

(b) $4x_1 - 10x_2 - 2x_3 = -6$

(c) $2x_1 + 3x_2 + x_3 = 23$

(d) $3x_2 - 6x_3 = 0$

B5 (a) $3x_1 + 49x_2 + 8x_3 = 121$

(b) $-8x_1 - 10x_2 - 4x_3 = -38$

(c) $3x_1 + 3x_2 + 6x_3 = 12$

(d) $14x_1 - 4x_2 - 5x_3 = 9$

B6 (a) $\vec{x} = \begin{bmatrix} 59/19 \\ 9/19 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 4 \\ -19 \end{bmatrix}, t \in \mathbb{R}$

(b) $\vec{x} = \begin{bmatrix} 14/11 \\ -10/11 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -7 \\ 11 \end{bmatrix}, t \in \mathbb{R}$

B7 (a) 117

(b) 6

(c) 40

(d) 48

Conceptual Problems

D1 If X is a point on the line through P and Q, then for some $t \in \mathbb{R}$, $\vec{x} = \vec{p} + t(\vec{q} - \vec{p})$ Hence,

$$\begin{aligned} \vec{x} \times (\vec{q} - \vec{p}) &= (\vec{p} + t(\vec{q} - \vec{p})) \times (\vec{q} - \vec{p}) \\ &= \vec{p} \times \vec{q} - \vec{p} \times \vec{p} + t(\vec{q} - \vec{p}) \times (\vec{q} - \vec{p}) = \vec{p} \times \vec{q} \end{aligned}$$

D2 The statement is false. For any non-zero vector $\vec{u}$ and any vector $\vec{v} \in \mathbb{R}^3$, let $\vec{w} = \vec{v} + t\vec{u}$ for any $t \in \mathbb{R}$, $t \neq 0$. Then

$$\vec{u} \times \vec{w} = \vec{u} \times (\vec{v} + t\vec{u}) = \vec{u} \times \vec{v}$$

but $\vec{v} \neq \vec{w}$.

D3 If $\vec{v}\times\vec{w} = \vec{0}$, then $\vec{u}\times(\vec{v}\times\vec{w}) = \vec{0}$ which clearly satisfies the equation $\vec{x} = s\vec{v} + t\vec{w}$. Assume $\vec{n} = \vec{v}\times\vec{w} \neq \vec{0}$. Then $\vec{n}$ is orthogonal to both $\vec{v}$ and $\vec{w}$ and hence it is a normal vector of the plane through the origin containing $\vec{v}$ and $\vec{w}$. Then, $\vec{u}\times(\vec{v}\times\vec{w}) = \vec{u}\times\vec{n}$ is orthogonal to $\vec{n}$ so it lies in the plane through the origin with normal vector $\vec{n}$. That is, it is in the plane containing $\vec{v}$ and $\vec{w}$. Hence, there exists $s, t \in \mathbb{R}$ such that $\vec{u}\times(\vec{v}\times\vec{w}) = s\vec{v} + t\vec{w}$.

D4 (a) We have $\vec{e}_1 \times (\vec{e}_2 \times \vec{e}_3) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = (\vec{e}_1 \times \vec{e}_2) \times \vec{e}_3$.

(b) Take $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Then $\vec{e}_1 \times (\vec{e}_2 \times \vec{w}) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ while $(\vec{e}_1 \times \vec{e}_2) \times \vec{w} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

Chapter 1 Quiz

Problems

E1 Any direction vector of the line is a non-zero scalar multiple of the directed line segment between P and Q. Thus, we can take $\vec{d} = \vec{PQ} = \begin{bmatrix} 5-(-2) \\ -2-1 \\ 1-(-4) \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ 5 \end{bmatrix}$. Thus, since $P(-2, 1, -4)$ is a point on the line we get that a vector equation of the line is

$$\vec{x} = \begin{bmatrix} -2 \\ 1 \\ -4 \end{bmatrix} + t\begin{bmatrix} 7 \\ -3 \\ 5 \end{bmatrix}, \quad t \in \mathbb{R}$$

E2 We have that the vectors $\vec{PQ} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$ and $\vec{PR} = \begin{bmatrix} -5 \\ 2 \\ 6 \end{bmatrix}$ are vectors in the plane. Hence, a normal vector for the plane is $\vec{n} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix} \times \begin{bmatrix} -5 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 16 \\ -2 \\ 14 \end{bmatrix}$. Then, since $P(1, -1, 0)$ is a point on the plane we get a scalar equation of the plane is

$$16x_1 - 2x_2 + 14x_3 = 16(1) - 2(-1) + 14(0) = 18$$

or $8x_1 - x_2 + 7x_3 = 9$.

E3 To show that $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right\}$ is a basis, we need to show that it spans $\mathbb{R}^2$ and that it is linearly independent. Consider

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t_1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + t_2\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} t_1 - t_2 \\ 2t_1 + 2t_2 \end{bmatrix}$$

This gives $x_1 = t_1 - t_2$ and $x_2 = 2t_1 + 2t_2$. Solving using substitution and elimination we get $t_1 = \frac{1}{4}(2x_1 + x_2)$ and $t_2 = \frac{1}{4}(-2x_1 + x_2)$. Hence, every vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ can be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{4}(2x_1 + x_2)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{4}(-2x_1 + x_2)\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

So, it spans $\mathbb{R}^2$. Moreover, if $x_1 = x_2 = 0$, then our calculations above show that $t_1 = t_2 = 0$, so the set is also linearly independent. Therefore, it is a basis for $\mathbb{R}^2$.

E4 If $d \neq 0$, then $a_1(0) + a_2(0) + a_3(0) = 0 \neq d$, so $\vec{0} \notin S$ and thus, S is not a subspace of $\mathbb{R}^3$.

On the other hand, assume $d = 0$. Observe that, by definition, S is a subset of $\mathbb{R}^3$ and that $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in S$ since taking $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$ satisfies $a_1x_1 + a_2x_2 + a_3x_3 = 0$.

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in S$. Then they must satisfy the condition of the set, so $a_1x_1 + a_2x_2 + a_3x_3 = 0$ and $a_1y_1 + a_2y_2 + a_3y_3 = 0$.

To show that S is closed under addition, we must show that $\vec{x} + \vec{y}$ satisfies the condition of S. We have $\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$ and

$$\begin{aligned} a_1(x_1 + y_1) + a_2(x_2 + y_2) + a_3(x_3 + y_3) &= a_1x_1 + a_2x_2 + a_3x_3 + a_1y_1 + a_2y_2 + a_3y_3 \\ &= 0 + 0 = 0 \end{aligned}$$

Hence, $\vec{x} + \vec{y} \in S$. Similarly, for any $t \in \mathbb{R}$ we have $t\vec{x} = \begin{bmatrix} tx_1 \\ tx_2 \\ tx_3 \end{bmatrix}$ and

$$a_1(tx_1) + a_2(tx_2) + a_3(tx_3) = t(a_1x_1 + a_2x_2 + a_3x_3) = t(0) = 0$$

So, S is closed under scalar multiplication. Therefore, S is a subspace of $\mathbb{R}^3$.

E5 The coordinate axes have direction vector given by the standard basis vectors. The cosine of the angle between $\vec{v}$ and $\vec{e}_1$ is

$$\cos\alpha = \frac{\vec{v} \cdot \vec{e}_1}{\|\vec{v}\|\|\vec{e}_1\|} = \frac{2}{\sqrt{14}}$$

The cosine of the angle between $\vec{v}$ and $\vec{e}_2$ is

$$\cos\beta = \frac{\vec{v} \cdot \vec{e}_2}{\|\vec{v}\|\|\vec{e}_2\|} = \frac{-3}{\sqrt{14}}$$

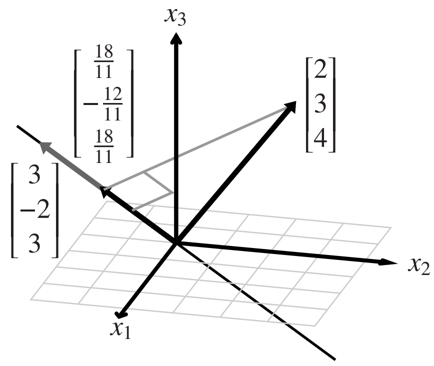
The cosine of the angle between $\vec{v}$ and $\vec{e}_3$ is

$$\cos\gamma = \frac{\vec{v} \cdot \vec{e}_3}{\|\vec{v}\|\|\vec{e}_3\|} = \frac{1}{\sqrt{14}}$$

E6 Since the origin $O(0,0,0)$ is on the line, we get that the point Q on the line closest to P is given by $\vec{OQ} = \text{proj}_{\vec{d}}\,\vec{OP}$, where $\vec{d} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}$ is a direction vector of the line. Hence,

$$\vec{OQ} = \frac{\vec{OP}\cdot\vec{d}}{\|\vec{d}\|^2}\vec{d} = \begin{bmatrix} 18/11 \\ -12/11 \\ 18/11 \end{bmatrix}$$

and the closest point is $Q(18/11, -12/11, 18/11)$.



E7 Let $Q(0,0,0,1)$ be a point in the hyperplane. We have that a normal vector to the plane is $\vec{n} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Then, the point R in the hyperplane closest to P satisfies $\vec{PR} = \text{proj}_{\vec{n}}\,\vec{PQ}$. Hence,

$$\vec{OR} = \vec{OP} + \text{proj}_{\vec{n}}\,\vec{PQ} = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 2 \end{bmatrix} - \frac{2}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/2 \\ -5/2 \\ -1/2 \\ 3/2 \end{bmatrix}$$

Then the distance from the point to the line is the length of $\vec{PR}$.

$$\|\vec{PR}\| = \left\|\begin{bmatrix} -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \end{bmatrix}\right\| = 1$$

E8 A vector orthogonal to both vectors is $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \times \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 7 \end{bmatrix}$.

E9 The volume of the parallelepiped determined by $\vec{u} + k\vec{v}$, $\vec{v}$, and $\vec{w}$ is

$$\begin{aligned} |(\vec{u} + k\vec{v}) \cdot (\vec{v} \times \vec{w})| &= |\vec{u} \cdot (\vec{v} \times \vec{w}) + k(\vec{v} \cdot (\vec{v} \times \vec{w}))| \\ &= |\vec{u} \cdot (\vec{v} \times \vec{w}) + k(0)| \end{aligned}$$

which equals the volume of the parallelepiped determined by $\vec{u}$, $\vec{v}$, and $\vec{w}$.

E10 (i) FALSE. The points $P(0,0,0)$, $Q(0,0,1)$, and $R(0,0,2)$ lie in every plane of the form $t_1x_1 + t_2x_2 = 0$ with t_1 and t_2 not both zero.

(ii) TRUE. This is the definition of a line reworded in terms of a spanning set.

(iii) TRUE. The set contains the zero vector and hence is linearly dependent.

(iv) FALSE. The dot product of the zero vector with itself is 0.

(v) FALSE. Let $\vec{x} = \begin{bmatrix}1\\0\end{bmatrix}$ and $\vec{y} = \begin{bmatrix}1\\1\end{bmatrix}$. Then, $\text{proj}_{\vec{x}}\,\vec{y} = \begin{bmatrix}1\\0\end{bmatrix}$, while $\text{proj}_{\vec{y}}\,\vec{x} = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}$.

(vi) FALSE. If $\vec{y} = \vec{0}$, then $\text{proj}_{\vec{x}}\,\vec{y} = \vec{0}$. Thus, $\{\text{proj}_{\vec{x}}\,\vec{y}, \text{perp}_{\vec{x}}\,\vec{y}\}$ contains the zero vector so it is linearly dependent.

(vii) TRUE. We have

$$\|\vec{u} \times (\vec{v} + 3\vec{u})\| = \|\vec{u} \times \vec{v} + 3(\vec{u} \times \vec{u})\| = \|\vec{u} \times \vec{v} + \vec{0}\| = \|\vec{u} \times \vec{v}\|$$

so the parallelograms have the same area.

Chapter 1 Further Problems

Problems

F1 The statement is true. Rewrite the conditions in the form

$$\vec{u} \cdot (\vec{v} - \vec{w}) = 0, \qquad \vec{u} \times (\vec{v} - \vec{w}) = \vec{0}$$

The first condition says that $\vec{v} - \vec{w}$ is orthogonal to $\vec{u}$, so the angle θ between $\vec{u}$ and $\vec{v} - \vec{w}$ is $\frac{\pi}{2}$ radians. Thus, $\sin\theta = 1$, so the second condition tells us that

$$0 = \|\vec{u} \times (\vec{v} - \vec{w})\| = \|\vec{u}\|\|\vec{v} - \vec{w}\| \sin\theta = \|\vec{u}\|\|\vec{v} - \vec{w}\|$$

Since $\|\vec{u}\| \neq 0$, it follows that $\|\vec{v} - \vec{w}\| = 0$ and hence $\vec{v} = \vec{w}$.

F2 Since $\vec{u}$ and $\vec{v}$ are orthogonal unit vectors, $\vec{u} \times \vec{v}$ is a unit vector orthogonal to the plane containing $\vec{u}$ and $\vec{v}$. Then $\text{perp}_{\vec{u}\times\vec{v}}\,\vec{x}$ is orthogonal to $\vec{u} \times \vec{v}$, so it lies in the plane containing $\vec{u}$ and $\vec{v}$. Therefore, for some $s, t \in \mathbb{R}$, $\text{perp}_{\vec{u}\times\vec{v}}\,\vec{x} = s\vec{u} + t\vec{v}$. Now since $\vec{u} \cdot \vec{u} = 1$, $\vec{u} \cdot \vec{v} = 0$, and $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$,

$$s = \vec{u} \cdot (s\vec{u} + t\vec{v}) = \vec{u} \cdot \text{perp}_{\vec{u}\times\vec{v}}\,\vec{x} = \vec{u} \cdot (\vec{x} - \text{proj}_{\vec{u}\times\vec{v}}\,\vec{x}) = \vec{u} \cdot \vec{x} - 0$$

Similarly, $t = \vec{v} \cdot \vec{x}$. Hence,

$$\text{perp}_{\vec{u}\times\vec{v}}\,\vec{x} = (\vec{u} \cdot \vec{x})\vec{u} + (\vec{v} \cdot \vec{x})\vec{v} = \text{proj}_{\vec{u}}\,\vec{x} + \text{proj}_{\vec{v}}\,\vec{x}$$

F3 (a) We can calculate that both sides of the equation are equal to

$$\begin{bmatrix} u_2v_1w_2 - u_2v_2w_1 + u_3v_1w_3 - u_3v_3w_1 \\ -u_1v_1w_2 + u_1v_2w_1 + u_3v_2w_3 - u_3v_3w_2 \\ -u_1v_1w_3 + u_1v_3w_1 - u_2v_2w_3 + u_2v_3w_2 \end{bmatrix}$$

(b)

$$\begin{aligned} &\vec{u}\times(\vec{v} \times \vec{w}) + \vec{v} \times (\vec{w} \times \vec{u}) + \vec{w} \times (\vec{u} \times \vec{v}) = \\ &\quad ((\vec{u} \times \vec{w})\vec{v} - (\vec{u} \times \vec{v})\vec{w}) + ((\vec{v} \cdot \vec{u})\vec{w} - (\vec{v} \cdot \vec{w})\vec{u}) + ((\vec{w} \cdot \vec{v})\vec{u} - (\vec{w} \cdot \vec{u})\vec{v}) = \vec{0} \end{aligned}$$

F4 (a) We have

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$$
$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 - 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$$

By subtraction,

$$\frac{1}{4}\|\vec{u}+\vec{v}\|^2 - \frac{1}{4}\|\vec{u}-\vec{v}\|^2 = \vec{u}\cdot\vec{v}$$

(b) By addition of the above expressions,

$$\|\vec{u}+\vec{v}\|^2 + \|\vec{u}-\vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2$$

(c) The vectors $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are the diagonal vectors of the parallelogram. Take the equation of part (a) and divide by $\|\vec{u}\|\|\vec{v}\|$ to obtain an expression for the cosine of the angle between $\vec{u}$ and $\vec{v}$, in terms of the lengths of $\vec{u}$, $\vec{v}$, and the diagonal vectors. The cosine is 0 if and only if the diagonals are of equal length. In this case, the parallelogram is a rectangle.

Part (b) says that the sum of the squares of the two diagonal lengths is the sum of the squares of the lengths of all four sides of the parallelogram. You can also see that this is true by using the cosine law and the fact that if the angle between $\vec{u}$ and $\vec{v}$ is θ, then the angle at the next vertex of the parallelogram is $\pi-\theta$.

F5 P, Q, and R are collinear if and only if for some scalar t, $\vec{PQ} = t\vec{PR}$. Thus, $\vec{q}-\vec{p} = t(\vec{r}-\vec{p})$, or $\vec{q} = (1-t)\vec{p} + t\vec{r}$. Then

$$\begin{aligned}(\vec{p}\times\vec{q}) + (\vec{q}\times\vec{r}) + (\vec{r}\times\vec{p}) &= \vec{p}\times((1-t)\vec{p}+t\vec{r}) + ((1-t)\vec{p}+t\vec{r})\times\vec{r} + \vec{r}\times\vec{p}\\ &= t\vec{p}\times\vec{r} + \vec{p}\times\vec{r} - t\vec{p}\times\vec{r} + \vec{r}\times\vec{p} = \vec{0}\end{aligned}$$

since $\vec{p}\times\vec{r} = -\vec{r}\times\vec{p}$.

F6 (a) Suppose that the skew lines are $\vec{x} = \vec{p} + s\vec{c}$ and $\vec{x} = \vec{q} + t\vec{d}$. Then the cross-product of the two direction vectors $\vec{n} = \vec{c}\times\vec{d}$ is perpendicular to both lines, so the plane through P with normal $\vec{n}$ contains the first line, and the plane through Q with normal $\vec{n}$ contains the second line. Since the two planes have the same normal vector, they are parallel planes.

(b) We find that $\vec{n} = \begin{bmatrix}2\\0\\1\end{bmatrix}\times\begin{bmatrix}1\\1\\3\end{bmatrix} = \begin{bmatrix}-1\\-5\\2\end{bmatrix}$. Thus, the equation of the plane passing through $P(1,4,2)$ is $-1x_1 - 5x_2 + 2x_3 = -17$. Now, find the distance from the point $Q(2,-3,1)$ to this plane is $\frac{32}{\sqrt{30}}$ which is the distance between the skew lines.

CHAPTER 2 Systems of Linear Equations

2.1 Systems of Linear Equations and Elimination

Practice Problems

A1 (a) From the last equation we have $x_2 = 4$. Substituting into the first equation yields $x_1 - 3(4) = 5$, or $x_1 = 17$. Hence, the general solution is $\vec{x} = \begin{bmatrix} 17 \\ 4 \end{bmatrix}$.

(b) From the last equation we get $x_3 = 6$. Also, x_2 does not appear as the leading variable in any equation so it is a free variable. Thus, we let $x_2 = t \in \mathbb{R}$. Then the first equation yields $x_1 = 7 - 2t + 6 = 13 - 2t$. Thus, the general solution is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 13 - 2t \\ t \\ 6 \end{bmatrix} = \begin{bmatrix} 13 \\ 0 \\ 6 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}$$

(c) From the last equation we get $x_3 = 2$. Substituting this into the second equation gives $x_2 = 2 - 5(2) = -8$. Now substitute x_2 and x_3 into the first equation to get $x_1 = 4 - 3(-8) + 2(2) = 32$. Thus, the general solution is $\vec{x} = \begin{bmatrix} 32 \\ -8 \\ 2 \end{bmatrix}$.

(d) Observe that x_4 is not the leading variable in any equation, so x_4 is a free variable. Let $x_4 = t \in \mathbb{R}$. Then the third equation gives $x_3 = 2 - t$ and the second equation gives $x_2 = -3 + t$. Substituting these into the first equation yields $x_1 = 7 + 2(-3 + t) - (2 - t) - 4t = -1 - t$. Thus, the general solution is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 - t \\ -3 + t \\ 2 - t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

A2 (a) A is in row echelon form.

(b) B is in row echelon form.

(c) C is not in row echelon form because the leading 1 in the third row is not further to the right than the leading 1 in the second row.

(d) D is not in row echelon form because the leading 1 in the third row is the left of the leading 1 in the second row.

A3 There are infinitely many different ways of row reducing each of the following matrices and infinitely many correct answers. Of course, the idea is to find a sequences of elementary row operations which takes as few steps as possible. Below is one possible sequence for each matrix. For practice, you should try to find others. Look for tricks which help reduce the number of steps.

(a) Row reducing gives

$$\begin{bmatrix} 4 & 1 & 1 \\ 1 & -3 & 2 \end{bmatrix} R_1 \updownarrow R_2 \sim \begin{bmatrix} 1 & -3 & 2 \\ 4 & 1 & 1 \end{bmatrix} \begin{matrix} \\ R_2 - 4R_1 \end{matrix} \sim \begin{bmatrix} 1 & -3 & 2 \\ 0 & 13 & -7 \end{bmatrix}$$

(b) Row reducing gives

$$\begin{bmatrix} 2 & -2 & 5 & 8 \\ 1 & -1 & 2 & 3 \\ -1 & 1 & 0 & 2 \end{bmatrix} R_1 \updownarrow R_2 \sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & -2 & 5 & 8 \\ -1 & 1 & 0 & 2 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 2R_2 \end{matrix} \sim \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) Row reducing gives

$$\begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & -2 \\ 5 & 0 & 0 \\ 3 & 4 & 5 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 5R_1 \\ R_4 - 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 5 & 5 \\ 0 & 7 & 8 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 5R_2 \\ R_4 - 7R_2 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 8 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 - \frac{8}{5}R_3 \end{matrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

(d) Row reducing gives

$$\begin{bmatrix} 2 & 0 & 2 & 0 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 3 & 6 & 13 & 20 \end{bmatrix} \begin{matrix} \\ R_2 - \frac{1}{2}R_1 \\ R_3 - \frac{1}{2}R_1 \\ R_4 - \frac{3}{2}R_1 \end{matrix} \sim \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 2 & 4 \\ 0 & 4 & 8 & 16 \\ 0 & 6 & 10 & 20 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 2R_2 \\ R_4 - 3R_2 \end{matrix} \sim$$

$$\begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 4 & 8 \\ 0 & 0 & 4 & 8 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 - R_3 \end{matrix} \sim \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(e) Row reducing gives

$$\left[\begin{array}{cccc} 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 \\ 3 & -1 & -4 & 1 \\ 2 & 1 & 3 & 6 \end{array}\right] \begin{array}{l} R_1 \updownarrow R_2 \\ \\ \\ \\ \end{array} \sim \left[\begin{array}{cccc} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 3 & -1 & -4 & 1 \\ 2 & 1 & 3 & 6 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 3R_1 \\ R_4 - 2R_1 \end{array} \sim$$

$$\left[\begin{array}{cccc} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & -7 & -7 & -2 \\ 0 & -3 & 1 & 4 \end{array}\right] \begin{array}{l} \\ \\ R_3 + 7R_2 \\ R_4 + 3R_2 \end{array} \sim \left[\begin{array}{cccc} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 5 \\ 0 & 0 & 7 & 7 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 - R_3 \end{array} \sim$$

$$\left[\begin{array}{cccc} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 5 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

(f) Row reducing gives

$$\left[\begin{array}{ccccc} 3 & 1 & 8 & 2 & 4 \\ 1 & 0 & 3 & 0 & 1 \\ 0 & 2 & -2 & 4 & 3 \\ -4 & 1 & 11 & 3 & 8 \end{array}\right] \begin{array}{l} \\ R_1 \updownarrow R_2 \\ \\ \\ \end{array} \sim \left[\begin{array}{ccccc} 1 & 0 & 3 & 0 & 1 \\ 3 & 1 & 8 & 2 & 4 \\ 0 & 2 & -2 & 4 & 3 \\ -4 & 1 & 11 & 3 & 8 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \\ \\ R_4 + 4R_1 \end{array} \sim$$

$$\left[\begin{array}{ccccc} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -1 & 2 & 1 \\ 0 & 2 & -2 & 4 & 3 \\ 0 & 1 & 23 & 3 & 12 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 2R_2 \\ R_4 - R_2 \end{array} \sim \left[\begin{array}{ccccc} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 24 & 1 & 11 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 \updownarrow R_3 \end{array} \sim$$

$$\left[\begin{array}{ccccc} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 24 & 1 & 11 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

A4 (a) Since the bottom row is of the form $0 = -5$, it is inconsistent.

(b) The system is consistent. Rewrite the augmented matrix in equation form to get the system of equations $x_1 = 2$ and $x_3 = 3$. We also have that x_2 is a free variable, so we let $x_2 = t \in \mathbb{R}$. Hence, the general solution is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ t \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}$$

(c) The system is consistent. Rewrite the augmented matrix in equation form to get the system of equations

$$\begin{aligned} x_1 + x_3 &= 1 \\ x_2 + x_3 + x_4 &= 2 \\ x_4 &= 3 \end{aligned}$$

Thus, x_3 is a free variable, so we let $x_3 = t \in \mathbb{R}$. Then, substituting the third equation into the second gives $x_2 = 2 - t - 3 = -1 - t$. Also, the first equation becomes $x_1 = 1 - t$. Hence, the general solution is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1-t \\ -1-t \\ t \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}$$

(d) The system is consistent. Rewrite the augmented matrix in equation form to get the system of equations

$$\begin{aligned} x_1 + x_2 - x_3 + 3x_4 &= 1 \\ 2x_3 + x_4 &= 3 \\ x_4 &= -2 \end{aligned}$$

Thus, x_2 is a free variable, so we let $x_2 = t \in \mathbb{R}$. Then, substituting the third equation into the second gives $2x_3 = 3-(-2) = 5$ or $x_3 = \frac{5}{2}$. Substituting everything into the first equations yields $x_1 = 1-t+\frac{5}{2}-3(-2) = \frac{19}{2} - t$. Hence, the general solution is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 19/2-t \\ t \\ 5/2 \\ -2 \end{bmatrix} = \begin{bmatrix} 19/2 \\ 0 \\ 5/2 \\ -2 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}$$

(e) The system is consistent. Rewrite the augmented matrix in equation form to get the system of equations

$$\begin{aligned} x_1 + x_3 - x_4 &= 0 \\ x_2 &= 0 \end{aligned}$$

We get that x_3 and x_4 are free variables, so we let $x_3 = s \in \mathbb{R}$ and $x_t = t \in \mathbb{R}$. Then, the general solution is

$$\vec{x} = \begin{bmatrix} -s+t \\ 0 \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

A5 (a) We have

i. $\left[\begin{array}{cc|c} 3 & -5 & 2 \\ 1 & 2 & 4 \end{array}\right]$

ii.

$$\left[\begin{array}{cc|c} 3 & -5 & 2 \\ 1 & 2 & 4 \end{array}\right] R_1 \updownarrow R_2 \sim \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 3 & -5 & 2 \end{array}\right] R_2 - 3R_1 \sim \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -11 & -10 \end{array}\right]$$

iii. The system is consistent and has no free variables, so there are no parameters in the general solution.

iv. By back-substitution, $x_2 = \frac{10}{11}$ and $x_1 = 4 - 2\left(\frac{10}{11}\right) = \frac{24}{11}$. Hence, the general solution is $\vec{x} = \begin{bmatrix} 24/11 \\ 10/11 \end{bmatrix}$.

(b) We have

i. $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 2 & -3 & 2 & 6 \end{array}\right]$

ii.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 2 & -3 & 2 & 6 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & -7 & 0 & -4 \end{array}\right]$$

iii. The system is consistent and has one free variable, so there is one parameter in the general solution.

iv. Let $x_3 = t \in \mathbb{R}$. By back-substitution, $x_2 = \frac{4}{7}$ and $x_1 = 5 - 2\left(\frac{4}{7}\right) - t = \frac{27}{7} - t$. Hence, the general solution is $\vec{x} = \begin{bmatrix} 27/7 \\ 4/7 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$.

(c) We have

i. $\left[\begin{array}{ccc|c} 1 & 2 & -3 & 8 \\ 1 & 3 & -5 & 11 \\ 2 & 5 & -8 & 19 \end{array}\right]$

ii.

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 8 \\ 1 & 3 & -5 & 11 \\ 2 & 5 & -8 & 19 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - 2R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 8 \\ 0 & 1 & -2 & 3 \\ 0 & 1 & -2 & 3 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \end{array} \sim$$
$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 8 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

iii. The system is consistent and has one free variable, so there is one parameter in the general solution.

iv. Let $x_3 = t \in \mathbb{R}$. By back-substitution, $x_2 = 3 + 2t$ and $x_1 = 8 - 2(3 + 2t) + 3t = 2 - t$. Hence, the general solution is $\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$.

(d) We have

i. $\left[\begin{array}{ccc|c} -3 & 6 & 16 & 36 \\ 1 & -2 & -5 & -11 \\ 2 & -3 & -8 & -17 \end{array}\right]$

ii.

$$\left[\begin{array}{ccc|c} -3 & 6 & 16 & 36 \\ 1 & -2 & -5 & -11 \\ 2 & -3 & -8 & -17 \end{array}\right] R_1 \updownarrow R_2 \sim \left[\begin{array}{ccc|c} 1 & -2 & -5 & -11 \\ -3 & 6 & 16 & 36 \\ 2 & -3 & -8 & -17 \end{array}\right] \begin{array}{l} \\ R_2 + 3R_1 \\ R_3 - 2R_1 \end{array} \sim$$
$$\left[\begin{array}{ccc|c} 1 & -2 & -5 & -11 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 2 & 5 \end{array}\right] \begin{array}{l} \\ \\ R_2 \updownarrow R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & -2 & -5 & -11 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

iii. The system is consistent and has no free variables, so there are no parameters in the general solution.

iv. By back-substitution, $x_3 = 3$, $x_2 = 5 - 2(3) = -1$, and $x_1 = -11 + 2(-1) + 5(3) = 2$. Hence, the general solution is $\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$.

(e) We have

i. $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 2 & 5 & 1 & 10 \\ 4 & 9 & -1 & 19 \end{array}\right]$

ii.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 2 & 5 & 1 & 10 \\ 4 & 9 & -1 & 19 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 4R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 3 & 3 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

iii. The system is inconsistent.

(f) We have

i. $\left[\begin{array}{cccc|c} 1 & 2 & -3 & 0 & -5 \\ 2 & 4 & -6 & 1 & -8 \\ 6 & 13 & -17 & 4 & -21 \end{array}\right]$

ii.

$$\left[\begin{array}{cccc|c} 1 & 2 & -3 & 0 & -5 \\ 2 & 4 & -6 & 1 & -8 \\ 6 & 13 & -17 & 4 & -21 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 6R_1 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 2 & -3 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 4 & 9 \end{array}\right] \begin{array}{l} \\ \\ R_2 \updownarrow R_3 \end{array} \sim$$
$$\left[\begin{array}{cccc|c} 1 & 2 & -3 & 0 & -5 \\ 0 & 1 & 1 & 4 & 9 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

iii. The system is consistent and has one free variable, so there is one parameter in the general solution.

iv. Let $x_3 = t \in \mathbb{R}$. By back-substitution, $x_4 = 2$, $x_2 = 9 - t - 4(2) = 1 - t$, and $x_1 = -5 - 2(1 - t) + 3t = -7 + 5t$. Hence, the general solution is $\vec{x} = \begin{bmatrix} -7 \\ 1 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 5 \\ -1 \\ 1 \\ 0 \end{bmatrix}$, $t \in \mathbb{R}$.

(g) We have

i. $\left[\begin{array}{ccccc|c} 0 & 2 & -2 & 0 & 1 & 2 \\ 1 & 2 & -3 & 1 & 4 & 1 \\ 2 & 4 & -5 & 3 & 8 & 3 \\ 2 & 5 & -7 & 3 & 10 & 5 \end{array}\right]$

ii.

$$\left[\begin{array}{ccccc|c} 0 & 2 & -2 & 0 & 1 & 2 \\ 1 & 2 & -3 & 1 & 4 & 1 \\ 2 & 4 & -5 & 3 & 8 & 3 \\ 2 & 5 & -7 & 3 & 10 & 5 \end{array}\right] R_1 \updownarrow R_2 \sim \left[\begin{array}{ccccc|c} 1 & 2 & -3 & 1 & 4 & 1 \\ 0 & 2 & -2 & 0 & 1 & 2 \\ 2 & 4 & -5 & 3 & 8 & 3 \\ 2 & 5 & -7 & 3 & 10 & 5 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 2R_1 \\ R_4 - 2R_1 \end{array} \sim$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & -3 & 1 & 4 & 1 \\ 0 & 2 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 & 2 & 3 \end{array}\right] \begin{array}{l} \\ \\ \\ R_4 - \frac{1}{2}R_2 \end{array} \sim \left[\begin{array}{ccccc|c} 1 & 2 & -3 & 1 & 4 & 1 \\ 0 & 2 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3/2 & 2 \end{array}\right]$$

iii. The system is consistent and has one free variable, so there is one parameter in the general solution.

iv. Let $x_5 = t \in \mathbb{R}$. By back-substitution, $x_4 = 2-\frac{3}{2}t$, $x_3 = 1-\left(2 - \frac{3}{2}t\right) = -1+\frac{3}{2}t$, $2x_2 = 2+2\left(-1 + \frac{3}{2}t\right)-t = 2t$, so $x_2 = t$, and $x_1 = 1 - 2(t) + 3\left(-1 + \frac{3}{2}t\right) - \left(2 - \frac{3}{2}t\right) - 4t = -4$. Hence, the general solution is $\vec{x} = \begin{bmatrix} -4 \\ 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 3/2 \\ -3/2 \\ 1 \end{bmatrix}, t \in \mathbb{R}$.

A6 (a) If $a \neq 0$ and $b \neq 0$, then the matrix is consistent and has no free variables, so the solution is unique. If $a = 0$, $b \neq 0$, the system is consistent, but x_3 is a free variable so the solution is not unique. If $a \neq 0$, $b = 0$, then row reducing gives

$$\left[\begin{array}{ccc|c} 2 & 4 & -3 & 6 \\ 0 & 0 & 7 & 2 \\ 0 & 0 & a & a \end{array}\right] \begin{array}{l} \\ \\ \frac{a}{7}R_2 - R_3 \end{array} \left[\begin{array}{ccc|c} 2 & 4 & -3 & 6 \\ 0 & 0 & 7 & 2 \\ 0 & 0 & 0 & 5a/7 \end{array}\right]$$

Hence, the system is inconsistent. If $a = 0$, $b = 0$, the system is consistent and x_2 is a free variable, so the solution is not unique.

(b) If $c \neq 0$, $d \neq 0$, the system is consistent and has no free variables, so the solution is unique. If $d = 0$, then the last row is $\left[\begin{array}{ccc|c} 0 & 0 & 0 & c \end{array}\right]$, so the system is consistent only if $c = 0$. If $c = 0$, the system is consistent for all values of d, but x_4 is a free variable so the solution is not unique.

A7 Let a be the number of apples, b be the number of bananas, and c be the number of oranges. Then

$$a + b + c = 1500$$

The second piece of information given is the weight of the fruit (in grams):

$$120a + 140b + 160c = 208000$$

Finally, the total selling price (in cents) is:

$$25a + 20b + 30c = 38000$$

Thus, a, b, and c satisfy the system of linear equations

$$\begin{aligned} a + b + c &= 1500 \\ 120a + 140b + 160c &= 208000 \\ 25a + 20b + 30c &= 38000 \end{aligned}$$

Row reducing the corresponding augmented matrix for this system gives

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1500 \\ 120 & 140 & 160 & 208000 \\ 25 & 20 & 30 & 38000 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1500 \\ 0 & 20 & 40 & 28000 \\ 0 & 0 & 15 & 7500 \end{array}\right]$$

By back-substitution $15x_3 = 7500$ so $x_3 = 500$, $20x_2 = 28000 - 40(500)$ hence $x_2 = 400$, and $x_1 = 1500 - 400 - 500 = 600$. Thus the fruit-seller has 600 apples, 400 bananas, and 500 oranges.

A8 Let A be her mark in algebra, C be her mark in calculus, and P be her mark in physics. Then for a physics prize,

$$0.2A + 0.3C + 0.5P = 84$$

For an applied mathematics prize,

$$\frac{1}{3}(A + C + P) = 83$$

For a pure mathematics prize,

$$0.5A + 0.5C = 82.5$$

Thus, A, C, and P satisfy the linear system of equations

$$\begin{aligned} \frac{1}{5}A + \frac{3}{10}C + \frac{1}{2}P &= 84 \\ \frac{1}{3}A + \frac{1}{3}C + \frac{1}{3}P &= 83 \\ \frac{1}{2}A + \frac{1}{2}C \phantom{+ \frac{1}{2}P} &= 82.5 \end{aligned}$$

To avoid working with fractions, we can multiply each equation by a non-zero scalar to get

$$\begin{aligned} 2A + 3C + 5P &= 840 \\ A + C + P &= 249 \\ A + C &= 165 \end{aligned}$$

Row Reducing the corresponding augmented matrix for this system gives

$$\left[\begin{array}{ccc|c} 2 & 3 & 5 & 840 \\ 1 & 1 & 1 & 249 \\ 1 & 1 & 0 & 165 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 165 \\ 0 & 1 & 5 & 510 \\ 0 & 0 & 1 & 84 \end{array}\right]$$

By back-substitution $P = 84$, $C = 510 - 5(84) = 90$, and $A = 165 - 90 = 75$. Therefore, the student has 75% in algebra, 90% in calculus, and 84% in physics.

Homework Problems

B1

(a) $\vec{x} = \begin{bmatrix} 23 \\ 10 \\ -2 \end{bmatrix}$

(b) $\vec{x} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -7 \\ -2 \\ 1 \end{bmatrix}, t \in \mathbb{R}$

(c) $\vec{x} = \begin{bmatrix} -4 \\ 2 \\ 3 \\ 0 \end{bmatrix} + t\begin{bmatrix} 10 \\ -5 \\ -3 \\ 1 \end{bmatrix}, t \in \mathbb{R}$

(d) $\vec{x} = \begin{bmatrix} -2 \\ -2 \\ -3 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, s, t \in \mathbb{R}$

B2 (a) The matrix is in row echelon form.

(b) The matrix is in row echelon form.

(c) The matrix is not in row echelon form as the leading entry in the second row is not to the right of the leading entry in the first row.

(d) The matrix is in row echelon form.

B3 Alternate correct answers are possible.

(a) $\begin{bmatrix} 1 & 3 & 3 & 0 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 1 & -7/2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 2 & 4 \\ 0 & 0 & 0 & -6 & 11 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 1 & 3 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$

B4 (a) Consistent. The solution is $\vec{x} = \begin{bmatrix} 9/2 \\ 2 \\ -3/2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ 1 \\ -2 \\ 1 \end{bmatrix}, t \in \mathbb{R}$.

(b) Inconsistent.

(c) Consistent. The solution is $\vec{x} = \begin{bmatrix} 1/2 \\ 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}, t \in \mathbb{R}$.

(d) Consistent. The solution is $\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, s, t \in \mathbb{R}$.

B5 (a) $\left[\begin{array}{ccc|c} 2 & 1 & 5 & -4 \\ 1 & 1 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & -2 \\ 0 & -1 & 3 & 0 \end{array}\right]$. Consistent with solution $\vec{x} = \begin{bmatrix} -2-4t \\ 3t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}, t \in \mathbb{R}$.

(b) $\left[\begin{array}{ccc|c} 2 & 1 & -1 & 6 \\ 1 & -2 & -2 & 1 \\ -1 & 12 & 8 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & -2 & 1 \\ 0 & 5 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$. Consistent with solution $\vec{x} = \begin{bmatrix} 13/5 \\ 4/5 \\ 0 \end{bmatrix} + t\begin{bmatrix} 4/5 \\ -3/5 \\ 1 \end{bmatrix}, t \in \mathbb{R}$

(c) $\left[\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 \\ 2 & 3 & 3 & 9 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$. The system is inconsistent.

(d) $\left[\begin{array}{ccc|c} 1 & 1 & 0 & -7 \\ 2 & 4 & 1 & -16 \\ 1 & 2 & 1 & 9 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & -7 \\ 0 & 1 & 1 & 16 \\ 0 & 0 & -1 & -34 \end{array}\right]$. Consistent with solution $\vec{x} = \begin{bmatrix} 11 \\ -18 \\ 34 \end{bmatrix}$.

(e) $\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 3 \\ 1 & 2 & 4 & 1 & 7 \\ 1 & 0 & 0 & 1 & -21 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 3 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & -20 \end{array}\right]$. The system is inconsistent.

(f) $\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 4 & 1 & -1 \\ 1 & 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$. Consistent with solution $\vec{x} = \begin{bmatrix} 3 \\ -2 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $s, t \in \mathbb{R}$.

B6 (a) If $b = 0$ the system is inconsistent. If $b \neq 0$ and $a^2 - 1 \neq 0$, then the system is consistent with a unique solution. If $a = 1$, the system is consistent only if $b = -1$. If $a = -1$, the system is consistent only if $b = 1$. In the case where $a = 1$, $b = -1$ and in the case where $a = -1$ and $b = 1$ there are infinitely many solutions.

(b) If $c \neq 0$ and $d \neq 0$, then the system is consistent with a unique solution. If $c = 0$ or $d = 0$, then the system is inconsistent.

B7 The price of an armchair is \$300, the price of a sofa bed is \$600, and the price of a double bed is \$400.

B8 The magnitude of F_1 is 8.89N, the magnitude of F_2 is 40N, and the magnitude of F_3 is 31.11N.

B9 The average of the Business students is 78%, the average of the Liberal Arts students is 84% and the average of the Science students is 93%.

Computer Problems

C1 Alternate correct answers are possible.

(a) $\begin{bmatrix} 35 & 45 & 18 & 13 \\ 0 & 302/7 & -2441/35 & 24/35 \\ 0 & 0 & -17303/755 & 24632/755 \end{bmatrix}$

(b) $\begin{bmatrix} -25 & -36 & 37 & 41 & 22 \\ 0 & -110 & 123 & 95 & 89 \\ 0 & 0 & -63891/2750 & -16589/550 & 1907/2750 \end{bmatrix}$

C2 See solutions of $A3$, $A5$, $B3$, and $B5$ respectively.

C3 (a) $\vec{x} = \begin{bmatrix} -33.720 \\ 0.738 \\ 20.600 \end{bmatrix}$

(b) $\vec{x} = \begin{bmatrix} 26.389 \\ -2.132 \\ -13.733 \end{bmatrix}$

Conceptual Problems

D1 Write the augmented matrix and row reduce:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & b \\ 2 & 3 & 1 & 5 & 6 \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 2 & 2 & a & 1 \end{array}\right] \begin{array}{c} \\ R_2 - 2R_1 \\ \\ \\ \end{array} \sim \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & b \\ 0 & 1 & 1 & 3 & 6-2b \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 2 & 2 & a & 1 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4 - 2R_2 \end{array} \sim$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & b \\ 0 & 1 & 1 & 3 & 6-2b \\ 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & a-6 & 4b-11 \end{array}\right]$$

(a) To be inconsistent, we must have a row of the form $\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & c \end{array}\right]$ where $c \neq 0$. The only way this is possible is by taking $a = 6$ and $4b - 11 \neq 0$. In all other cases the system is consistent.

(b) By Theorem 2.1.1, to have a consistent system with a unique solution, we cannot have a row of the form $\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & c \end{array}\right]$ where $c \neq 0$ and we need the number of pivots to equal the number of variables. Since x, y, and z always have a pivot, we just require that w also has a pivot. This will happen whenever $a \neq 6$ which also implies that the system in consistent.

(c) By Theorem 2.1.1, to have a consistent system with infinitely many solutions, we cannot have a row of the form $\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & c \end{array}\right]$ where $c \neq 0$ and we need the number of pivots to be less than the number of variables. To have the number of pivots less than the number of variables, we need $a = 6$. But, then for the system to be consistent we must have $4b - 11 = 0$.

D2 Since the two planes are parallel, $\vec{m} = t\vec{n}$ for some non-zero real number t. To determine a point (or points) of intersection, we solve the system of equations with augmented matrix

$$\left[\begin{array}{ccc|c} n_1 & n_2 & n_2 & c \\ m_1 & m_2 & m_3 & d \end{array}\right]$$

Add $-t$ times the first row to the second to obtain

$$\left[\begin{array}{ccc|c} n_1 & n_2 & n_2 & c \\ 0 & 0 & 0 & d - tc \end{array}\right]$$

because $\vec{m} = t\vec{n}$. If $d \neq tc$, the system of equation is inconsistent, and there are no points of intersection. If $d = tc$, then the second equation is a non-zero multiple of the first. That is, the planes coincide.

2.2 Reduced Row Echelon Form, Rank, and Homogeneous Systems

Practice Problems

A1 (a)

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 3 & 2 \end{bmatrix} R_1 \updownarrow R_2 \sim \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \sim$$
$$\begin{bmatrix} 1 & -1 \\ 0 & 3 \\ 0 & 5 \end{bmatrix} \tfrac{1}{3}R_2 \sim \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 5 \end{bmatrix} \begin{array}{l} R_1 + R_2 \\ \\ R_3 - 5R_2 \end{array} \sim$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Thus, the rank is 2.

(b)

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{matrix} R_1 - R_3 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_1 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{matrix} R_1 + R_2 \\ \\ R_3 - 2R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix} \begin{matrix} \\ \\ -\frac{1}{3}R_3 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 - 2R_3 \\ R_2 - 2R_3 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the rank is 3.

(c)

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -4 \\ 0 & -1 & -2 \end{bmatrix} \begin{matrix} \\ R_2 - 4R_3 \\ \\ \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & -1 & -2 \end{bmatrix} \begin{matrix} R_1 - 2R_2 \\ \\ R_3 + R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{matrix} \\ \\ \frac{1}{2}R_3 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 + 5R_3 \\ R_2 - 4R_3 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the rank is 3.

(d)

$$\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 6 \\ 0 & 3 & 8 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 3R_2 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 6 \\ 0 & 0 & -10 \end{bmatrix} \begin{matrix} \\ \\ -\frac{1}{10}R_3 \end{matrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 + 2R_3 \\ R_2 - 6R_3 \\ \\ \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the rank is 3.

(e)

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ -1 & -2 & 3 \\ 2 & 4 & 3 \end{bmatrix} \begin{array}{l} \\ R_2 - R_1 \\ R_3 + R_1 \\ R_4 - 2R_1 \end{array} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{l} \\ \frac{1}{2}R_2 \\ \\ \\ \end{array} \sim$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 4R_2 \\ R_4 - R_2 \end{array} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the rank is 2.

(f)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ R_2 \updownarrow R_3 \\ \\ \end{array} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \sim$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{array}{l} \\ \\ R_3 - R_2 \end{array} \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, the rank is 3.

(g)

$$\begin{bmatrix} 2 & -1 & 2 & 8 \\ 1 & -1 & 0 & 2 \\ 3 & -2 & 3 & 13 \end{bmatrix} \begin{array}{l} \\ R_1 \updownarrow R_2 \\ \\ \end{array} \sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 2 & -1 & 2 & 8 \\ 3 & -2 & 3 & 13 \end{bmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \sim$$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 3 & 7 \end{bmatrix} \begin{array}{l} R_1 + R_2 \\ \\ R_3 - R_2 \end{array} \sim \begin{bmatrix} 1 & 0 & 2 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \begin{array}{l} R_1 - 2R_3 \\ R_2 - 2R_3 \\ \\ \end{array} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Thus, the rank is 3.

(h)

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 3 & 1 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{array}{l} \\ \\ R_3 - 2R_1 \\ R_4 - R_1 \end{array} \sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 3 & 3 \end{bmatrix} \begin{array}{l} R_1 - R_2 \\ \\ R_3 - R_2 \\ R_4 - R_2 \end{array} \sim$$

$$\begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \begin{array}{l} \\ \\ \\ \frac{1}{2}R_4 \end{array} \sim \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \end{bmatrix} \begin{array}{l} R_1 + R_4 \\ R_2 - R_4 \\ \\ \\ \end{array} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \end{bmatrix} \begin{matrix} \\ \\ \\ R_3 \updownarrow R_4 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, the rank is 3.

(i)

$$\begin{bmatrix} 0 & 1 & 0 & 2 & 5 \\ 3 & 1 & 8 & 5 & 3 \\ 1 & 0 & 3 & 2 & 1 \\ 2 & 1 & 6 & 7 & 1 \end{bmatrix} R_1 \updownarrow R_3 \sim \begin{bmatrix} 1 & 0 & 3 & 2 & 1 \\ 3 & 1 & 8 & 5 & 3 \\ 0 & 1 & 0 & 2 & 5 \\ 2 & 1 & 6 & 7 & 1 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ \\ R_4 - 2R_1 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 3 & 2 & 1 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 1 & 0 & 3 & -1 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \\ R_4 - R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 3 & 2 & 1 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 4 & -1 \end{bmatrix} \begin{matrix} R_1 - 3R_3 \\ R_2 + R_3 \\ \\ R_4 - R_3 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & -7 & -14 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix} \begin{matrix} R_1 + 7R_4 \\ R_2 - 2R_4 \\ R_3 - 3R_4 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & -56 \\ 0 & 1 & 0 & 0 & 17 \\ 0 & 0 & 1 & 0 & 23 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}$$

Thus, the rank is 4.

A2 (a) There is 1 parameter. Let $x_3 = t \in \mathbb{R}$. Then we have $x_1 = -2t$, $x_2 = t$, and $x_4 = 0$. Thus, the general solution is

$$\vec{x} = \begin{bmatrix} -2t \\ t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}$$

(b) There are 2 parameters. Let $x_1 = s \in \mathbb{R}$ and $x_3 = t \in \mathbb{R}$. Then we have $x_2 = -2t$ and $x_4 = 0$. Thus, the general solution is

$$\vec{x} = \begin{bmatrix} s \\ -2t \\ t \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

(c) There are 2 parameters. Let $x_2 = s \in \mathbb{R}$ and $x_3 = t \in \mathbb{R}$. Then we have $x_1 = 3s - 2t$ and $x_4 = 0$. Thus, the general solution is

$$\vec{x} = \begin{bmatrix} 3s - 2t \\ s \\ t \\ 0 \end{bmatrix} = s \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

(d) There are 2 parameters. Let $x_3 = s \in \mathbb{R}$ and $x_5 = t \in \mathbb{R}$. Then we have $x_1 = -2s$, $x_2 = s + 2t$, and $x_4 = -t$. Thus, the general solution is

$$\vec{x} = \begin{bmatrix} -2s \\ s+2t \\ s \\ -t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

(e) There are 2 parameters. Let $x_2 = s \in \mathbb{R}$ and $x_4 = t \in \mathbb{R}$. Then we have $x_1 = -4t$, $x_3 = 5t$, and $x_4 = 0$. Thus, the general solution is

$$\vec{x} = \begin{bmatrix} -4t \\ s \\ 5t \\ t \\ 0 \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -4 \\ 0 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

(f) There is 1 parameter. Let $x_3 = t \in \mathbb{R}$. Then we have $x_1 = 0$, $x_2 = -t$, $x_4 = 0$, and $x_5 = 0$. Thus, the general solution is

$$\vec{x} = \begin{bmatrix} 0 \\ -t \\ t \\ 0 \\ 0 \end{bmatrix} = t\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}$$

A3 (a) The coefficient matrix is $\begin{bmatrix} 0 & 2 & -5 \\ 1 & 2 & 3 \\ 1 & 4 & -3 \end{bmatrix}$. To find the rank, we row reduce the coefficient matrix to RREF.

$$\begin{bmatrix} 0 & 2 & -5 \\ 1 & 2 & 3 \\ 1 & 4 & -3 \end{bmatrix} \begin{matrix} R_1 \updownarrow R_2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -5 \\ 1 & 4 & -3 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_1 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -5 \\ 0 & 2 & -6 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \\ R_3 - R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 8 \\ 0 & 2 & -5 \\ 0 & 0 & -1 \end{bmatrix} \begin{matrix} R_1 + 8R_3 \\ R_2 - 5R_3 \\ \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{matrix} \\ \frac{1}{2}R_2 \\ (-1)R_3 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the rank is 3. The number of parameters in the general solution to the homogeneous system with coefficient A is the number of variables minus the rank, which is $3 - 3 = 0$. Therefore, the only solution is $\vec{x} = \vec{0}$.

(b) The coefficient matrix is $\begin{bmatrix} 3 & 1 & -9 \\ 1 & 1 & -5 \\ 2 & 1 & -7 \end{bmatrix}$. To find the rank, we row reduce the coefficient matrix to RREF.

$$\begin{bmatrix} 3 & 1 & -9 \\ 1 & 1 & -5 \\ 2 & 1 & -7 \end{bmatrix} \begin{matrix} R_1 \updownarrow R_2 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 1 & -5 \\ 3 & 1 & -9 \\ 2 & 1 & -7 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{matrix} \sim$$
$$\begin{bmatrix} 1 & 1 & -5 \\ 0 & -2 & 6 \\ 0 & -1 & 3 \end{bmatrix} \begin{matrix} \\ -\frac{1}{2}R_2 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 1 & -5 \\ 0 & 1 & -3 \\ 0 & -1 & 3 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \\ R_3 + R_2 \end{matrix} \sim$$
$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the rank is 2. The number of parameters in the general solution to the homogeneous system with coefficient A is the number of variables minus the rank, which is $3 - 2 = 1$. Therefore, there are infinitely many solutions. To find the general solution, we rewrite the RREF back into equation form to get

$$\begin{aligned} x_1 - 2x_3 &= 0 \\ x_2 - 3x_3 &= 0 \end{aligned}$$

Thus, x_3 is the free variable. So, we let $x_3 = t \in \mathbb{R}$ and get the general solution is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2t \\ 3t \\ t \end{bmatrix} = t\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

(c) The coefficient matrix is $\begin{bmatrix} 1 & -1 & 2 & -3 \\ 3 & -3 & 8 & -5 \\ 2 & -2 & 5 & -4 \\ 3 & -3 & 7 & -7 \end{bmatrix}$. To find the rank, we row reduce the coefficient matrix to RREF.

$$\begin{bmatrix} 1 & -1 & 2 & -3 \\ 3 & -3 & 8 & -5 \\ 2 & -2 & 5 & -4 \\ 3 & -3 & 7 & -7 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \\ R_4 - 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \\ R_3 - \frac{1}{2}R_2 \\ R_4 - \frac{1}{2}R_2 \end{matrix} \sim$$
$$\begin{bmatrix} 1 & -1 & 0 & -7 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \frac{1}{2}R_2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & -1 & 0 & -7 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, the rank is 2. The number of parameters in the general solution to the homogeneous system with coefficient A is the number of variables minus the rank, which is $4 - 2 = 2$. Therefore, there are infinitely many solutions. To find the general solution, we rewrite the RREF back into equation form to get

$$\begin{aligned} x_1 - x_2 - 7x_4 &= 0 \\ x_3 + 2x_4 &= 0 \end{aligned}$$

Thus, x_2 and x_4 are free variables. So, we let $x_2 = s \in \mathbb{R}$ and $x_4 = t \in \mathbb{R}$. Then, the general solution is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} s+7t \\ s \\ -2t \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 7 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \quad s,t \in \mathbb{R}$$

(d) The coefficient matrix is $\begin{bmatrix} 0 & 1 & 2 & 2 & 0 \\ 1 & 2 & 5 & 3 & -1 \\ 2 & 1 & 5 & 1 & -3 \\ 1 & 1 & 4 & 2 & -2 \end{bmatrix}$. To find the rank, we row reduce the coefficient matrix to RREF.

$$\begin{bmatrix} 0 & 1 & 2 & 2 & 0 \\ 1 & 2 & 5 & 3 & -1 \\ 2 & 1 & 5 & 1 & -3 \\ 1 & 1 & 4 & 2 & -2 \end{bmatrix} \begin{matrix} \\ \\ R_1 \updownarrow R_2 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 2 & 5 & 3 & -1 \\ 0 & 1 & 2 & 2 & 0 \\ 2 & 1 & 5 & 1 & -3 \\ 1 & 1 & 4 & 2 & -2 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 2R_1 \\ R_4 - R_1 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 2 & 5 & 3 & -1 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & -3 & -5 & -5 & -1 \\ 0 & -1 & -1 & -1 & -1 \end{bmatrix} \begin{matrix} R_1 - 2R_2 \\ \\ R_3 + 3R_2 \\ R_4 + R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 1 & -1 & -1 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix} \begin{matrix} R_1 - R_3 \\ R_2 - 2R_3 \\ \\ R_4 - R_3 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, the rank is 3. The number of parameters in the general solution to the homogeneous system with coefficient A is the number of variables minus the rank, which is $5 - 3 = 2$. Therefore, there are infinitely many solutions. To find the general solution, we rewrite the RREF back into equation form to get

$$\begin{aligned} x_1 - 2x_4 &= 0 \\ x_2 + 2x_5 &= 0 \\ x_3 + x_4 - x_5 &= 0 \end{aligned}$$

Thus, x_4 and x_5 are free variables. So, we let $x_4 = s \in \mathbb{R}$ and $x_5 = t \in \mathbb{R}$. Then, the general solution is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2s \\ -2t \\ -s+t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad s,t \in \mathbb{R}$$

A4 (a) We have

$$\left[\begin{array}{cc|c} 3 & -5 & 2 \\ 1 & 2 & 4 \end{array}\right] R_1 \updownarrow R_2 \sim \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 3 & -5 & 2 \end{array}\right] \quad R_2 - 3R_1 \sim$$
$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -11 & -10 \end{array}\right] -\tfrac{1}{11}R_2 \sim \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 10/11 \end{array}\right] R_1 - 2R_2 \sim$$
$$\left[\begin{array}{cc|c} 1 & 0 & 24/11 \\ 0 & 1 & 10/11 \end{array}\right]$$

Hence, the solution is $\vec{x} = \begin{bmatrix} 24/11 \\ 10/11 \end{bmatrix}$.

(b) We have

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 2 & -3 & 2 & 6 \end{array}\right] R_2 - 2R_1 \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & -7 & 0 & -4 \end{array}\right] -\tfrac{1}{7}R_2 \sim$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 5 \\ 0 & 1 & 0 & 4/7 \end{array}\right] R_2 - 2R_2 \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 27/7 \\ 0 & 1 & 0 & 4/7 \end{array}\right]$$

We have $x_1 + x_3 = \frac{27}{7}$ and $x_2 = \frac{4}{7}$. Then x_3 is a free variable, so let $x_3 = t \in \mathbb{R}$ and we get that the general solution is

$$\vec{x} = \begin{bmatrix} 27/7 \\ 4/7 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

(c) We have

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 8 \\ 1 & 3 & -5 & 11 \\ 2 & 5 & -8 & 19 \end{array}\right] \begin{array}{c} \\ R_2 - R_1 \\ R_3 - 2R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 8 \\ 0 & 1 & -2 & 3 \\ 0 & 1 & -2 & 3 \end{array}\right] \begin{array}{c} \\ \\ R_3 - R_2 \end{array} \sim$$
$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 8 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} R_1 - 2R_2 \\ \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We have $x_1 + x_3 = 2$ and $x_2 - 2x_3 = 3$. Then x_3 is a free variable, so let $x_3 = t \in \mathbb{R}$ and we get that the general solution is

$$\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

(d) We have

$$\left[\begin{array}{ccc|c} -3 & 6 & 16 & 36 \\ 1 & -2 & -5 & -11 \\ 2 & -3 & -8 & -17 \end{array}\right] R_1 \updownarrow R_2 \sim \left[\begin{array}{ccc|c} 1 & -2 & -5 & -11 \\ -3 & 6 & 16 & 36 \\ 2 & -3 & -8 & -17 \end{array}\right] \begin{array}{l} \\ R_2 + 3R_1 \\ R_3 - 2R_1 \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -5 & -11 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 2 & 5 \end{array}\right] \begin{array}{l} \\ \\ R_2 \updownarrow R_3 \end{array} \sim \left[\begin{array}{ccc|c} 1 & -2 & -5 & -11 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} R_1 + 2R_2 \\ \\ \\ \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} R_1 + R_3 \\ R_2 - 2R_3 \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Thus the only solution is $\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$.

(e) We have

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 2 & 5 & 1 & 10 \\ 4 & 9 & -1 & 19 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 4R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 1 & 3 & 3 \end{array}\right] \begin{array}{l} R_1 - 2R_2 \\ \\ R_3 - R_2 \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -7 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right] R_2 - 2R_3 \sim \left[\begin{array}{ccc|c} 1 & 0 & -7 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Hence, the last equation is $0 = 1$ so the system is inconsistent.

(f) We have

$$\left[\begin{array}{cccc|c} 1 & 2 & -3 & 0 & -5 \\ 2 & 4 & -6 & 1 & -8 \\ 6 & 13 & -17 & 4 & -21 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 6R_1 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 2 & -3 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 4 & 9 \end{array}\right] \begin{array}{l} \\ \\ R_2 \updownarrow R_3 \end{array} \sim$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -3 & 0 & -5 \\ 0 & 1 & 1 & 4 & 9 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] R_2 - 4R_3 \sim \left[\begin{array}{cccc|c} 1 & 2 & -3 & 0 & -5 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} R_1 - 2R_2 \\ \\ \\ \end{array} \sim$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -5 & 0 & -7 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

In equation form this is $x_1 - 5x_3 = -7$, $x_2 + x_3 = 1$, and $x_4 = 2$. Thus, x_3 is a free variable, so we let $x_3 = t \in \mathbb{R}$. Then the general solution is

$$\vec{x} = \begin{bmatrix} -7 \\ 1 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 5 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}$$

(g) We have

$$\left[\begin{array}{ccccc|c} 0 & 2 & -2 & 0 & 1 & 2 \\ 1 & 2 & -3 & 1 & 4 & 1 \\ 2 & 4 & -5 & 3 & 8 & 3 \\ 2 & 5 & -7 & 3 & 10 & 5 \end{array}\right] \begin{array}{c} R_1 \updownarrow R_2 \\ \\ \\ \\ \end{array} \sim \left[\begin{array}{ccccc|c} 1 & 2 & -3 & 1 & 4 & 1 \\ 0 & 2 & -2 & 0 & 1 & 2 \\ 2 & 4 & -5 & 3 & 8 & 3 \\ 2 & 5 & -7 & 3 & 10 & 5 \end{array}\right] \begin{array}{c} \\ \\ R_3 - 2R_1 \\ R_4 - 2R_1 \end{array} \sim$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & -3 & 1 & 4 & 1 \\ 0 & 2 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 & 2 & 3 \end{array}\right] \begin{array}{c} \\ \\ \\ R_2 \updownarrow R_4 \end{array} \sim \left[\begin{array}{ccccc|c} 1 & 2 & -3 & 1 & 4 & 1 \\ 0 & 1 & -1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 2 & -2 & 0 & 1 & 2 \end{array}\right] \begin{array}{c} R_1 - 2R_2 \\ \\ \\ R_4 - 2R_2 \end{array} \sim$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & -1 & -1 & 0 & -5 \\ 0 & 1 & -1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -2 & -3 & -4 \end{array}\right] \begin{array}{c} R_1 + R_3 \\ R_2 + R_3 \\ \\ -\frac{1}{2}R_4 \end{array} \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 2 & 2 & 4 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3/2 & 2 \end{array}\right] \begin{array}{c} \\ R_2 - 2R_4 \\ R_3 - R_4 \\ \\ \end{array} \sim$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -3/2 & -1 \\ 0 & 0 & 0 & 1 & 3/2 & 2 \end{array}\right]$$

In equation form this is $x_1 = -4$, $x_2 - x_5 = 0$, $x_3 - \frac{3}{2}x_5 = -1$, and $x_4 + \frac{3}{2}x_5 = 2$. Thus, x_5 is a free variable, so we let $x_5 = t \in \mathbb{R}$. Then the general solution is

$$\vec{x} = \begin{bmatrix} -4 \\ 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 3/2 \\ -3/2 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

A5 (a) We have

$$\left[\begin{array}{ccc|c} 2 & -1 & 4 & 1 \\ 1 & 3 & 0 & 0 \\ 1 & 1 & 2 & 2 \end{array}\right] \begin{array}{c} \\ R_1 \updownarrow R_3 \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 1 & 3 & 0 & 0 \\ 2 & -1 & 4 & 1 \end{array}\right] \begin{array}{c} \\ R_2 - R_1 \\ R_3 - 2R_1 \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 2 & -2 & -2 \\ 0 & -3 & 0 & -3 \end{array}\right] \begin{array}{c} R_1 - \frac{1}{2}R_2 \\ \\ R_3 + \frac{3}{2}R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 2 & -2 & -2 \\ 0 & 0 & -3 & -6 \end{array}\right] \begin{array}{c} \\ \frac{1}{2}R_2 \\ -\frac{1}{3}R_3 \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{c} R_1 - 3R_3 \\ R_2 + R_3 \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Therefore, the solution to $\left[A \mid \vec{b}\,\right]$ is $\vec{x} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$. If we replace $\vec{b}$ be $\vec{0}$, we get that the solution to the homogeneous system is $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(b) We have

$$\left[\begin{array}{ccc|c} 1 & 7 & 5 & 5 \\ 1 & 0 & 5 & -2 \\ -1 & 2 & -5 & 4 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 + R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 7 & 5 & 5 \\ 0 & -7 & 0 & -7 \\ 0 & 9 & 0 & 9 \end{array}\right] \begin{array}{l} \\ -\frac{1}{7}R_2 \\ \frac{1}{9}R_3 \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 7 & 5 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 - 7R_2 \\ \\ R_3 - R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Writing this back into equation form gives $x_1 + 5x_3 = -2$ and $x_2 = 1$. Let $x_3 = t \in \mathbb{R}$. Then the general solution to $\left[A \mid \vec{b}\,\right]$ is

$$\vec{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

Replacing $\vec{b}$ with $\vec{0}$, we get the general solution to the homogeneous system is

$$\vec{x} = t\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

(c) We have

$$\left[\begin{array}{cccc|c} 0 & -1 & 5 & -2 & -1 \\ -1 & -1 & -4 & -1 & 4 \end{array}\right] R_1 \updownarrow R_2 \sim \left[\begin{array}{cccc|c} -1 & -1 & -4 & -1 & 4 \\ 0 & -1 & 5 & -2 & -1 \end{array}\right] \begin{array}{l} (-1)R_1 \\ (-1)R_2 \end{array} \sim$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 4 & 1 & -4 \\ 0 & 1 & -5 & 2 & 1 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 9 & -1 & -5 \\ 0 & 1 & -5 & 2 & 1 \end{array}\right]$$

Writing this back into equation form gives $x_1 + 9x_3 - x_4 = -5$ and $x_2 - 5x_3 + 2x_4 = 1$. Let $x_3 = s \in \mathbb{R}$ and $x_4 = t \in \mathbb{R}$. Then the general solution to $\left[A \mid \vec{b}\,\right]$ is

$$\vec{x} = \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -9 \\ 5 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

Replacing $\vec{b}$ with $\vec{0}$, we get the general solution to the homogeneous system is

$$\vec{x} = s\begin{bmatrix} -9 \\ 5 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

(d) We have

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 3 \\ 4 & 3 & 2 & -4 & 3 \\ -1 & -4 & -3 & 5 & 5 \end{array}\right] \begin{array}{c} \\ R_2 - 4R_1 \\ R_3 + R_1 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 3 \\ 0 & 3 & 6 & 0 & -9 \\ 0 & -4 & -4 & 4 & 8 \end{array}\right] \begin{array}{c} \\ \frac{1}{3}R_2 \\ -\frac{1}{4}R_3 \end{array} \sim$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 3 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 1 & 1 & -1 & -2 \end{array}\right] \begin{array}{c} \\ \\ R_3 - R_2 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 3 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & -1 & -1 & 1 \end{array}\right] \begin{array}{c} \\ \\ (-1)R_3 \end{array} \sim$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 3 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right] \begin{array}{c} R_1 + R_3 \\ R_2 - 2R_3 \\ \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & -2 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

Writing this back into equation form gives $x_1 = -2$, $x_2 - 2x_4 = -1$, and $x_3 + x_4 = -1$. Let $x_4 = t \in \mathbb{R}$. Then the general solution to $\left[A \mid \vec{b}\,\right]$ is

$$\vec{x} = \vec{x} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

Replacing $\vec{b}$ with $\vec{0}$, we get the general solution to the homogeneous system is

$$\vec{x} = t \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

Homework Problems

B1

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$; the rank is 2.

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; the rank is 3.

(c) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$; the rank is 2.

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$; the rank is 3.

(e) $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$; the rank is 3.

(f) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$; the rank is 4.

(g) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$; the rank is 3.

B2 (a) There are 2 parameters. The general solution is $\vec{x} = s \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$, $s, t \in \mathbb{R}$.

(b) There are 2 parameters. The general solution is $\vec{x} = s\begin{bmatrix}-1\\1\\0\\0\\0\end{bmatrix} + t\begin{bmatrix}2\\0\\-1\\1\\0\end{bmatrix}$, $s, t \in \mathbb{R}$.

(c) There are 3 parameters. The general solution is $\vec{x} = r\begin{bmatrix}-2\\1\\0\\0\\0\\0\end{bmatrix} + s\begin{bmatrix}0\\0\\5\\1\\0\\0\end{bmatrix} + t\begin{bmatrix}3\\0\\-4\\0\\-1\\1\end{bmatrix}$, $r, s, t \in \mathbb{R}$.

B3 (a) $\begin{bmatrix}1 & 5 & -3\\3 & 5 & -9\\1 & 1 & -3\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -3\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$; the rank is 2; there is 1 parameter. The general solution is $\vec{x} = t\begin{bmatrix}3\\0\\1\end{bmatrix}$, $t \in \mathbb{R}$.

(b) $\begin{bmatrix}1 & 4 & -2\\2 & 0 & -3\\4 & 8 & -7\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -3/2\\0 & 1 & -1/8\\0 & 0 & 0\end{bmatrix}$; the rank is 2; there is 1 parameter. The general solution is $\vec{x} = t\begin{bmatrix}3/2\\1/8\\1\end{bmatrix}$, $t \in \mathbb{R}$.

(c) $\begin{bmatrix}1 & 1 & 1 & -2\\2 & 7 & 0 & -14\\1 & 3 & 0 & -6\\1 & 4 & 0 & -8\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & -2\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix}$; the rank is 3; there is 1 parameter. The general solution is $\vec{x} = t\begin{bmatrix}0\\2\\0\\1\end{bmatrix}$, $t \in \mathbb{R}$.

(d) $\begin{bmatrix}1 & 3 & 1 & 1 & 2\\0 & 2 & 1 & 0 & -1\\1 & 2 & 2 & 1 & 0\\1 & 2 & 1 & 1 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0 & 1 & 0\\0 & 1 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 0 & 1\end{bmatrix}$; the rank is 4; there is 1 parameter. The general solution is $\vec{x} = t\begin{bmatrix}-1\\0\\0\\1\\0\end{bmatrix}$, $t \in \mathbb{R}$.

B4 Alternative correct answers are possible.

(a) $\left[\begin{array}{ccc|c}4 & 0 & 6 & 0\\6 & 6 & 3 & -6\\-2 & 1 & -4 & -1\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 0 & 3/2 & 0\\0 & 1 & -1 & -1\\0 & 0 & 0 & 0\end{array}\right]$. The solution to $\left[A \mid \vec{b}\,\right]$ is $\vec{x} = \begin{bmatrix}0\\-1\\0\end{bmatrix} + t\begin{bmatrix}-3/2\\1\\1\end{bmatrix}$, $t \in \mathbb{R}$. The solution to the homogeneous system is $\vec{x} = t\begin{bmatrix}-3/2\\1\\1\end{bmatrix}$, $t \in \mathbb{R}$.

(b) $\left[\begin{array}{ccc|c}1 & 2 & -4 & 10\\-1 & -5 & -5 & -1\\-4 & 1 & 9 & 1\\-5 & -4 & 0 & 8\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 0 & 0 & -4\\0 & 1 & 0 & 3\\0 & 0 & 1 & -2\\0 & 0 & 0 & 0\end{array}\right]$. The solution to $\left[A \mid \vec{b}\,\right]$ is $\vec{x} = \begin{bmatrix}-4\\3\\-2\end{bmatrix}$. The solution to the homogeneous system is $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(c) $\left[\begin{array}{cccc|c} 1 & -1 & 4 & -1 & 4 \\ -1 & -2 & 5 & -2 & 5 \\ -4 & -1 & 2 & 2 & -4 \\ 5 & 4 & 1 & 8 & 5 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 & 14 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & -5 \end{array}\right]$. The solution to $\left[A \mid \vec{b}\right]$ is $\vec{x} = \begin{bmatrix} -3 \\ 14 \\ 4 \\ -5 \end{bmatrix}$. The solution to the homogeneous system is $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

(d) $\left[\begin{array}{ccccc|c} 1 & 1 & 3 & 1 & 4 & 2 \\ 4 & 4 & 6 & -8 & 4 & -4 \\ 1 & 1 & 4 & -2 & 1 & -6 \\ 3 & 3 & 2 & -4 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 3 & 6 \\ 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$. The solution to $\left[A \mid \vec{b}\right]$ is $\vec{x} = \begin{bmatrix} 6 \\ 0 \\ -2 \\ 2 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$, $s, t \in \mathbb{R}$. The solution to the homogeneous system is $\vec{x} = s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$, $s, t \in \mathbb{R}$.

Computer Problems

C1 $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5.742880954 \\ 0 & 1 & 0 & -1.935982515 \\ 0 & 0 & 1 & -0.3291708706 \end{array}\right]$

C2 (a) $\begin{bmatrix} 1 & 0 & 0 & 4.041900246 \\ 0 & 1 & 0 & -2.285384036 \\ 0 & 0 & 1 & -1.423568167 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 & -0.5653691443 & 0.2889765382 \\ 0 & 1 & 0 & 0.5880170916 & -0.8424660750 \\ 0 & 0 & 1 & 1.298226667 & -0.02984770938 \end{bmatrix}$

Conceptual Problems

D1 (a) If $\vec{x}$ is orthogonal to $\vec{a}$, $\vec{b}$, and $\vec{c}$, then it must satisfy the equations $\vec{a} \cdot \vec{x} = 0$, $\vec{b} \cdot \vec{x} = 0$, and $\vec{c} \cdot \vec{x} = 0$.

(b) Since a homogeneous system is always consistent, Theorem 2.2.2 tells us that a homogeneous system in 3 variables has non-trivial solutions if and only if the rank of the matrix is less than 3.

D2 (a) Let the variables be x_1, x_2, and x_3. From the reduced row echelon form of the coefficient matrix, we see that x_3 is arbitrary, say $x_3 = t \in \mathbb{R}$. Then $x_2 = x_3 = t$ and $x_1 = -2t$, so the general solution is $\vec{x} = t\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$. Since it is the span of one non-zero vector it is a line.

(b) Since the rank of A is 2 and the homogeneous system has 3 variables, there must be 1 parameter in the general solution of the system. The general solution is of the form $\vec{x} = t\vec{d}$, $t \in \mathbb{R}$, with $\vec{d} \neq \vec{0}$, so it represents a line through the origin. If the rank was 1, there would be 2 parameters and the general solution would be of the form $\vec{x} = s\vec{u} + t\vec{v}$, $s, t \in \mathbb{R}$, with $\{\vec{u}, \vec{v}\}$ linearly independent. This describes a plane through the origin in $\mathbb{R}^3$.

(c) This homogeneous system has the coefficient matrix $C = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{bmatrix}$.

If the rank of C is 3, the set of vectors $\vec{x}$ orthogonal to the three given vectors is a line through the origin in $\mathbb{R}^4$, because the general solution has 1 parameter. If the rank is 2, the set is a 2-plane, and if the rank is 1, the set is a 3-plane.

D3 (a) We are not told enough to know whether the system is consistent. If the augmented matrix also has rank 4, the system is consistent and there are 3 parameters in the general solution. If the augmented matrix has rank 5, the system is inconsistent.

(b) Since the rank of the coefficient matrix equals the number of rows the system is consistent. There are 3 parameters in the general solution.

(c) We need to know the rank of the coefficient matrix too. If the rank is also 4, then the system is consistent, with a unique solution. If the coefficient matrix has rank 3, then the system is inconsistent.

D4 Row reducing gives

$$\left[\begin{array}{ccc|c} 1 & a & b & 1 \\ 1 & 1 & 0 & a \\ 1 & 0 & 1 & b \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & b \\ 0 & 1 & -1 & a-b \\ 0 & 0 & a+b-1 & 1-b-a^2+ab \end{array}\right]$$

If $a + b \neq 1$, the system is consistent with the unique solution

$$\vec{x} = \frac{1}{a+b-1}\begin{bmatrix} b^2-1+a^2 \\ ab+1-b^2-a \\ 1-b-a^2+ab \end{bmatrix}$$

If $a + b = 1$, the system is consistent only if $1 - b - a^2 + ab = 0$ also. This implies that either $a = 0$ and $b = 1$, or $a = 1$ and $b = 0$ is required for consistency. If $a = 1$ and $b = 0$, then the reduced augmented matrix becomes $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and the general solution is

$$\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

If $a = 0$, $b = 1$, the matrix becomes $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and the general solution is

$$\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

2.3 Application to Spanning and Linear Independence

Practice Problems

A1 (a) We need to determine if there are values c_1, c_2, and c_3 such that

$$\begin{bmatrix}-3\\2\\8\\4\end{bmatrix} = c_1\begin{bmatrix}1\\0\\1\\1\end{bmatrix} + c_2\begin{bmatrix}2\\1\\0\\1\end{bmatrix} + c_3\begin{bmatrix}-1\\1\\2\\1\end{bmatrix} = \begin{bmatrix}c_1+2c_2-c_3\\c_2+c_3\\c_1+2c_3\\c_1+c_2+c_3\end{bmatrix}$$

Vectors are equal if and only if there corresponding entries are equal. Thus, equating corresponding entries we get the system of linear equations

$$\begin{aligned}c_1+2c_2-c_3&=-3\\c_2+c_3&=2\\c_1+2c_3&=8\\c_1+c_2+c_3&=4\end{aligned}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{ccc|c}1&2&-1&-3\\0&1&1&2\\1&0&2&8\\1&1&1&4\end{array}\right] \sim \left[\begin{array}{ccc|c}1&0&0&2\\0&1&0&-1\\0&0&1&3\\0&0&0&0\end{array}\right]$$

Thus, the solution is $c_1=2$, $c_2=-1$, and $c_3=3$. Thus the vector is in the span. In particular,

$$2\begin{bmatrix}1\\0\\1\\1\end{bmatrix} + (-1)\begin{bmatrix}2\\1\\0\\1\end{bmatrix} + 3\begin{bmatrix}-1\\1\\2\\1\end{bmatrix} = \begin{bmatrix}-3\\2\\8\\4\end{bmatrix}$$

(b) We need to determine if there are values c_1, c_2, and c_3 such that

$$\begin{bmatrix}5\\4\\6\\7\end{bmatrix} = c_1\begin{bmatrix}1\\0\\1\\1\end{bmatrix} + c_2\begin{bmatrix}2\\1\\0\\1\end{bmatrix} + c_3\begin{bmatrix}-1\\1\\2\\1\end{bmatrix} = \begin{bmatrix}c_1+2c_2-c_3\\c_2+c_3\\c_1+2c_3\\c_1+c_2+c_3\end{bmatrix}$$

Equating corresponding entries we get the system of linear equations

$$\begin{aligned}c_1+2c_2-c_3&=5\\c_2+c_3&=4\\c_1+2c_3&=6\\c_1+c_2+c_3&=7\end{aligned}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & 1 & 1 & 4 \\ 1 & 0 & 2 & 6 \\ 1 & 1 & 1 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 5 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 5 & 9 \\ 0 & 0 & 0 & 3/5 \end{array}\right]$$

The system is inconsistent. Therefore, the vector is not in the span.

(c) We need to determine if there are values c_1, c_2, and c_3 such that

$$\begin{bmatrix} 2 \\ -2 \\ 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + 2c_2 - c_3 \\ c_2 + c_3 \\ c_1 + 2c_3 \\ c_1 + c_2 + c_3 \end{bmatrix}$$

Vectors are equal if and only if there corresponding entries are equal. Thus, equating corresponding entries we get the system of linear equations

$$\begin{aligned} c_1 + 2c_2 - c_3 &= 2 \\ c_2 + c_3 &= -2 \\ c_1 + 2c_3 &= 1 \\ c_1 + c_2 + c_3 &= 1 \end{aligned}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, the solution is $c_1 = 3$, $c_2 = -1$, and $c_3 = -1$. Thus the vector is in the span. In particular,

$$3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

A2 (a) We need to determine if there are values c_1, c_2, and c_3 such that

$$\begin{bmatrix} 3 \\ 2 \\ -1 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 - c_2 + c_3 \\ -c_1 + c_2 + c_3 \\ c_1 - c_3 \\ 2c_2 - c_3 \end{bmatrix}$$

Equating corresponding entries we get the system of linear equations

$$\begin{aligned} c_1 - c_2 + c_3 &= 3 \\ -c_1 + c_2 + c_3 &= 2 \\ c_1 - c_3 &= -1 \\ 2c_2 - c_3 &= -1 \end{aligned}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ -1 & 1 & 1 & 2 \\ 1 & 0 & -1 & -1 \\ 0 & 2 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 1 & -2 & -4 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & -1/2 \end{array}\right]$$

The system is inconsistent. Therefore, the vector is not in the span.

(b) We need to determine if there are values c_1, c_2, and c_3 such that

$$\begin{bmatrix} -7 \\ 3 \\ 0 \\ 8 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 - c_2 + c_3 \\ -c_1 + c_2 + c_3 \\ c_1 - c_3 \\ 2c_2 - c_3 \end{bmatrix}$$

Equating corresponding entries we get the system of linear equations

$$\begin{aligned} c_1 - c_2 + c_3 &= -7 \\ -c_1 + c_2 + c_3 &= 3 \\ c_1 - c_3 &= 0 \\ 2c_2 - c_3 &= 8 \end{aligned}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -7 \\ -1 & 1 & 1 & 3 \\ 1 & 0 & -1 & 0 \\ 0 & 2 & 1 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, the solution is $c_1 = -2$, $c_2 = 3$, and $c_3 = -2$. Thus the vector is in the span. In particular,

$$(-2) \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + (-2) \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ 3 \\ 0 \\ 8 \end{bmatrix}$$

(c) We need to determine if there are values c_1, c_2, and c_3 such that

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} c_1 - c_2 + c_3 \\ -c_1 + c_2 + c_3 \\ c_1 - c_3 \\ 2c_2 - c_3 \end{bmatrix}$$

Equating corresponding entries we get the system of linear equations

$$\begin{aligned} c_1 - c_2 + c_3 &= 1 \\ -c_1 + c_2 + c_3 &= 1 \\ c_1 - c_3 &= 1 \\ 2c_2 - c_3 &= 1 \end{aligned}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & -2 \end{array}\right]$$

The system is inconsistent. Therefore, the vector is not in the span.

A3 (a) A vector $\vec{x}$ is in the set if and only if for some t_1, t_2, and t_3

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \\ 0 \end{bmatrix}$$

Equating corresponding entries we get the system $x_1 = t_1$, $x_2 = t_2$, and $x_3 = 0$. This is consistent if and only if $x_3 = 0$. Thus, the homogeneous equation $x_3 = 0$ defines the set.

(b) A vector $\vec{x}$ is in the set if and only if

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2t_1 \\ t_1 \\ 0 \end{bmatrix}$$

for some real numbers t_1, t_2. Thus, we get $x_1 = 2t_1$, $x_2 = t_1$, and $x_3 = 0$. Substituting the second equation into the first we get $x_1 = 2x_2$ or $x_1 - 2x_2 = 0$. Therefore, the homogeneous system $x_1 - 2x_2 = 0$, $x_3 = 0$ defines the set.

(c) A vector $\vec{x}$ is in the set if and only if

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + t_2 \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} t_1 + 3t_2 \\ t_1 - t_2 \\ 2t_1 \end{bmatrix}$$

for some real numbers t_1 and t_2. Equating corresponding entries we get the system

$$\begin{aligned} t_1 + 3t_2 &= x_1 \\ t_1 - t_2 &= x_2 \\ 2t_1 &= x_3 \end{aligned}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{cc|c} 1 & 3 & x_1 \\ 1 & -1 & x_2 \\ 2 & 0 & x_3 \end{array}\right] \sim \left[\begin{array}{cc|r} 1 & 3 & x_1 \\ 0 & -4 & x_2 - x_1 \\ 0 & 0 & x_1 + 3x_2 - 2x_3 \end{array}\right]$$

Therefore, the homogeneous equation $x_1 + 3x_2 - 2x_3 = 0$ defines the set.

(d) A vector $\vec{x}$ is in the set if and only if

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t_1 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2t_1 + t_2 \\ t_1 - 2t_2 \\ -t_1 + t_2 \end{bmatrix}$$

for some real numbers t_1 and t_2. Equating corresponding entries we get the system

$$\begin{aligned} 2t_1 + t_2 &= x_1 \\ t_1 - 2t_2 &= x_2 \\ -t_1 + t_2 &= x_3 \end{aligned}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{rr|r} 2 & 1 & x_1 \\ 1 & -2 & x_2 \\ -1 & 1 & x_3 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & -2 & x_2 \\ 0 & 5 & x_1 - 2x_2 \\ 0 & 0 & x_1 + 3x_2 + 5x_3 \end{array}\right]$$

Therefore, the homogeneous equation $x_1 + 3x_2 + 5x_3 = 0$ defines the set.

(e) A vector $\vec{x}$ is in the set if and only if

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} t_1 + 2t_2 \\ -t_2 \\ t_1 + t_2 \\ t_2 \end{bmatrix}$$

for some real numbers t_1 and t_2. Equating corresponding entries we get the system

$$\begin{aligned} t_1 + 2t_2 &= x_1 \\ -t_2 &= x_2 \\ t_1 + t_2 &= x_3 \\ t_2 &= x_4 \end{aligned}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{rr|r} 1 & 2 & x_1 \\ 0 & -1 & x_2 \\ 1 & 1 & x_3 \\ 0 & 1 & x_4 \end{array}\right] \sim \left[\begin{array}{rr|r} 1 & 2 & x_1 \\ 0 & -1 & x_2 \\ 0 & 0 & -x_1 - x_2 + x_3 \\ 0 & 0 & x_2 + x_4 \end{array}\right]$$

Therefore, the homogeneous system $-x_1 - x_2 + x_3 = 0$, $x_2 + x_4 = 0$ defines the set.

(f) A vector $\vec{x}$ is in the set if and only if

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ 3 \\ -2 \end{bmatrix} + t_3 \begin{bmatrix} -2 \\ 0 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} t_1 - 2t_3 \\ -t_1 + t_2 \\ t_1 + 3t_2 + 4t_3 \\ 2t_1 - 2t_2 - 3t_3 \end{bmatrix}$$

for some real numbers t_1, t_2, and t_3. Equating corresponding entries we get the system

$$\begin{aligned} t_1 - 2t_3 &= x_1 \\ -t_1 + t_2 &= x_2 \\ t_1 + 3t_2 + 4t_3 &= x_3 \\ 2t_1 - 2t_2 - 3t_3 &= x_4 \end{aligned}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & x_1 \\ -1 & 1 & 0 & x_2 \\ 1 & 3 & 4 & x_3 \\ 2 & -2 & -3 & x_4 \end{array}\right] \sim \left[\begin{array}{ccc|r} 1 & 0 & -2 & x_1 \\ 0 & 1 & -2 & x_1 + x_2 \\ 0 & 0 & -3 & 2x_2 + x_4 \\ 0 & 0 & 0 & -4x_1 + 5x_2 + x_3 + 4x_4 \end{array}\right]$$

Therefore, the homogeneous equation $-4x_1 + 5x_2 + x_3 + 4x_4 = 0$ defines the set.

A4 (a) To show that the set is a basis for the plane, we need to prove that it is both linearly independent and spans the plane. Since neither vector in the set is a scalar multiple of the other, the set is linearly independent. For spanning, we need to show that every vector in the plane can be written as a linear combination of the basis vectors. To do this, we first find the general form of a vector $\vec{x}$ in the plane. We solve the equation of the plane for x_1 to get $x_1 = -x_2 + x_3$. Hence, every vector in the plane has the form $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix}$.

Therefore, we now just need to show that the system corresponding to

$$\begin{bmatrix} -x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} t_1 + t_2 \\ t_1 \\ 2t_1 + t_2 \end{bmatrix}$$

is always consistent. Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{cc|r} 1 & 1 & -x_2 + x_3 \\ 1 & 0 & x_2 \\ 2 & 1 & x_3 \end{array}\right] \sim \left[\begin{array}{cc|r} 1 & 0 & x_2 \\ 0 & 1 & x_3 - 2x_2 \\ 0 & 0 & 0 \end{array}\right]$$

So, the system is always consistent and consequently is a basis for the plane.

(b) Observe that neither vector in the set is a scalar multiple of the other, so the set is linearly independent. Write the equation of the plane in the form $x_3 = -2x_1 + 3x_2$. Then every vector in the plane has the form $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ -2x_1 + 3x_2 \end{bmatrix}$. Consider

$$\begin{bmatrix} x_1 \\ x_2 \\ -2x_1 + 3x_2 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} t_1 + t_2 \\ t_1 + 2t_1 \\ t_1 - 3t_2 \end{bmatrix}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{cc|r} 1 & 1 & x_1 \\ 1 & 2 & x_2 \\ 1 & -3 & -2x_1 + 3x_2 \end{array}\right] \sim \left[\begin{array}{cc|r} 1 & 1 & x_1 \\ 0 & 1 & -x_1 + x_2 \\ 0 & 0 & x_1 - x_2 \end{array}\right]$$

Therefore, it is not consistent whenever $x_1 - x_2 \neq 0$. Hence, it is not a basis for the plane.

(c) None of the vectors in the set can be written as a linear combination of the other vectors, so the set is linearly independent. Write the equation of the hyperplane in the form $x_1 = x_2 - 2x_3 - 2x_4$. Then every vector in the plane has the form $\vec{x} = \begin{bmatrix} x_2 - 2x_3 - 2x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. Consider

$$\begin{bmatrix} x_2 - 2x_3 - 2x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + t_3 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} t_1 + 3t_4 \\ t_1 + t_3 \\ t_2 \\ t_2 + t_3 \end{bmatrix}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & x_2 - 2x_3 - 2x_4 \\ 1 & 0 & 1 & x_2 \\ 0 & 1 & 0 & x_3 \\ 0 & 1 & 1 & x_4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & x_2 \\ 0 & 1 & 0 & x_3 \\ 0 & 0 & 1 & -x_3 + x_4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So, the system is always consistent and consequently is a basis for the hyperplane.

A5 (a) To determine whether a set of vectors is linearly dependent of independent, we find all linear combinations of the vectors which equals the zero vector. If there is only one such linear combination, the combination where all coefficients are zero, then the set is linearly independent.

Consider

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} + t_3 \begin{bmatrix} 1 \\ -3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} t_1 + t_2 + t_3 \\ 2t_1 + 2t_2 - 3t_3 \\ t_1 + 3t_2 + 2t_3 \\ -t_1 + t_2 + t_3 \end{bmatrix}$$

Row reducing the coefficient matrix to the corresponding homogeneous system gives

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & -3 \\ 1 & 3 & 2 \\ -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the only solution is $t_1 = t_2 = t_3 = 0$. Therefore, the set is linearly independent.

(b) Consider

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + t_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + t_4 \begin{bmatrix} 3 \\ 2 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} t_1 + 3t_4 \\ t_2 + 2t_4 \\ t_1 + t_2 + t_3 + 6t_4 \\ t_2 + t_3 + 3t_4 \end{bmatrix}$$

Row reducing the coefficient matrix to the corresponding homogeneous system gives

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, t_4 is a free variable, so let $t_4 = t \in \mathbb{R}$. Then, the general solution of the homogeneous system is

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} = \begin{bmatrix} -3t \\ -2t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -3 \\ -2 \\ -1 \\ 1 \end{bmatrix}$$

Since there are infinitely many solutions, the set is linearly dependent. Moreover, we have that for any $t \in \mathbb{R}$,

$$-3t \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - 2t \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - t \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(c) Consider

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 2 \\ 3 \\ 1 \\ 3 \\ 3 \end{bmatrix} + t_3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} t_1 + 2t_2 \\ t_1 + 3t_2 + t_3 \\ t_2 + t_3 \\ t_1 + 3t_2 + t_3 \\ t_1 + 3t_2 + t_3 \end{bmatrix}$$

Row reducing the coefficient matrix to the corresponding homogeneous system gives

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, t_3 is a free variable, so let $t_3 = t \in \mathbb{R}$. Then, the general solution of the homogeneous system is

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} 2t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Since there are infinitely many solutions, the set is linearly dependent. Moreover, we have that for any $t \in \mathbb{R}$,

$$2t \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - t \begin{bmatrix} 2 \\ 3 \\ 1 \\ 3 \\ 3 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

A6 (a) Consider

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + t_3 \begin{bmatrix} 2 \\ -3 \\ -1 \\ k \end{bmatrix} = \begin{bmatrix} t_1 + 2t_3 \\ t_2 - 3t_3 \\ t_1 + t_2 - t_3 \\ t_2 + kt_3 \end{bmatrix}$$

Row reducing the coefficient matrixto the corresponding homogeneous system gives

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 1 & 1 & -1 \\ 0 & 1 & k \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & k+3 \\ 0 & 0 & 0 \end{bmatrix}$$

By Lemma 2.3.3, the set is linearly independent if and only if the rank of the coefficient matrix equals the number of vectors in the set. Thus, the set is linearly independent if and only if we have 3 leading ones in the matrix. Therefore, we see that the set is linearly independent if and only if $k \neq -3$.

(b) Consider

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} -1 \\ 2 \\ k \\ 1 \end{bmatrix} = \begin{bmatrix} t_1 + t_2 - t_3 \\ t_1 - t_2 + 2t_3 \\ t_1 + 2t_2 + kt_3 \\ 2t_1 + t_3 \end{bmatrix}$$

Row reducing the coefficient matrix to the corresponding homogeneous system gives

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 2 \\ 1 & 2 & k \\ 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -3/2 \\ 0 & 0 & k+5/2 \\ 0 & 0 & 0 \end{bmatrix}$$

By Lemma 2.3.3, the set is linearly independent if and only if the rank of the coefficient matrix equals the number of vectors in the set. Therefore, the set is linearly independent if and only if $k \neq -5/2$.

A7 (a) To show the set is a basis, we must show it is linearly independent and spans $\mathbb{R}^3$. By Theorem 2.3.5, we can show this by showing that the rank of the coefficient matrix of the system corresponding to

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + t_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} t_1 + t_2 + 2t_3 \\ t_1 - t_2 + t_3 \\ 2t_1 - t_2 + t_3 \end{bmatrix}$$

is n. Row reducing the coefficient matrix gives

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 2 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore, the rank of the coefficient matrix is 3, so the set is a basis for $\mathbb{R}^3$.

(b) To be a basis, the set must span $\mathbb{R}^3$. But, Theorem 2.3.2 says that we cannot span $\mathbb{R}^3$ with fewer than 3 vectors. So, since this set has only 2 vectors, it cannot span $\mathbb{R}^3$. Therefore, it is not a basis.

(c) To be a basis, the set must be linearly independent. But, Theorem 2.3.4 says that a set with greater than 3 vectors in $\mathbb{R}^3$ cannot be linearly independent. Consequently, the given set is linearly dependent and so is not a basis.

(d) To show the set is a basis, we must show it is linearly independent and spans $\mathbb{R}^3$. By Theorem 2.3.5, we can show this by showing that the rank of the coefficient matrix of the system corresponding to

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} t_1 + t_2 + 3t_3 \\ -t_1 + 2t_2 \\ t_1 - t_2 + t_3 \end{bmatrix}$$

is n. Row reducing the coefficient matrix gives

$$\begin{bmatrix} 1 & 1 & 3 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the rank of the coefficient matrix is 2, the set is not a basis.

Homework Problems

B1 (a) $\begin{bmatrix} -4 \\ -2 \\ 2 \\ -6 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}$

(b) $\begin{bmatrix} 6 \\ 0 \\ 0 \\ 3 \end{bmatrix}$ is not in the span.

(c) $\begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + (0)\begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}$

B2 (a) $\begin{bmatrix} 0 \\ 1 \\ 4 \\ 2 \end{bmatrix}$ is not in the span.

(b) $\begin{bmatrix} 6 \\ 4 \\ 10 \\ 3 \end{bmatrix}$ is not in the span.

(c) $\begin{bmatrix} 2 \\ 3 \\ 1 \\ -1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

B3 (a) $-3x_1 - x_2 + x_3 = 0$

(b) $x_1 - x_3 = 0, x_2 + 3x_3 = 0$

(c) $x_1 + 2x_3 = 0$

(d) $10x_1 - 4x_2 - x_4 = 0$, $9x_1 - 3x_2 - x_3 = 0$

(e) $x_1 - \frac{2}{3}x_2 + \frac{2}{3}x_3 + x_4 = 0$

(f) $3x_2 + x_3 + x_4 = 0$

B4 (a) It is a basis for the plane.

(b) It is not a basis for the plane.

(c) It is not a basis for the hyperplane.

(d) It is a basis for the hyperplane.

B5 (a) Linearly independent.

(b) Linearly dependent. $-2s\begin{bmatrix}1\\0\\1\\1\\0\end{bmatrix} + s\begin{bmatrix}1\\2\\0\\0\\1\end{bmatrix} + s\begin{bmatrix}0\\1\\1\\2\\-2\end{bmatrix} + s\begin{bmatrix}1\\-3\\1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$, $s \in \mathbb{R}$

B6 (a) Linearly independent for all $k \neq 0$.

(b) Linearly independent for all $k \neq 12$.

B7 (a) It is linearly dependent, so it is not a basis.

(b) It has 4 vectors in $\mathbb{R}^3$ so it is linearly dependent. So, not a basis.

(c) It is a basis.

(d) Only 2 vectors, so it cannot span $\mathbb{R}^3$. So, not a basis.

Computer Problems

C1 (a) It is linearly independent.

(b) $\vec{v}$ is not in the span.

Conceptual Problems

D1 Let $\vec{x} = \begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix}$ be any vector in $\mathbb{R}^n$. Observe that $\begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix} = x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n$.

Hence, Span $B = \mathbb{R}^n$. Moreover, the only solution to $\vec{0} = t_1\vec{e}_1 + \cdots + t_n\vec{e}_n$ is $t_1 = \cdots = t_n = 0$, so B is linearly independent.

D2 (a) Consider $t_1\vec{v}_1 + \cdots + t_k\vec{v}_k = \vec{b}$ for $\vec{b} \in \mathbb{R}^n$. This corresponds to a system of n linear equations (one for each entry in the vectors) in the k-unknowns $t_1, \dots, t_k$. If this system is consistent for all $\vec{b} \in \mathbb{R}^n$, then the rank of the coefficient matrix equals the number of rows which is n. But, then we must have $n \geq k$ which is a contradiction. Hence, the system cannot be consistent for all $\vec{b} \in \mathbb{R}^n$. In particular, there must exists a $\vec{v} \in \mathbb{R}^n$ such that $\vec{v} \notin$ Span B.

(b) Consider $t_1\vec{v}_1 + \cdots + t_k\vec{v}_k = \vec{0}$ for $\vec{b} \in \mathbb{R}^n$. This corresponds to a system of n linear equations (one for each entry in the vectors) in the k-unknowns $t_1, \ldots, t_k$. Since $k > n$, the rank of the coefficient matrix A is at most n. Hence, there are at least $n - \operatorname{rank} A = n - k > 0$ parameters in the general solution. Thus, B is linearly dependent.

(c) Assume $\operatorname{Span} B = \mathbb{R}^n$. Then, $t_1\vec{v}_1 + \cdots + t_n\vec{v}_n = \vec{b}$ is consistent for all $\vec{b} \in \mathbb{R}^n$. Hence, the rank of the coefficient matrix A is n. Since A also has n columns, this means that the number of parameters in the general solution is $n - \operatorname{rank} A = n - n = 0$. Thus, the system $t_1\vec{v}_1 + \cdots + t_n\vec{v}_n = \vec{0}$ has a unique solution, so B is linearly independent.

Similarly, if B is linearly independent, then the system $t_1\vec{v}_1 + \cdots + t_n\vec{v}_n = \vec{0}$ has a unique solution, and thus $\operatorname{rank} A = n$. Therefore, $t_1\vec{v}_1 + \cdots + t_n\vec{v}_n = \vec{b}$ is consistent for all $\vec{b} \in \mathbb{R}^n$ as required.

2.4 Applications of Systems of Linear Equations

Practice Problems

A1 To simplify writing, let $\alpha = \frac{1}{\sqrt{2}}$.
Total horizontal force: $R_1 + R_2 = 0$.
Total vertical force: $R_V - F_V = 0$.
Total moment about A: $R_1 s + F_V(2s) = 0$.
The horizontal and vertical equations at the joints A, B, C, D, E are
$\alpha N_2 + R_2 = 0$ and $N_1 + \alpha N_2 + R_V = 0$;
$N_3 + \alpha N_4 + R_1 = 0$ and $-N_1 + \alpha N_4 = 0$;
$-N_3 + \alpha N_6 = 0$ and $-\alpha N_2 + N_5 + \alpha N_6 = 0$;
$-\alpha N_4 + N_7 = 0$ and $-\alpha N_4 - N_5 = 0$;
$-N_7 - \alpha N_6 = 0$ and $-\alpha N_6 - F_V = 0$.

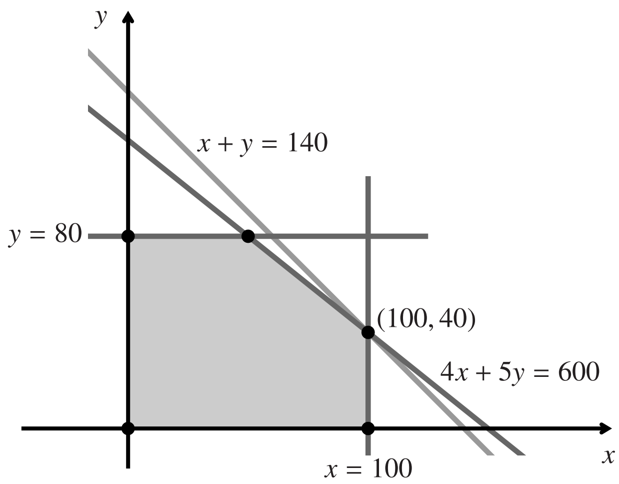
A2 To determine the currents in the loops, it is necessary to write down Kirchoff's second law with Ohm's law describing the voltage drop across the resistors. From the diagram, the resulting equations are:
For the top left loop: $R_1 i_1 + R_2(i_1 - i_2) = E_1$
For the top middle loop: $R_2(i_2 - i_1) + R_3(i_2 - i_3) = 0$
For the top right loop: $R_3(i_3 - i_2) + R_4 i_3 + R_8(i_3 - i_5) = 0$
For the bottom left loop: $R_5 i_4 + R_6(i_4 - i_5) = 0$
For the bottom right loop: $R_6(i_5 - i_4) + R_8(i_5 - i_3) + R_7 i_5 = E_2$
Multiplying out and collecting terms to display the equations as a system in the variables i_1, i_2, i_3, i_4, and i_5 yields

$$\begin{aligned}
(R_1 + R_2)i_1 - R_2 i_2 &= E_1 \\
-R_2 i_1 + (R_2 + R_3)i_2 - R_3 i_3 &= 0 \\
-R_3 i_2 + (R_3 + R_4 + R_8)i_3 - R_8 i_5 &= 0 \\
(R_5 + R_6)i_4 - R_6 i_5 &= 0 \\
-R_8 i_3 - R_6 i_4 + (R_6 + R_7 + R_8)i_5 &= E_2
\end{aligned}$$

The augmented matrix of this system is

$$\left[\begin{array}{ccccc|c} R_1+R_2 & -R_2 & 0 & 0 & 0 & E_1 \\ -R_2 & R_2+R_3 & -R_3 & 0 & 0 & 0 \\ 0 & -R_3 & R_3+R_4+R_8 & 0 & -R_8 & 0 \\ 0 & 0 & 0 & R_5+R_6 & -R_6 & 0 \\ 0 & 0 & -R_8 & -R_6 & R_6+R_7+R_8 & E_2 \end{array}\right]$$

A3 Let $R(x,y) = x+y$ be the function to be maximized. The vertices are found to be $(0,0)$, $(100,0)$, $(100,40)$, $(0,80)$, $(50,80)$. Now compare the value of $R(x,y)$ at all the vertices: $R(0,0) = 0$, $R(100,0) = 100$, $R(100,40) = 140$, $R(0,80) = 80$, $R(50,80) = 130$. So the maximum value is 140 (occurring at the vertex $(100,40)$).



Chapter 2 Quiz

Problems

E1 Row reducing the augmented matrix gives

$$\left[\begin{array}{cccc|c} 0 & 1 & -2 & 1 & 2 \\ 2 & -2 & 4 & -1 & 10 \\ 1 & -1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 0 & 9 \end{array}\right] R_1 \updownarrow R_3 \sim \left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 2 \\ 2 & -2 & 4 & -1 & 10 \\ 0 & 1 & -2 & 1 & 2 \\ 1 & 0 & 1 & 0 & 9 \end{array}\right] \begin{array}{c} \\ R_2 - 2R_1 \\ \\ R_4 - R_1 \end{array} \sim$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 2 \\ 0 & 0 & 2 & -1 & 6 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 1 & 0 & 0 & 7 \end{array}\right] R_2 \updownarrow R_4 \sim \left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 7 \\ 0 & 1 & -2 & 1 & 2 \\ 0 & 0 & 2 & -1 & 6 \end{array}\right] \begin{array}{c} \\ \\ R_3 - R_2 \\ \\ \end{array} \sim$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & -2 & 1 & -5 \\ 0 & 0 & 2 & -1 & 6 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4 + R_3 \end{array} \sim \left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 7 \\ 0 & 0 & -2 & 1 & -5 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Therefore the system is inconsistent.

E2 Row reducing we get

$$\left[\begin{array}{ccccc} 0 & 3 & 3 & 0 & -1 \\ 1 & 1 & 3 & 3 & 1 \\ 2 & 4 & 9 & 6 & 1 \\ -2 & -4 & -6 & -3 & -1 \end{array}\right] \begin{array}{c} R_1 \updownarrow R_2 \\ \\ \\ R_4 + R_3 \end{array} \sim \left[\begin{array}{ccccc} 1 & 1 & 3 & 3 & 1 \\ 0 & 3 & 3 & 0 & -1 \\ 2 & 4 & 9 & 6 & 1 \\ 0 & 0 & 3 & 3 & 0 \end{array}\right] \begin{array}{c} \\ \frac{1}{3}R_2 \\ R_3 - 2R_1 \\ \frac{1}{3}R_4 \end{array} \sim$$

$$\left[\begin{array}{ccccc} 1 & 1 & 3 & 3 & 1 \\ 0 & 1 & 1 & 0 & -1/3 \\ 0 & 2 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \begin{array}{c} R_1 - R_2 \\ \\ R_3 - 2R_2 \\ \\ \end{array} \sim \left[\begin{array}{ccccc} 1 & 0 & 2 & 3 & 4/3 \\ 0 & 1 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \begin{array}{c} R_1 - 2R_3 \\ R_2 - R_3 \\ \\ R_4 - R_3 \end{array} \sim$$

$$\left[\begin{array}{ccccc} 1 & 0 & 0 & 3 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1/3 \\ 0 & 0 & 0 & 1 & 1/3 \end{array}\right] \begin{array}{c} R_1 - 3R_4 \\ \\ \\ \\ \end{array} \sim \left[\begin{array}{ccccc} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1/3 \\ 0 & 0 & 0 & 1 & 1/3 \end{array}\right]$$

E3 (a) The system is inconsistent if and only if some row of A is of the form $\begin{bmatrix} 0 & \cdots & 0 & c \end{bmatrix}$ with $c \neq 0$. Thus, the system is inconsistent when either $b+2 = 0$ and $b \neq 0$, or when $c^2 - 1 =$ and $c+1 \neq 0$; or equivalently, when $b = -2$ or $c = 1$. Thus, the system is inconsistent for all (a, b, c) of the form $(a, b, 1)$ or $(a, -2, c)$, and is consistent for all (a, b, c) where $b \neq -2$ and $c \neq 1$.

(b) The system has a unique solution if the number of leading ones in the RREF of A equals the number of variables in the system. So, to have a unique solution we require $b + 2 \neq 0$ and $c^2 - 1 \neq 0$. Hence, the system has a unique solution if and only if $b \neq -2$, $c \neq 1$, and $c \neq -1$.

E4 (a) Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ be a vector in $\mathbb{R}^5$ which is orthogonal to all three of the vectors. Then we have

$$0 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \\ 4 \end{bmatrix} = x_1 + x_2 + 3x_3 + x_4 + 4x_5$$

$$0 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 5 \\ 0 \\ 0 \end{bmatrix} = 2x_1 + x_2 + 5x_3$$

$$0 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 8 \\ 5 \\ 9 \end{bmatrix} = 3x_1 + 2x_2 + 8x_3 + 5x_4 + 9x_5$$

which is a homogeneous system of three equations in five variables. Row reducing the coefficient matrix gives

$$\begin{bmatrix} 1 & 1 & 3 & 1 & 4 \\ 2 & 1 & 5 & 0 & 0 \\ 3 & 2 & 8 & 5 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 0 & -11/4 \\ 0 & 1 & 1 & 0 & 11/2 \\ 0 & 0 & 0 & 1 & 5/4 \end{bmatrix}$$

Let $x_3 = s \in \mathbb{R}$ and $x_5 = t \in \mathbb{R}$, then we get that the general solution is

$$\vec{x} = s\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 11/4 \\ -11/2 \\ 0 \\ -5/4 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

These are all vectors in $\mathbb{R}^5$ which are orthogonal to the three vectors.

(b) If there exists a vector $\vec{x} \in \mathbb{R}^5$ which is orthogonal to $\vec{u}$, $\vec{v}$, and $\vec{w}$, then $\vec{x} \cdot \vec{u} = 0$, $\vec{x} \cdot \vec{v} = 0$, and $\vec{x} \cdot \vec{w} = 0$, yields a homogeneous system of three linear equations with five variables. Hence, the rank of the matrix is at most three and thus there are at least # of variables - rank = $5 - 3 = 2$ parameters. So, there are in fact infinitely many vectors orthogonal to the three vectors.

E5 Consider $\vec{x} = t_1\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + t_2\begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix} + t_3\begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}$. Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix} 3 & 1 & 4 \\ 1 & 1 & 1 \\ 2 & 6 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, it is a basis for $\mathbb{R}^3$ by Theorem 2.3.5.

E6 (a) False. The system may have infinitely many solutions.

(b) False. The system $x_1 = 1$, $2x_1 = 2$ has more equations than variables, but is consistent.

(c) True. The system $x_1 = 0$ has a unique solution.

(d) True. If there are more variables than equations and the system is consistent, then there must be parameters and hence the system cannot have a unique solution. Of course, the system may be inconsistent.

Chapter 2 Further Problems

Problems

F1 (a) Solve the homogeneous system by back-substitution to get solution

$$\vec{x}_H = t\begin{bmatrix} -r_{13} \\ -r_{23} \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

For the non-homogeneous system we get solution

$$\vec{x}_N = \begin{bmatrix} c_1 \\ c_2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -r_{13} \\ -r_{23} \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

(b) For the homogeneous system we get

$$\vec{x}_H = t_1 \begin{bmatrix} -r_{12} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -r_{15} \\ 0 \\ -r_{25} \\ -r_{35} \\ 1 \end{bmatrix}, \quad t_1, t_2 \in \mathbb{R}$$

For the non-homogeneous system we get solution

$$\vec{x}_N = \begin{bmatrix} c_1 \\ 0 \\ c_2 \\ c_3 \\ 0 \end{bmatrix} + s \begin{bmatrix} -r_{12} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -r_{15} \\ 0 \\ -r_{25} \\ -r_{35} \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

(c) Let $L(i)$ be the column index of the leading 1 in the i-th row of R; then $x_{L(i)}$ is the i-th leading variable. Since the rank of R is k, $L(i)$ is defined only for $i = 1, \ldots, k$. Let $x_{N(j)}$ denote the j-th non-leading variable. In terms of this notation, the fact that R is in RREF implies that entries above and below the leading 1s are zero, so that $(R)_{iL(i)} = 1$, but $(R)_{hL(i)} = 0$ if $h \neq i$. It is a little more awkward to express the fact that the entries in the lower left of R must also be zero:

$$(R)_{hN(j)} = 0 \quad \text{if } h > [\text{the greatest } L(i) \text{ less than } N(j)].$$

We also need some way of keeping coordinates in the correct slot, and we use the standard basis vectors $\vec{e}_i$ for this. Now we construct the general solution by back-substitution: let $x_{N(j)} = t_j$, solve, and collect terms in t_j:

$$\begin{aligned} \vec{x}_H = {} & t_1\big(- (R)_{1N(1)}\vec{e}_{L(1)} - \cdots - (R)_{kN(1)}\vec{e}_{L(k)} + \vec{e}_{N(1)}\big) \\ & + t_2\big(- (R)_{1N(2)}\vec{e}_{L(1)} - \cdots - (R)_{kN(2)}\vec{e}_{L(k)} + \vec{e}_{N(2)}\big) + \cdots \\ & + t_{n-k}\big(- (R)_{1N(n-k)}\vec{e}_{L(1)} - \cdots - (R)_{kN(n-k)}\vec{e}_{L(k)} + \vec{e}_{N(n-k)}\big) \end{aligned}$$

Recall that some of these $(R)_{ij}$ are equal to zero. Thus, if we let

$$\vec{v}_j = -(R)_{1N(j)}\vec{e}_{L(1)} - \cdots - (R)_{kN(j)}\vec{e}_{L(k)} + \vec{e}_{N(j)}$$

we have obtained the required expression for the solution $\vec{x}_H$ of the corresponding homogeneous problem. For the non-homogeneous system, note that the entries on the right-hand side of the equations are associated with leading variables. Thus we proceed as above and find that

$$\vec{x}_N = \big(c_1\vec{e}_{L(1)} + \cdots + c_k\vec{e}_{L(k)}\big) + \vec{x}_H$$

Thus

$$\vec{p} = c_1\vec{e}_{L(1)} + \cdots + c_k\vec{e}_{L(k)}$$

We refer to $\vec{p}$ as a particular solution for the inhomogeneous system.

(d) With the stated assumptions, $\left[A \mid \vec{b}\right]$ is row equivalent to some $[R \mid \vec{c}\,]$, where R is in RREF. Therefore, we know that the solution to the system $\left[A \mid \vec{b}\right]$ has a solution of the form found in part (c): the general solution of the inhomogeneous system can be written as a sum of a particular solution $\vec{p}$ and the general solution of the corresponding homogeneous system $\left[A \mid \vec{0}\right]$.

F2 It will be convenient to have notation for keeping track of multiplications and divisions. We shall write statements such as $M = n^2$ to indicate the number of multiplications and divisions to this point in the argument is n^2.

(a) We first calculate a_{21}/a_{11}, $(M = 1)$. We know that $a_{21} - (a_{21}/a_{11})a_{11} = 0$, so no calculation is required in the first column, but we must then calculate $a_{2j} - (a_{21}/a_{11})a_{1j}$ for $2 \leq j \leq n$, $(M = 1 + (n-1) = n)$, and also $b_2 - (a_{21}/a_{11})b_1$, $M = n + 1$.

We now have a zero as the entry in the first column, second row. We must do exactly the same number of operations to get zeros in the first column and rows, $3, \ldots, n$. Thus we do $(n+1)$ operations on each of $(n-1)$ rows, so $M = (n-1)(n+1) = n^2 - 1$.

Now we ignore the first row and first column and turn our attention to the remaining $(n-1) \times (n-1)$ coefficient matrix and its corresponding augmenting column. Since the $n \times n$ case requires $n^2 - 1$ operations, the $(n-1) \times (n-1)$ case requires $(n-1)^2 - 1$ operations. Similarly, the $(n-i) \times (n-i)$ case requires $(n-i)^2 - 1$ operations.

When we have done all this, we have the required form $[R \mid \vec{c}\,]$, and

$$\begin{aligned} M &= (n^2 - 1) + ((n-1)^2 - 1) + \cdots + (2^2 - 1) + (1^2 - 1) \\ &= \left(\sum_{k=1}^{n} k^2\right) - n = \frac{n(n+1)(2n+1)}{6} - n \\ &= \frac{1}{3}n^3 + \frac{1}{2}n^2 - \frac{5}{6}n \end{aligned}$$

(b) We use M_B for our counter for back-substitution. We begin by solving for the n-th variable by $x_n = d_n/c_{nn}$, $(M_B = 1)$. Next we substitute this value into the $(n-1)$-st equation:

$$x_{n-1} = \frac{1}{c_{(n-1)(n-1)}}(d_{n-1} - c_{(n-1),n}x_n)$$

This requires one multiplication and one division, so $M_B = 1 + 2$. Continuing in this way, we find that the total for the procedure of back-substitution is

$$M_B = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

(c) For the extra steps in the Gauss-Jordan procedure, we denote our counter by M_J. We again begin with the matrix $\left[C \mid \vec{d}\,\right]$. We start by dividing the last row by c_{nn}; we know that the entry in the n-th row and n-th column becomes 1, so no operation is required. We do calculate d_1/c_{nn}, $M_J = 1$. Next we obtain zeros in all the entries of the n-th column above the last row: no actual operations are required for these entries themselves, but we must apply the appropriate operation in the augmenting column: $d_j - c_{jn}/dn$. We require one operation on each of $(n-1)$ rows, so $M_j = 1 + (n-1)$. Now we turn to the $(n-1) \times (n-1)$ coefficient matrix consisting of the first $(n-1)$ rows and $(n-1)$ columns with the appropriate augmenting column. By an argument similar to that just given, we require a further $1 + (n-2)$ operations, so now

$M_J = [1 + (n-1)] + [1 + (n-2)]$. Continuing in this way we find that to fully row reducing the matrix, we require

$$M_J = [1 + (n-1)] + [1 + (n-2)] \cdots + [1 + 0] = \frac{n(n+1)}{2}$$

operations, the same number as required for back-substitution.

Notice that the total number of operations required by elimination and back-substitution (or by Gauss-Jordan elimination) is

$$\frac{1}{3}n^3 + n^2 - \frac{1}{3}n$$

For large n, the important term is $\frac{1}{3}n^3$.

(d) Here our counter is M_C. We begin as with standard elimination, so to obtain zeros below the first entry in the first column, we have $M_C = (n+1)(n-1)$. Next we must obtain zeros above and below the entry in the second row and second column. Since the entries have all been changed, we don't really have a correct notation for the next calculation, but it is of the form: calculate

$$\frac{a_{j2}}{a_{22}} \quad \text{(one division)}$$

then calculate

$$a_{jk} - \frac{a_{j2}}{a_{22}}a_{2k} \quad \text{(one multiplication)}$$

We must do this for each entry in $(n-1)$ rows (all the rows except the second), and $n + 1 - 1 = n$ columns (include the augmenting column, but omit the first column). Thus we require an additional $n(n-1)$ operations, so to this point, $M_C + (n+1)(n-1) + n(n-1)$.

Continuing in this way, we find that

$$\begin{aligned} M_C &= (n+1)(n-1) + n(n-1) + \cdots + 2(n-1) \\ &= (n-1)\sum_{k=2}^{n+1} k = (n-1)\left(\frac{(n+1)(n+2)}{2} - 1\right) \\ &= \frac{1}{2}n^3 + n^2 - \frac{3}{2}n \end{aligned}$$

CHAPTER 3 Matrices, Linear Mappings, and Inverses

3.1 Operations on Matrices

Practice Problems

A1 (a) $\begin{bmatrix}2 & -2 & 3\\4 & 1 & -1\end{bmatrix}+\begin{bmatrix}-3 & -4 & 1\\2 & -5 & 3\end{bmatrix}=\begin{bmatrix}-1 & -6 & 4\\6 & -4 & 2\end{bmatrix}$

(b) $(-3)\begin{bmatrix}1 & -2\\2 & 1\\4 & -2\end{bmatrix}=\begin{bmatrix}-3 & 6\\-6 & -3\\-12 & 6\end{bmatrix}$

(c) $\begin{bmatrix}2 & 3\\1 & -2\end{bmatrix}-3\begin{bmatrix}1 & -2\\4 & 5\end{bmatrix}=\begin{bmatrix}-1 & 9\\-11 & -17\end{bmatrix}$

A2 (a) $\begin{bmatrix}-2 & 3\\3 & 4\end{bmatrix}\begin{bmatrix}4 & 5 & -3\\-1 & 3 & 2\end{bmatrix}=\begin{bmatrix}-11 & -1 & 12\\8 & 27 & -1\end{bmatrix}$

(b) $\begin{bmatrix}2 & 0 & 3\\1 & 1 & 1\\-1 & 3 & 2\end{bmatrix}\begin{bmatrix}6 & -2\\3 & 1\\0 & 5\end{bmatrix}=\begin{bmatrix}12 & 11\\9 & 4\\3 & 15\end{bmatrix}$

(c) $\begin{bmatrix}2 & 3\\-1 & -1\\5 & 3\end{bmatrix}\begin{bmatrix}4 & 3 & 2 & 1\\-4 & 0 & 3 & -2\end{bmatrix}=\begin{bmatrix}-4 & 6 & 13 & -4\\0 & -3 & -5 & 1\\8 & 15 & 19 & -1\end{bmatrix}$

(d) The product is not defined since the number of columns of the first matrix does not equal the number of rows of the second matrix.

A3 (a) We have

$$A(B+C)=\begin{bmatrix}2 & 3\\-3 & 1\end{bmatrix}\begin{bmatrix}-5 & -1\\-1 & 4\end{bmatrix}=\begin{bmatrix}-13 & 10\\14 & 7\end{bmatrix}$$

$$AB+AC=\begin{bmatrix}-3 & 4\\10 & -6\end{bmatrix}+\begin{bmatrix}-10 & 6\\4 & 13\end{bmatrix}=\begin{bmatrix}-13 & 10\\14 & 7\end{bmatrix}$$

$$A(3B)=\begin{bmatrix}2 & 3\\-3 & 1\end{bmatrix}\begin{bmatrix}-9 & 6\\3 & 0\end{bmatrix}=\begin{bmatrix}-9 & 12\\30 & -18\end{bmatrix}$$

$$3(AB)=3\begin{bmatrix}-3 & 4\\10 & -6\end{bmatrix}=\begin{bmatrix}-9 & 12\\30 & -18\end{bmatrix}$$

(b) We have

$$A(B+C) = \begin{bmatrix} 2 & -1 & -2 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ -6 & 4 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 6 & -16 \\ 4 & 14 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} 1 & -6 \\ -9 & 17 \end{bmatrix} + \begin{bmatrix} 5 & -10 \\ 13 & -3 \end{bmatrix} = \begin{bmatrix} 6 & -16 \\ 4 & 14 \end{bmatrix}$$

$$A(3B) = \begin{bmatrix} 2 & -1 & -2 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -3 & 6 \\ -9 & 12 \\ 0 & 9 \end{bmatrix} = \begin{bmatrix} 3 & -18 \\ -27 & 51 \end{bmatrix}$$

$$3(AB) = 3\begin{bmatrix} 1 & -6 \\ -9 & 17 \end{bmatrix} = \begin{bmatrix} 3 & -18 \\ -27 & 51 \end{bmatrix}$$

A4 (a) Since A and B have the same size $A+B$ is defined. The number of columns of A does not equal the number of rows of B so AB is not defined. We have

$$(A+B)^T = \begin{bmatrix} -3 & -1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}^T = \begin{bmatrix} -3 & 2 & 1 \\ -1 & 2 & 3 \end{bmatrix}$$

$$A^T + B^T = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 3 & 1 \end{bmatrix} + \begin{bmatrix} -4 & 1 & 3 \\ -3 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 1 \\ -1 & 2 & 3 \end{bmatrix}$$

(b) Since A and B are different sizes $A+B$ is not defined. The number of columns of A equals the number of rows of B so AB is defined. We have

$$(AB)^T = \begin{bmatrix} -21 & 15 \\ -10 & -27 \end{bmatrix}^T = \begin{bmatrix} -21 & -10 \\ 15 & -27 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} -3 & 5 & 1 \\ -4 & -2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ -4 & 1 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -21 & -10 \\ 15 & -27 \end{bmatrix}$$

A5 (a) $AB = \begin{bmatrix} 13 & 31 & 2 \\ 10 & 12 & 10 \end{bmatrix}$.

(b) BA is not defined since the number of rows of A does not equal the number of columns of B.

(c) AC is not defined since the number of rows of C does not equal the number of columns of A.

(d) DC is not defined since the number of rows of C does not equal the number of columns of D.

(e) CD is not defined since the number of rows of D does not equal the number of columns of C.

(f) $C^T D = \begin{bmatrix} 1 & 1 & 4 \\ 4 & 3 & -3 \end{bmatrix} \begin{bmatrix} 4 & 3 & 2 & 1 \\ -1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 7 & 3 & 15 \\ 7 & 9 & 11 & 1 \end{bmatrix}$

(g) We have

$$A(BC) = \begin{bmatrix} 2 & 5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -14 & 17 \\ 16 & 21 \end{bmatrix} = \begin{bmatrix} 52 & 139 \\ 62 & 46 \end{bmatrix}$$

$$(AB)C = \begin{bmatrix} 13 & 31 & 2 \\ 10 & 12 & 10 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 1 & 3 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 52 & 139 \\ 62 & 46 \end{bmatrix}$$

(h) We have

$$(AB)^T = \begin{bmatrix} 13 & 10 \\ 31 & 12 \\ 2 & 10 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} -1 & 3 \\ 3 & 5 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 13 & 10 \\ 31 & 12 \\ 2 & 10 \end{bmatrix}$$

(i) Using the result from (f) we get

$$D^T C = (C^T D)^T = \begin{bmatrix} 11 & 7 \\ 7 & 9 \\ 3 & 11 \\ 15 & 1 \end{bmatrix}$$

A6 (a) $A\vec{x} = \begin{bmatrix} 12 \\ 17 \\ 3 \end{bmatrix}$, $A\vec{y} = \begin{bmatrix} 8 \\ 4 \\ -4 \end{bmatrix}$, $A\vec{z} = \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}$

(b) We have

$$A \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & -1 \\ 4 & -1 & 1 \end{bmatrix} = A \begin{bmatrix} \vec{x} & \vec{y} & \vec{z} \end{bmatrix} = \begin{bmatrix} A\vec{x} & A\vec{y} & A\vec{z} \end{bmatrix} = \begin{bmatrix} 12 & 8 & -2 \\ 17 & 4 & 5 \\ 3 & -4 & 1 \end{bmatrix}$$

A7 Using the second view of matrix-vector multiplication we get

$$A\vec{x} = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

(a) Since $\vec{b}$ is the first column of A, we see that taking $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ gives $A\vec{x} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$.

(b) Since $\vec{b}$ is two times the first column of A, we see that taking $\vec{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ gives $A\vec{x} = \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix}$.

(c) Since $\vec{b}$ is three times the third column of A, we see that taking $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$ gives $A\vec{x} = \begin{bmatrix} 3 \\ 12 \\ 3 \end{bmatrix}$.

(d) Observe that $\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$ is the sum of the first two columns of A. Hence, taking $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ gives $A\vec{x} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$.

A8 (a) $\begin{bmatrix} -3 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 6 \end{bmatrix} = \begin{bmatrix} -6 & -18 \\ 2 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 6 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = [0]$

(c) $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\begin{bmatrix} 5 & 4 & -3 \end{bmatrix} = \begin{bmatrix} 10 & 8 & -6 \\ -5 & -4 & 3 \\ 15 & 12 & -9 \end{bmatrix}$

(d) $\begin{bmatrix} 5 & 4 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = [-3]$

A9 We have

$$\begin{bmatrix} 2 & 3 & -4 & 5 \\ -4 & 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 6 & 3 \\ -2 & 4 \\ 1 & 3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -13 & 16 \\ -27 & 0 \end{bmatrix}$$

$$\begin{aligned}\begin{bmatrix} 2 & 3 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} 6 & 3 \\ -2 & 4 \end{bmatrix} + \begin{bmatrix} -4 & 5 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -3 & 2 \end{bmatrix} &= \begin{bmatrix} 6 & 18 \\ -26 & -8 \end{bmatrix} + \begin{bmatrix} -19 & -2 \\ -1 & 8 \end{bmatrix} \\ &= \begin{bmatrix} -13 & 16 \\ -27 & 0 \end{bmatrix}\end{aligned}$$

A10 We need to determine if there exists t_1, t_2, and t_3 such that

$$\begin{bmatrix} 2 & 3 \\ 2 & -3 \end{bmatrix} = t_1\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} + t_2\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} + t_3\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} t_1 + t_3 & 2t_1 + t_2 + t_3 \\ t_1 - t_2 + 3t_3 & 2t_2 - t_3 \end{bmatrix}$$

Comparing corresponding entries gives the system of linear equations

$$\begin{aligned} t_1 + t_3 &= 2 \\ 2t_1 + t_2 + t_3 &= 3 \\ t_1 - t_2 + 3t_3 &= 2 \\ 2t_2 - t_3 &= -3 \end{aligned}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 3 \\ 1 & -1 & 3 & 2 \\ 0 & 2 & -1 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system is consistent and so the matrix is in the span. In particular, we have $t_1 = 3$, $t_2 = -2$, and $t_3 = -1$, so

$$3\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} + (-2)\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} + (-1)\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & -3 \end{bmatrix}$$

A11 Consider

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = t_1\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} + t_2\begin{bmatrix} 0 & 1 \\ 2 & -2 \end{bmatrix} + t_3\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} -t_1 + t_3 & t_1 + t_2 - t_3 \\ t_1 + 2t_2 + 3t_3 & -t_1 - 2t_2 - 3t_3 \end{bmatrix}$$

Row reducing the coefficient matrix of the corresponding homogeneous system gives

$$\begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the only solution is the trivial solution, so the set is linearly independent.

A12 Using the second view of matrix-vector multiplication and the fact that the i-th component of $\vec{e}_i$ is 1 and all other components are 0, we get

$$A\vec{e}_i = 0\vec{a}_1 + \cdots + 0\vec{a}_{i-1} + 1\vec{a}_i + 0\vec{a}_{i+1} + \cdots + 0\vec{a}_n = \vec{a}_i$$

Homework Problems

B1 (a) $\begin{bmatrix} 8 & 2 \\ -3 & 5 \\ -3 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} -10 & -15 & 30 & 10 \\ 35 & -5 & 0 & -25 \end{bmatrix}$ (c) $\begin{bmatrix} 12 & 6 & -17 \\ -26 & -27 & 1 \end{bmatrix}$

B2 (a) $\begin{bmatrix} -5 & -9 & 4 \\ 13 & 8 & -9 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -3 \\ -8 & 1 \\ 24 & -4 \end{bmatrix}$

(c) $\begin{bmatrix} 11 & 2 & -4 & 3 \\ -2 & 6 & -12 & 16 \\ -8 & 9 & -18 & 25 \end{bmatrix}$

(d) It is not defined since the number of columns of the first matrix does not equal the number of rows of the second matrix.

B3 (a) $A(B+C) = \begin{bmatrix} 21 & -11 & 34 \\ -7 & -2 & -11 \end{bmatrix} = AB + AC$ $\quad A(3B) = \begin{bmatrix} 51 & -72 & 111 \\ -27 & 6 & -18 \end{bmatrix} = 3(AB)$

(b) $A(B+C) = \begin{bmatrix} 10 & -3 \\ -2 & -9 \\ 26 & 57 \end{bmatrix} = AB + AC$ $\quad A(3B) = \begin{bmatrix} 30 & 33 \\ 6 & -9 \\ -3 & 102 \end{bmatrix} = 3(AB)$

B4 (a) $A + B$ is not defined. AB is defined. $(AB)^T = \begin{bmatrix} 3 & -12 \\ 3 & -11 \end{bmatrix} = B^T A^T$.

(b) $A + B$ is defined. $(A+B)^T = \begin{bmatrix} 7 & -1 & 3 \\ 1 & 2 & -3 \\ 3 & 7 & -7 \end{bmatrix} = A^T + B^T$.

AB is defined. $(AB)^T = \begin{bmatrix} 11 & -3 & 0 \\ -2 & -7 & 7 \\ 7 & -17 & 14 \end{bmatrix} = B^T A^T$.

(a) $AB = \begin{bmatrix} 10 & -3 \\ -11 & -6 \\ 0 & -31 \end{bmatrix}$ (b) BA is not defined.

(c) $AC = \begin{bmatrix} -6 & 11 & -4 \\ 6 & -7 & 2 \\ -2 & 17 & -8 \end{bmatrix}$ (d) DC is not defined.

B5 (e) $DA = \begin{bmatrix} -21 & 10 \\ 2 & -7 \\ -4 & -10 \\ -3 & 2 \end{bmatrix}$ (f) $CD^T = \begin{bmatrix} 23 & 8 & 14 & 5 \\ 1 & 2 & 8 & -1 \end{bmatrix}$

(g) $B(CA) = \begin{bmatrix} -31 & -2 \\ -93 & -44 \end{bmatrix} = (BC)A$ (h) $(AB)^T = \begin{bmatrix} 10 & -11 & 0 \\ -3 & -6 & -31 \end{bmatrix} = B^T A^T$

(i) $DC^T = (CD^T)^T = \begin{bmatrix} 23 & 1 \\ 8 & 2 \\ 14 & 8 \\ 5 & -1 \end{bmatrix}$

B6 (a) $A\vec{x} = \begin{bmatrix} 0 \\ -10 \\ -4 \end{bmatrix}, A\vec{y} = \begin{bmatrix} 3 \\ 9 \\ 2 \end{bmatrix}, A\vec{z} = \begin{bmatrix} 11 \\ 6 \\ -4 \end{bmatrix}$. (b) $\begin{bmatrix} 0 & -3 & -11 \\ -10 & -9 & -6 \\ -4 & -2 & 4 \end{bmatrix}$

B7 (a) $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

B8 (a) $\begin{bmatrix} -6 & 12 \\ -4 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \end{bmatrix}$ (c) $\begin{bmatrix} -6 & 2 & 4 \\ -3 & 1 & 2 \\ -15 & 5 & 10 \end{bmatrix}$ (d) $\begin{bmatrix} 5 \end{bmatrix}$

B9 Both sides give $\begin{bmatrix} -14 & 8 \\ 19 & 29 \end{bmatrix}$.

B10 It is not in the span.

B11 The set is linearly independent.

Computer Problems

C1 See solutions of $A2$ and $A5$ respectively.

C2 (a) $\begin{bmatrix} 15.65 & 38.55 & 1.779 \\ 13.86 & 13.92 & 17.88 \end{bmatrix}, \begin{bmatrix} 15.65 & 13.86 \\ 38.55 & 13.92 \\ 1.779 & 17.88 \end{bmatrix}$

(b) $\begin{bmatrix} -19.61 & 25.82 \\ 25.99 & 25.32 \end{bmatrix}, \begin{bmatrix} -19.61 & 25.99 \\ 25.82 & 25.32 \end{bmatrix}$

Conceptual Problems

D1 According to the hints, we only need to prove that if $(A-B)\vec{x} = \vec{0}$ for every $\vec{x} \in \mathbb{R}^n$, then $A - B$ is the zero matrix. We let $A - B = C = \begin{bmatrix} \vec{c}_1 & \cdots & \vec{c}_n \end{bmatrix}$. If $C\vec{x} = \vec{0}$ for every $\vec{x} \in \mathbb{R}^n$, then taking $\vec{x} = \vec{e}_i$, the i-th standard basis vector, we get $\vec{0} = C\vec{e}_i = \vec{c}_i$ since $\vec{e}_i$ has 1 as its i-th component and 0s elsewhere. Thus, every column of C is the zero vector and hence C is the zero matrix. Therefore $A = B$ as required.

D2 Since $(I_m)_{ij} = 0$ for $i \neq j$ and $(I_m)_{ii} = 1$, we have

$$\begin{aligned}(I_mA)_{ij} &= \sum_{k=1}^{n} (I_m)_{ik} a_{kj} \\ &= (I_m)_{i1}a_{1j} + \cdots + (I_m)_{i(i-1)}a_{(i-1)j} + (I_m)_{ii}a_{ij} + (I_m)_{i(i+1)}a_{(i+1)j} + \cdots + (I_m)_{im}a_{mj} \\ &= 0 + \cdots + 0 + 1a_{ij} + 0 + \cdots + 0 \\ &= a_{ij}\end{aligned}$$

Thus, $I_mA = A$. Similarly,

$$(AI_n)_{ij} = \sum_{k=1}^{n} A_{ik}(I_n)_{kj} = 0 + \cdots + 0 + 1a_{ij} + 0 + \cdots + 0 = a_{ij}$$

so $AI_n = A$.

D3 By definition of matrix multiplication, the ij-th entry of AA^T is the dot product of the i-th row of A and the j-th column of A^T. But, the j-th column of A^T is just the j-th row of A. Hence, the ij-th entry of AA^T is the dot product of the i-th row and j-th row of A.
Similarly, the ij-th entry of A^TA is the dot product of the i-th row of A^T and the j-th column of A. But, the i-th row of A^T is just the i-th column of A. Hence, the ij-th entry of A^TA is the dot product of the i-th column and j-th column of A.
In particular, observe that the entries $(AA^T)_{ii}$ are given by the dot product of the i-th row of A with itself. So, if AA^T is the zero matrix, then this entry is 0, so the dot product of the i-th row of A with itself is 0, and hence the i-th row of A is zero. This is true for all $1 \leq i \leq n$, hence A is the zero matrix. The argument for A^TA is similar.

D4 (a) There are many possible choices. A couple of possibilities are: $\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}$, or $\begin{bmatrix} a & -a \\ a & -a \end{bmatrix}$, for any $a \in \mathbb{R}$.

(b) $A^2 - AB - BA + B^2 = (A-B)^2$. To get $(A-B)^2 = O_{2,2}$ we can pick any A and B such that $A - B$ satisfies one of the choices from (a). One possibility is $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$.

D5 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We want A to satisfy

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & b(a+d) \\ c(a+d) & bc + d^2 \end{bmatrix}$$

To get zeros on the off-diagonal, we require either $b = c = 0$, or $a + d = 0$. If $b = c = 0$, then $a^2 = 1$ and $d^2 = 1$, so the possible matrices are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

If $a + d = 0$, we may write $d = -a$; we still require $a^2 + bc = 1$. If we take $a = \pm 1$ we get the previous case. If we take $a = 0$, we get $\begin{bmatrix} 0 & b \\ 1/b & 0 \end{bmatrix}$ for $b \neq 0$.

The general 2×2 matrix A satisfying $A^2 = I$ is of the form

$$A = \begin{bmatrix} a & b \\ \frac{1-a^2}{b} & -a \end{bmatrix}, \quad b \neq 0$$

$$A = \begin{bmatrix} a & \frac{1-a^2}{c} \\ c & -a \end{bmatrix}, \quad c \neq 0$$

3.2 Matrix Mappings and Linear Mappings

Practice Problems

A1 (a) Since A has two columns, $A\vec{x}$ is defined only if $\vec{x}$ has two rows. Thus, the domain of f_A is $\mathbb{R}^2$. Since A has four rows the product $A\vec{x}$ will have four entries, thus the codomain of f_A is $\mathbb{R}^4$.

(b) We have

$$f_A(2,-5) = \begin{bmatrix} -2 & 3 \\ 3 & 0 \\ 1 & 5 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} -19 \\ 6 \\ -23 \\ 38 \end{bmatrix}$$

$$f_A(-3,4) = \begin{bmatrix} -2 & 3 \\ 3 & 0 \\ 1 & 5 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 18 \\ -9 \\ 17 \\ -36 \end{bmatrix}$$

(c) We have

$$f_A(1,0) = \begin{bmatrix} -2 & 3 \\ 3 & 0 \\ 1 & 5 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 1 \\ 4 \end{bmatrix}$$

$$f_A(0,1) = \begin{bmatrix} -2 & 3 \\ 3 & 0 \\ 1 & 5 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 5 \\ -6 \end{bmatrix}$$

(d) $f_A(\vec{x}) = \begin{bmatrix} -2 & 3 \\ 3 & 0 \\ 1 & 5 \\ 4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_1 + 3x_2 \\ 3x_1 + 0x_2 \\ x_1 + 5x_2 \\ 4x_1 - 6x_2 \end{bmatrix}$

(e) The standard matrix of f_A is

$$[f_A] = \begin{bmatrix} f_A(1,0) & f_A(0,1) \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 3 & 0 \\ 1 & 5 \\ 4 & -6 \end{bmatrix}$$

A2 (a) Since A has four columns, $A\vec{x}$ is defined only if $\vec{x}$ has four rows. Thus, the domain of f_A is $\mathbb{R}^4$. Since A has three rows the product $A\vec{x}$ will have three entries, thus the codomain of f_A is $\mathbb{R}^3$.

(b) We have

$$f_A(2,-2,3,1) = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & -1 & 0 & 3 \\ 1 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -11 \\ 9 \\ 7 \end{bmatrix}$$

$$f_A(-3,1,4,2) = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & -1 & 0 & 3 \\ 1 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -13 \\ -1 \\ 3 \end{bmatrix}$$

(c) We have

$$f_A(\vec{e}_1) = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & -1 & 0 & 3 \\ 1 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$f_A(\vec{e}_2) = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & -1 & 0 & 3 \\ 1 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

$$f_A(\vec{e}_3) = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & -1 & 0 & 3 \\ 1 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$$

$$f_A(\vec{e}_4) = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & -1 & 0 & 3 \\ 1 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$$

(d) $f_A(\vec{x}) = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & -1 & 0 & 3 \\ 1 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 - 3x_3 \\ 2x_1 - x_2 + 3x_4 \\ x_1 + 2x_3 - x_4 \end{bmatrix}$

(e) The standard matrix of f_A is

$$[f_A] = \begin{bmatrix} f_A(\vec{e}_1) & f_A(\vec{e}_2) & f_A(\vec{e}_3) & f_A(\vec{e}_4) \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 2 & -1 & 0 & 3 \\ 1 & 0 & 2 & -1 \end{bmatrix}$$

A3 (a) The domain of f is $\mathbb{R}^2$ and the codomain of f is $\mathbb{R}^2$. Since the components of the definition of f involve non-linear functions, we suspect that f is not linear. To prove that it is not linear, we just need to find one example where f does not preserve addition or scalar multiplication. If we take $x_1 = 0$ and $x_2 = 1$, then

$$2f(0,1) = 2\begin{bmatrix}\sin 0\\ e^1\end{bmatrix} = \begin{bmatrix}0\\ 2e\end{bmatrix}$$

but,

$$f(2(0,1)) = f(0,2) = \begin{bmatrix}\sin 0\\ e^{2^2}\end{bmatrix} = \begin{bmatrix}0\\ e^4\end{bmatrix}$$

Thus, $2f(0,1) \neq f(2(0,1))$ so f does not preserve scalar multiplication. Therefore f is not linear.

(b) The domain of g is $\mathbb{R}^2$ and the codomain of g is $\mathbb{R}^2$. The components of g are linear, so we suspect that g is linear. To prove this we need to show that g preserves addition and scalar multiplication. Let $\vec{x} = \begin{bmatrix}x_1\\ x_2\end{bmatrix}$, $\vec{y} = \begin{bmatrix}y_1\\ y_2\end{bmatrix}$, and $t \in \mathbb{R}$. Then

$$\begin{aligned}
g(t\vec{x}+\vec{y}) &= g(tx_1 + y_1, tx_2 + y_2)\\
&= \begin{bmatrix}2(tx_1 + y_1) + 3(tx_2 + y_2)\\ (tx_1 + y_1) - (tx_2 + y_2)\end{bmatrix}\\
&= \begin{bmatrix}2tx_1 + 2y_1 + 3tx_2 + 3y_2\\ tx_1 + y_1 - tx_2 - y_2\end{bmatrix}\\
&= \begin{bmatrix}2tx_1 + 3tx_2\\ tx_1 - tx_2\end{bmatrix} + \begin{bmatrix}2y_1 + 3y_2\\ y_1 - y_2\end{bmatrix}\\
&= t\begin{bmatrix}2x_1 + 3x_2\\ x_1 - x_2\end{bmatrix} + \begin{bmatrix}2y_1 + 3y_2\\ y_1 - y_2\end{bmatrix}\\
&= tg(\vec{x}) + g(\vec{y})
\end{aligned}$$

Thus, g is linear.

(c) The domain of h is $\mathbb{R}^2$ and the codomain of h is $\mathbb{R}^3$. Since the third component of h involves a multiplication of variables, we suspect that h is not linear. Let $\vec{x} = \begin{bmatrix}1\\ 2\end{bmatrix}$. Then

$$2h(1,2) = 2\begin{bmatrix}2(1)+3(2)\\ 1-2\\ 1(2)\end{bmatrix} = 2\begin{bmatrix}8\\ -1\\ 2\end{bmatrix}$$

but

$$h(2(1,2)) = h(2,4) = \begin{bmatrix}2(2)+3(4)\\ 2-4\\ 2(4)\end{bmatrix} = \begin{bmatrix}16\\ -2\\ 8\end{bmatrix} \neq 2h(1,2)$$

Therefore, h does not preserve scalar multiplication and so is not linear.

(d) The domain of k is $\mathbb{R}^3$ and the codomain of k is $\mathbb{R}^3$. The components of k are linear, so we suspect that k is linear. Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, and $t \in \mathbb{R}$. Then

$$\begin{aligned}
k(t\vec{x} + \vec{y}) &= k(tx_1 + y_1, tx_2 + y_2, tx_3 + y_3) \\
&= \begin{bmatrix} tx_1 + y_1 + tx_2 + y_2 \\ 0 \\ tx_2 + y_2 - (tx_3 + y_3) \end{bmatrix} \\
&= \begin{bmatrix} tx_1 + tx_2 \\ 0 \\ tx_2 - tx_3 \end{bmatrix} + \begin{bmatrix} y_1 + y_2 \\ 0 \\ y_2 - y_3 \end{bmatrix} \\
&= t\begin{bmatrix} x_1 + x_2 \\ 0 \\ x_2 - x_3 \end{bmatrix} + \begin{bmatrix} y_1 + y_2 \\ 0 \\ y_2 - y_3 \end{bmatrix} \\
&= tk(\vec{x}) + t(\vec{y})
\end{aligned}$$

Therefore, k is linear.

(e) The domain of ℓ is $\mathbb{R}^3$ and the codomain of ℓ is $\mathbb{R}^2$. Since one component of ℓ involves absolute values, we suspect this is not linear. Let $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Then

$$(-1)\ell(1, 0, 0) = (-1)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

but

$$\ell((-1)(1, 0, 0)) = \ell(-1, 0, 0) = \begin{bmatrix} 0 \\ |-1| \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq (-1)\ell(1, 0, 0)$$

Therefore, ℓ does not preserve scalar multiplication and so is not linear.

(f) The domain of m is $\mathbb{R}$ and the codomain of f is $\mathbb{R}^3$. Since the middle component of m is 1, m cannot be linear. For any $x_1 \in \mathbb{R}$ we have

$$0m(x_1) = 0\begin{bmatrix} x_1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \neq m(0(x_1, 1, 0))$$

Therefore, m does not preserve scalar multiplication and so is not linear.

REMARK: In this last problem, we used the property that for any linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ that

$$L(\vec{0}) = L(0\vec{x}) = 0L(\vec{x}) = \vec{0}$$

Thus, since $m(x_1)$ could not be the zero vector in $\mathbb{R}^3$, m could not be a linear mapping.

A4 (a) The domain is $\mathbb{R}^3$ and the codomain is $\mathbb{R}^2$. The columns of the standard matrix of L are the images of the standard basis vectors under L. We have

$$L(1,0,0) = \begin{bmatrix} 2(1) - 3(0) + 0 \\ 0 - 5(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$
$$L(0,1,0) = \begin{bmatrix} 2(0) - 3(1) + 0 \\ 1 - 5(0) \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$
$$L(0,0,1) = \begin{bmatrix} 2(0) - 3(0) + 1 \\ 0 - 5(1) \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

Thus,

$$[L] = \begin{bmatrix} L(1,0,0) & L(0,1,0) & L(0,0,1) \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & -5 \end{bmatrix}$$

(b) The domain is $\mathbb{R}^4$ and the codomain is $\mathbb{R}^2$. We have

$$K(1,0,0,0) = \begin{bmatrix} 5(1) + 3(0) - 0 \\ 0 - 7(0) + 3(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$
$$K(0,1,0,0) = \begin{bmatrix} 5(0) + 3(0) - 0 \\ 1 - 7(0) + 3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$K(0,0,1,0) = \begin{bmatrix} 5(0) + 3(1) - 0 \\ 0 - 7(1) + 3(0) \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$$
$$K(0,0,0,1) = \begin{bmatrix} 5(0) + 3(0) - 1 \\ 0 - 7(0) + 3(1) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Thus,

$$[K] = \begin{bmatrix} K(1,0,0,0) & K(0,1,0,0) & K(0,0,1,0) & K(0,0,0,1) \end{bmatrix} = \begin{bmatrix} 5 & 0 & 3 & -1 \\ 0 & 1 & -7 & 3 \end{bmatrix}$$

(c) The domain is $\mathbb{R}^4$ and the codomain is $\mathbb{R}^4$. We have

$$M(1,0,0,0) = \begin{bmatrix} 1 - 0 + 0 \\ 1 + 2(0) - 0 - 3(0) \\ 0 + 0 \\ 1 - 0 + 0 - 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$
$$M(0,1,0,0) = \begin{bmatrix} 0 - 0 + 0 \\ 0 + 2(1) - 0 - 3(0) \\ 1 + 0 \\ 0 - 1 + 0 - 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}$$
$$M(0,0,1,0) = \begin{bmatrix} 0 - 1 + 0 \\ 0 + 2(0) - 1 - 3(0) \\ 0 + 1 \\ 0 - 0 + 1 - 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$
$$M(0,0,0,1) = \begin{bmatrix} 0 - 0 + 1 \\ 0 + 2(0) - 0 - 3(1) \\ 0 + 0 \\ 0 - 0 + 0 - 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \\ -1 \end{bmatrix}$$

Thus,

$$[M] = \begin{bmatrix} M(1,0,0,0) & M(0,1,0,0) & M(0,0,1,0) & M(0,0,0,1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 1 & 2 & -1 & -3 \\ 0 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

A5 (a) Since the standard matrix of each mapping has three columns and two rows, the domain is $\mathbb{R}^3$ and codomain is $\mathbb{R}^2$ for both mappings.

(b) By Theorem 3.2.5 we get

$$[S+T] = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 5 \end{bmatrix}$$
$$[2S-3T] = 2\begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 9 \\ -8 & -6 & -5 \end{bmatrix}$$

A6 (a) The standard matrix of S has four columns and two rows, so the domain of S is $\mathbb{R}^4$ and the codomain of S is $\mathbb{R}^2$. The standard matrix of T has two columns and four rows, so the domain of T is $\mathbb{R}^2$ and the codomain of T is $\mathbb{R}^4$.

(b) By Theorem 3.2.5 we get

$$[S \circ T] = \begin{bmatrix} -3 & -3 & 0 & 1 \\ 0 & 2 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -2 & 1 \\ 2 & -1 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 6 & -19 \\ 10 & -10 \end{bmatrix}$$
$$[T \circ S] = \begin{bmatrix} 1 & 4 \\ -2 & 1 \\ 2 & -1 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} -3 & -3 & 0 & 1 \\ 0 & 2 & 4 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 5 & 16 & 9 \\ 6 & 8 & 4 & 0 \\ -6 & -8 & -4 & 0 \\ -9 & -17 & -16 & -5 \end{bmatrix}$$

A7 A composition of mappings $f \circ g$ is defined only if the domain of f contains the codomain of g. If it is defined, then by Theorem 3.2.5, the domain of $f \circ g$ is the domain of g and the codomain of $f \circ g$ is the codomain of f. This, of course, exactly matches the definition of matrix multiplication.

(a) The domain of L is $\mathbb{R}^2$ and the codomain of M is $\mathbb{R}^2$. Thus, $L \circ M$ is defined. The domain and codomain of $L \circ M$ are both $\mathbb{R}^3$.

(b) The domain of M is $\mathbb{R}^3$ and the codomain of L is $\mathbb{R}^3$. Thus, $M \circ L$ is defined. The domain and codomain of $M \circ L$ are both $\mathbb{R}^2$.

(c) The domain of L is $\mathbb{R}^2$, but the codomain of N is $\mathbb{R}^4$. Hence, this composition is not defined.

(d) The domain of N is $\mathbb{R}^2$, but the codomain of L is $\mathbb{R}^3$. Hence, this composition is not defined.

(e) The domain of M is $\mathbb{R}^3$, but the codomain of N is $\mathbb{R}^4$. Hence, this composition is not defined.

(f) The domain of N is $\mathbb{R}^2$ and the codomain of M is $\mathbb{R}^2$. Thus, $N \circ M$ is defined. The domain of $N \circ M$ is $\mathbb{R}^3$ and the codomain is $\mathbb{R}^4$.

A8 To find the standard matrix of the projection, we need to find the image of the standard basis vectors for $\mathbb{R}^2$ under the projection. That is, we need to project the standard basis vectors on to $\vec{v}$. We get

$$\begin{aligned}
\text{proj}_{\vec{v}}\,\vec{e}_1 &= \frac{\vec{e}_1\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{-2}{5}\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}4/5\\-2/5\end{bmatrix}\\
\text{proj}_{\vec{v}}\,\vec{e}_2 &= \frac{\vec{e}_2\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{1}{5}\begin{bmatrix}-2\\1\end{bmatrix} = \begin{bmatrix}-2/5\\1/5\end{bmatrix}
\end{aligned}$$

Thus,

$$[\text{proj}_{\vec{v}}] = \begin{bmatrix}\text{proj}_{\vec{v}}\,\vec{e}_1 & \text{proj}_{\vec{v}}\,\vec{e}_2\end{bmatrix} = \begin{bmatrix}4/5 & -2/5\\-2/5 & 1/5\end{bmatrix}$$

A9 To find the standard matrix of the perpendicular of the projection, we need to find the image of the standard basis vectors for $\mathbb{R}^2$ under the mapping. We get

$$\begin{aligned}
\text{perp}_{\vec{v}}\,\vec{e}_1 &= \vec{e}_1 - \frac{\vec{e}_1\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \begin{bmatrix}1\\0\end{bmatrix} - \frac{1}{17}\begin{bmatrix}1\\4\end{bmatrix} = \begin{bmatrix}16/17\\-4/17\end{bmatrix}\\
\text{perp}_{\vec{v}}\,\vec{e}_2 &= \vec{e}_2 - \frac{\vec{e}_2\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \begin{bmatrix}0\\1\end{bmatrix} - \frac{4}{17}\begin{bmatrix}1\\4\end{bmatrix} = \begin{bmatrix}-4/17\\1/17\end{bmatrix}
\end{aligned}$$

Thus,

$$[\text{perp}_{\vec{v}}] = \begin{bmatrix}\text{perp}_{\vec{v}}\,\vec{e}_1 & \text{perp}_{\vec{v}}\,\vec{e}_2\end{bmatrix} = \begin{bmatrix}16/17 & -4/17\\-4/17 & 1/17\end{bmatrix}$$

A10 To find the standard matrix of the projection, we need to find the image of the standard basis vectors for $\mathbb{R}^2$ under the projection. We get

$$\begin{aligned}
\text{proj}_{\vec{v}}\,\vec{e}_1 &= \frac{\vec{e}_1\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{2}{9}\begin{bmatrix}2\\2\\-1\end{bmatrix} = \begin{bmatrix}4/9\\4/9\\-2/9\end{bmatrix}\\
\text{proj}_{\vec{v}}\,\vec{e}_2 &= \frac{\vec{e}_2\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{2}{9}\begin{bmatrix}2\\2\\-1\end{bmatrix} = \begin{bmatrix}4/9\\4/9\\-2/9\end{bmatrix}\\
\text{proj}_{\vec{v}}\,\vec{e}_3 &= \frac{\vec{e}_3\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{-1}{9}\begin{bmatrix}2\\2\\-1\end{bmatrix} = \begin{bmatrix}-2/9\\-2/9\\1/9\end{bmatrix}
\end{aligned}$$

Thus,

$$[\text{proj}_{\vec{v}}] = \begin{bmatrix}\text{proj}_{\vec{v}}\,\vec{e}_1 & \text{proj}_{\vec{v}}\,\vec{e}_2 & \text{proj}_{\vec{v}}\,\vec{e}_3\end{bmatrix} = \begin{bmatrix}4/9 & 4/9 & -2/9\\4/9 & 4/9 & -2/9\\-2/9 & -2/9 & 1/9\end{bmatrix}$$

Homework Problems

B1 (a) The domain of f_A is $\mathbb{R}^3$ and the codomain of f_A is $\mathbb{R}^2$.

(b) $f_A(3,4,-5) = \begin{bmatrix}23\\-8\end{bmatrix}$, $f_A(-2,1,-4) = \begin{bmatrix}14\\9\end{bmatrix}$

(c) $f_A(\vec{e}_1) = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$, $f_A(\vec{e}_2) = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$, $f_A(\vec{e}_3) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

(d) $f_A(x_1, x_2, x_3) = \begin{bmatrix} -x_1 + 4x_2 - 2x_3 \\ -5x_1 + 3x_2 + x_3 \end{bmatrix}$

(e) The standard matrix of f_A is

$$[f_A] = \begin{bmatrix} f_A(1,0,0) & f_A(0,1,0) & f_A(0,0,1) \end{bmatrix} = \begin{bmatrix} -1 & 4 & -2 \\ -5 & 3 & 1 \end{bmatrix}$$

B2 (a) The domain of f_A is $\mathbb{R}^3$ and the codomain is $\mathbb{R}^4$.

(b) $f_A(-4, 2, 1) = \begin{bmatrix} -6 \\ 7 \\ 3 \\ 8 \end{bmatrix}$, $f_A(3, -3, 2) = \begin{bmatrix} 3 \\ 0 \\ 12 \\ 10 \end{bmatrix}$

(c) $f_A(\vec{e}_1) = \begin{bmatrix} 2 \\ 0 \\ 5 \\ 2 \end{bmatrix}$, $f_A(\vec{e}_2) = \begin{bmatrix} 1 \\ 2 \\ 7 \\ 4 \end{bmatrix}$, $f_A(\vec{e}_3) = \begin{bmatrix} 0 \\ 3 \\ 9 \\ 8 \end{bmatrix}$

(d) $f_A(x_1, x_2, x_3) = \begin{bmatrix} 2x_1 + x_2 \\ 2x_2 + 3x_3 \\ 5x_1 + 7x_2 + 9x_3 \\ 2x_1 + 4x_2 + 8x_3 \end{bmatrix}$

(e) The standard matrix of f_A is $[f_A] = \begin{bmatrix} f_A(1,0,0) & f_A(0,1,0) & f_A(0,0,1) \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 3 \\ 5 & 7 & 9 \\ 2 & 4 & 8 \end{bmatrix}$.

B3 (a) f is linear. (b) g is not linear. (c) h is linear.
(d) k is linear. (e) ℓ is not linear. (f) m is linear.

B4 (a) Domain $\mathbb{R}^3$, codomain $\mathbb{R}^3$, $[L] = \begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 0 \\ 4 & -5 & 0 \end{bmatrix}$

(b) Domain $\mathbb{R}^4$, codomain $\mathbb{R}^3$, $[K] = \begin{bmatrix} 2 & 0 & -1 & 3 \\ -1 & -2 & 2 & 1 \\ 0 & 3 & 1 & 0 \end{bmatrix}$

(c) Domain $\mathbb{R}^3$, codomain $\mathbb{R}^3$, $[M] = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 5 & 1 & 0 \end{bmatrix}$

B5 (a) The domain of both S and T is $\mathbb{R}^2$ and the codomain of both S and T is $\mathbb{R}^3$.

(b) $[T + S] = \begin{bmatrix} 5 & 2 \\ 2 & -4 \\ 4 & 1 \end{bmatrix}$, $[-S + 2T] = \begin{bmatrix} 4 & 1 \\ 1 & -2 \\ -1 & -1 \end{bmatrix}$

B6 (a) The domain of S is $\mathbb{R}^2$ and the codomain of S is $\mathbb{R}^4$. The domain of T is $\mathbb{R}^4$ and the codomain of T is $\mathbb{R}^2$.

(b) $[S \circ T] = \begin{bmatrix} 22 & -3 & -1 & 12 \\ 2 & 1 & 5 & 3 \\ 26 & -3 & 1 & 15 \\ -8 & 0 & -4 & -6 \end{bmatrix}$, $[T \circ S] = \begin{bmatrix} 20 & -18 \\ 8 & -2 \end{bmatrix}$

B7 (a) The domain of $L \circ M$ is $\mathbb{R}^4$ and the codomain of $L \circ M$ is $\mathbb{R}^3$.

(b) not defined

(c) not defined

(d) The domain of $N \circ L$ is $\mathbb{R}^2$ and the codomain of $N \circ L$ is $\mathbb{R}^4$.

(e) The domain of $M \circ N$ is $\mathbb{R}^3$ and the codomain of $M \circ N$ is $\mathbb{R}^2$.

(f) not defined

B8 $[\text{proj}_{\vec{v}}] = \frac{1}{10}\begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$

B9 $[\text{perp}_{\vec{v}}] = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

B10 $[\text{proj}_{\vec{v}}] = \frac{1}{18}\begin{bmatrix} 1 & -1 & 4 \\ -1 & 1 & -4 \\ 4 & -4 & 16 \end{bmatrix}$

B11 $[\text{perp}_{\vec{v}}] = \frac{1}{9}\begin{bmatrix} 8 & 2 & -2 \\ 2 & 5 & 4 \\ -2 & 4 & 5 \end{bmatrix}$

Conceptual Problems

D1 If $L(\vec{x}+\vec{y}) = L(\vec{x}) + L(\vec{y})$ and $L(t\vec{x}) = tL(\vec{x})$, then

$$L(t\vec{x}+\vec{y}) = L(t\vec{x}) + L(\vec{y}) = tL(\vec{x}) + L(\vec{y})$$

On the other hand, if $L(t\vec{x}) + L(\vec{y}) = tL(\vec{x}) + L(\vec{y})$, then taking $t = 1$ gives $L(\vec{x}+\vec{y}) = L(\vec{x}) + L(\vec{y})$. We also have

$$L(t\vec{x}) = L(t\vec{x}+\vec{0}) = tL(\vec{x}) + L(\vec{0}) = tL(\vec{x}) + \vec{0} = tL(\vec{x})$$

D2 We have

$$\begin{aligned}(L+M)(\vec{x}+\vec{y}) &= L(\vec{x}+\vec{y}) + M(\vec{x}+\vec{y}) = L(\vec{x}) + L(\vec{y}) + M(\vec{x}) + M(\vec{y}) \\ &= L(\vec{x}) + M(\vec{x}) + L(\vec{y}) + M(\vec{y}) = (L+M)(\vec{x}) + (L+M)(\vec{y})\end{aligned}$$

So, $(L+M)$ preserves addition.
Similarly

$$(L+M)(t\vec{x}) = L(t\vec{x}) + M(t\vec{x}) = tL(\vec{x}) + tM(\vec{x}) = t[L(\vec{x}) + M(\vec{x})] = t(L+M)(\vec{x})$$

Thus, it also preserves scalar multiplication and hence is a linear mapping.
For all $\vec{x} \in \mathbb{R}^n$ we have

$$[L+M]\vec{x} = (L+M)(\vec{x}) = L(\vec{x}) + M(\vec{x}) = [L]\vec{x} + [M]\vec{x} = ([L] + [M])\vec{x}$$

Thus, $[L+M] = [L] + [M]$.

D3 For any $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $t \in \mathbb{R}$ we have

$$\mathrm{DOT}_{\vec{v}}(t\vec{x}+\vec{y}) = \vec{v}\cdot(t\vec{x}+\vec{y}) = \vec{v}\cdot(t\vec{x}) + \vec{v}\cdot\vec{y} = t(\vec{v}\cdot\vec{x}) + \vec{v}\cdot\vec{y} = t\,\mathrm{DOT}_{\vec{v}}\,\vec{x} + \mathrm{DOT}_{\vec{v}}\,\vec{y}$$

so $\mathrm{DOT}_{\vec{v}}$ is linear.
Since $\mathrm{DOT}_{\vec{v}}$ is a real number, its codomain is $\mathbb{R}$.
For $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ We have

$$[\mathrm{DOT}_{\vec{v}}] = \begin{bmatrix} \mathrm{DOT}_{\vec{v}}\,\vec{e}_1 & \cdots & \mathrm{DOT}_{\vec{v}}\,\vec{e}_n \end{bmatrix} = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} = \vec{v}^T$$

D4 Let $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$. Observe that for any $\vec{x} \in \mathbb{R}^n$ we have $\mathrm{proj}_{\vec{u}}(\vec{x}) = \dfrac{\vec{x}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u} = (\vec{x}\cdot\vec{u})\vec{u}$ since $\vec{u}$ is a unit vector. Thus, by definition of $[\mathrm{proj}_{\vec{u}}]$ we have

$$\begin{aligned}
[\mathrm{proj}_{\vec{u}}] &= \begin{bmatrix} \mathrm{proj}_{\vec{u}}(\vec{e}_1) & \cdots & \mathrm{proj}_{\vec{u}}(\vec{e}_n) \end{bmatrix} \\
&= \begin{bmatrix} (\vec{e}_1\cdot\vec{u})\vec{u} & \cdots & (\vec{e}_n\cdot\vec{u})\vec{u}) \end{bmatrix} \\
&= \begin{bmatrix} u_1\vec{u} & u_2\vec{u} & \cdots & u_n\vec{u} \end{bmatrix} \\
&= \begin{bmatrix} u_1u_1 & u_1u_2 & \cdots & u_1u_n \\ u_2u_1 & u_2u_2 & \cdots & u_2u_n \\ \vdots & & \ddots & \vdots \\ u_nu_1 & u_nu_2 & \cdots & u_nu_n \end{bmatrix} \\
&= \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} = \vec{u}\vec{u}^T
\end{aligned}$$

D5 For any $\vec{x}, \vec{y} \in \mathbb{R}^3$ and $t \in \mathbb{R}$ we have

$$\mathrm{CROSS}_{\vec{v}}(t\vec{x}+\vec{y}) = \vec{v}\times(t\vec{x}+\vec{y}) = \vec{v}\times(t\vec{x}) + \vec{v}\times\vec{y} = t\,\mathrm{CROSS}_{\vec{v}}(\vec{x}) + \mathrm{CROSS}_{\vec{v}}(\vec{y})$$

Thus, $\mathrm{CROSS}_{\vec{v}}$ is linear.
The cross product $\vec{v}\times\vec{x}$ is only defined if $\vec{x}\in\mathbb{R}^3$, so the domain of the $\mathrm{CROSS}_{\vec{v}}$ is $\mathbb{R}^3$. The result of $\vec{v}\times\vec{x}$ is the vector in $\mathbb{R}^3$ orthogonal to both $\vec{v}$ and $\vec{x}$ so the codomain is $\mathbb{R}^3$.
We have

$$\begin{aligned}
\mathrm{CROSS}_{\vec{v}}(1,0,0) &= \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ v_3 \\ -v_2 \end{bmatrix} \\
\mathrm{CROSS}_{\vec{v}}(0,1,0) &= \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -v_3 \\ 0 \\ v_1 \end{bmatrix} \\
\mathrm{CROSS}_{\vec{v}}(0,0,1) &= \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} v_2 \\ -v_1 \\ 0 \end{bmatrix}
\end{aligned}$$

Hence $[\text{CROSS}_{\vec{v}}] = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}$.

3.3 Geometrical Transformations

Practice Problems

A1 The matrix of rotation of $\mathbb{R}^2$ through angle θ is $[R_\theta] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

(a) $[R_{\pi/2}] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (b) $[R_\pi] = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(c) $[R_{-\pi/4}] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ (d) $[R_{2\pi/5}] = \begin{bmatrix} 0.309 & -0.951 \\ 0.951 & 0.309 \end{bmatrix}$

A2 (a) The matrix of stretch by a factor of 5 in the x_2 direction is $[S] = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$.

(b) The composition of S followed by a rotation through angle θ is given by $R_\theta \circ S$. The matrix of $R_\theta \circ S$ is

$$[R_\theta \circ S] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} \cos\theta & -5\sin\theta \\ \sin\theta & 5\cos\theta \end{bmatrix}$$

(c) The composition of S following a rotation through angle θ is given by $S \circ R_\theta$. The matrix of $S \circ R_\theta$ is

$$[S \circ R_\theta] = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ 5\sin\theta & 5\cos\theta \end{bmatrix}$$

A3 (a) A normal vector to the line $x_1 + 3x_2 = 0$ is $\vec{n} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. To find the standard matrix of R, we need to reflect the standard basis vectors over the line. We get

$$\text{refl}_{\vec{n}}\,\vec{e}_1 = \vec{e}_1 - 2\frac{\vec{e}_1 \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2\frac{1}{10}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}$$
$$\text{refl}_{\vec{n}}\,\vec{e}_2 = \vec{e}_2 - 2\frac{\vec{e}_2 \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - 2\frac{3}{10}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3/5 \\ -4/5 \end{bmatrix}$$

Thus, $[R] = \begin{bmatrix} 4/5 & -3/5 \\ -3/5 & -4/5 \end{bmatrix}$.

(b) A normal vector to the line $2x_1 = x_2$ is $\vec{n} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. To find the standard matrix of S, we need to reflect the standard basis vectors over the line. We get

$$\text{refl}_{\vec{n}}\,\vec{e}_1 = \vec{e}_1 - 2\frac{\vec{e}_1 \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2\frac{2}{5}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$$
$$\text{refl}_{\vec{n}}\,\vec{e}_2 = \vec{e}_2 - 2\frac{\vec{e}_2 \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - 2\frac{-1}{5}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$$

Thus, $[S] = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$.

A4 (a) A normal vector to the plane $x_1 + x_2 + x_3 = 0$ is $\vec{n} = \begin{bmatrix}1\\1\\1\end{bmatrix}$. To find the standard matrix of the reflection, we need to reflect the standard basis vectors for $\mathbb{R}^3$ over the plane.

$$\text{refl}_{\vec{n}}\,\vec{e}_1 = \vec{e}_1 - 2\frac{\vec{e}_1\cdot\vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix}1\\0\\0\end{bmatrix} - 2\frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1/3\\-2/3\\-2/3\end{bmatrix}$$

$$\text{refl}_{\vec{n}}\,\vec{e}_2 = \vec{e}_2 - 2\frac{\vec{e}_2\cdot\vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix}0\\1\\0\end{bmatrix} - 2\frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}-2/3\\1/3\\-2/3\end{bmatrix}$$

$$\text{refl}_{\vec{n}}\,\vec{e}_2 = \vec{e}_2 - 2\frac{\vec{e}_2\cdot\vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix}0\\0\\1\end{bmatrix} - 2\frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}-2/3\\-2/3\\1/3\end{bmatrix}$$

Thus, $[\text{refl}_{\vec{n}}] = \frac{1}{3}\begin{bmatrix}1 & -2 & -2\\-2 & 1 & -2\\-2 & -2 & 1\end{bmatrix}$.

(b) A normal vector to the plane $2x_1 - 2x_2 - x_3 = 0$ is $\vec{n} = \begin{bmatrix}2\\-2\\-1\end{bmatrix}$. To find the standard matrix of the reflection, we need to reflect the standard basis vectors for $\mathbb{R}^3$ over the plane.

$$\text{refl}_{\vec{n}}\,\vec{e}_1 = \vec{e}_1 - 2\frac{\vec{e}_1\cdot\vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix}1\\0\\0\end{bmatrix} - 2\frac{2}{9}\begin{bmatrix}2\\-2\\-1\end{bmatrix} = \begin{bmatrix}1/9\\8/9\\4/9\end{bmatrix}$$

$$\text{refl}_{\vec{n}}\,\vec{e}_2 = \vec{e}_2 - 2\frac{\vec{e}_2\cdot\vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix}0\\1\\0\end{bmatrix} - 2\frac{-2}{9}\begin{bmatrix}2\\-2\\-1\end{bmatrix} = \begin{bmatrix}8/9\\1/9\\-4/9\end{bmatrix}$$

$$\text{refl}_{\vec{n}}\,\vec{e}_3 = \vec{e}_3 - 2\frac{\vec{e}_3\cdot\vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix}0\\0\\1\end{bmatrix} - 2\frac{-1}{9}\begin{bmatrix}2\\-2\\-1\end{bmatrix} = \begin{bmatrix}4/9\\-4/9\\7/9\end{bmatrix}$$

Thus, $[\text{refl}_{\vec{n}}] = \frac{1}{9}\begin{bmatrix}1 & 8 & 4\\8 & 1 & -4\\4 & -4 & 7\end{bmatrix}$

A5 (a) Both the domain and codomain of D are $\mathbb{R}^3$, so the standard matrix $[D]$ has three columns and three rows. A dilation by factor 5 stretches all vectors in $\mathbb{R}^3$ by a factor of 5. Thus,

$$[D] = \begin{bmatrix}5 & 0 & 0\\0 & 5 & 0\\0 & 0 & 5\end{bmatrix}$$

The domain of inj is $\mathbb{R}^3$ and the codomain of inj is $\mathbb{R}^4$, so the standard matrix [inj] has three columns and four rows. We have

$$\mathrm{inj}(1,0,0) = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \qquad \mathrm{inj}(0,1,0) = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \qquad \mathrm{inj}(0,0,1) = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

so $[\mathrm{inj}] = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\\0&0&1\end{bmatrix}$. Since the domain of inj is $\mathbb{R}^3$ and the codomain of D is $\mathbb{R}^3$ we have that $\mathrm{inj}\circ D$ is defined, and, by Theorem 3.2.5, we get

$$[\mathrm{inj}\circ D] = [\mathrm{inj}][D] = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix} = \begin{bmatrix}5&0&0\\0&5&0\\0&0&0\\0&0&5\end{bmatrix}$$

(b) We have

$$P(1,0,0) = \begin{bmatrix}0\\0\end{bmatrix} \qquad P(0,1,0) = \begin{bmatrix}1\\0\end{bmatrix} \qquad P(0,0,1) = \begin{bmatrix}0\\1\end{bmatrix}$$

Thus, $[P] = \begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}$. We also have

$$S(1,0,0) = \begin{bmatrix}1\\0\\2\end{bmatrix} \qquad S(0,1,0) = \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad S(0,0,1) = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Hence, $[S] = \begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}$. The domain of P and the codomain of S are both $\mathbb{R}^3$, so $P\circ S$ is defined. By Theorem 3.2.5 we get

$$[P\circ S] = \begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix} = \begin{bmatrix}0&1&0\\2&0&1\end{bmatrix}$$

(c) The matrix of a shear T in $\mathbb{R}^2$ in the direction of x_1 by amount s is given by $[T] = \begin{bmatrix}1&s\\0&1\end{bmatrix}$. Assume that $T\circ P = P\circ S$. Then we have

$$[T][P] = [P][S]$$
$$\begin{bmatrix}1&s\\0&1\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}$$
$$\begin{bmatrix}0&1&s\\0&0&1\end{bmatrix} = \begin{bmatrix}0&1&0\\2&0&1\end{bmatrix}$$

But, these matrices are not equal and so we have a contradiction. Thus our assumption that $T\circ P = P\circ S$ must be false, so there is no shear T such that $T\circ P = P\circ S$.

(d) We have

$$Q(1,0,0) = \begin{bmatrix}1\\0\end{bmatrix} \qquad Q(0,1,0) = \begin{bmatrix}0\\1\end{bmatrix} \qquad Q(0,0,1) = \begin{bmatrix}0\\0\end{bmatrix}$$

Thus, $[Q] = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\end{bmatrix}$. Since the domain of Q and the codomain of S are both $\mathbb{R}^3$ we get that $Q \circ S$ is defined and

$$[Q \circ S] = [Q][S] = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\2 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\end{bmatrix}$$

Homework Problems

B1 (a) $\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$ (b) $\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$ (c) $\begin{bmatrix}-\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\end{bmatrix}$ (d) $\begin{bmatrix}0.809 & 0.588\\-0.588 & 0.809\end{bmatrix}$

B2 (a) $[S] = \begin{bmatrix}1 & 0\\0 & 0.6\end{bmatrix}$ (b) $[R_\theta \circ S] = \begin{bmatrix}\cos\theta & -0.6\sin\theta\\ \sin\theta & 0.6\cos\theta\end{bmatrix}$ (c) $[S \circ R_{\pi/4}] = \begin{bmatrix}\sqrt{2}/2 & -\sqrt{2}/2\\ 3\sqrt{2}/10 & 3\sqrt{2}/10\end{bmatrix}$

B3 (a) $[R] = \begin{bmatrix}12/13 & 5/13\\5/13 & -12/13\end{bmatrix}$ (b) $[S] = \begin{bmatrix}7/25 & -24/25\\-24/25 & -7/25\end{bmatrix}$

B4 (a) $\frac{1}{11}\begin{bmatrix}9 & 6 & 2\\6 & -7 & -6\\2 & -6 & 9\end{bmatrix}$ (b) $\frac{1}{3}\begin{bmatrix}-1 & -2 & 2\\-2 & 2 & 1\\2 & 1 & 2\end{bmatrix}$

B5 (a) $[\mathrm{inj} \circ C] = \begin{bmatrix}0 & 0 & 0\\1/3 & 0 & 0\\0 & 0 & 0\\0 & 1/3 & 0\\0 & 0 & 1/3\end{bmatrix}$

(b) $[C \circ S] = \begin{bmatrix}1/3 & 0 & 0\\0 & 1/3 & -2/3\\0 & 0 & 1/3\end{bmatrix}$, $[S \circ C] = \begin{bmatrix}1/3 & 0 & 0\\0 & 1/3 & -2/3\\0 & 0 & 1/3\end{bmatrix}$.

(c) $[T \circ S] = \begin{bmatrix}1 & 3 & -6\\0 & 1 & -2\\0 & 0 & 1\end{bmatrix}$, $[S \circ T] = \begin{bmatrix}1 & 3 & 0\\0 & 1 & -2\\0 & 0 & 1\end{bmatrix}$

Conceptual Problems

D1 Use standard trigonometric identities we get

$$\begin{aligned}[R_\alpha \circ R_\theta] = [R_\alpha][R_\theta] &= \begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\\ &= \begin{bmatrix}\cos\alpha\cos\theta - \sin\alpha\sin\theta & -\cos\alpha\sin\theta - \sin\alpha\cos\theta\\ \sin\alpha\cos\theta + \cos\alpha\sin\theta & -\sin\alpha\sin\theta + \cos\alpha\cos\theta\end{bmatrix}\\ &= \begin{bmatrix}\cos(\alpha+\theta) & -\sin(\alpha+\theta)\\ \sin(\alpha+\theta) & \cos(\alpha+\theta)\end{bmatrix} = [R_{\alpha+\theta}]\end{aligned}$$

D2 We have

$$[R \circ S] = [R][S] = \begin{bmatrix} 4/5 & -3/5 \\ -3/5 & -4/5 \end{bmatrix} \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix} = \begin{bmatrix} -24/25 & 7/25 \\ -7/25 & -24/25 \end{bmatrix}.$$

This is the matrix of a rotation by angle θ where $\cos\theta = -\frac{24}{25}$ and $\sin\theta = -\frac{7}{25}$ hence θ is an angle of about 3.425 radians.

D3 Let R_{23} denote the reflection in the x_2x_3-plane. Then $R_{23}(\vec{e}_1) = -\vec{e}_1$, $R_{23}(\vec{e}_2) = \vec{e}_2$, and $R_{23}(\vec{e}_3) = \vec{e}_3$, so

$$[R_{23}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly, if R_{12} denotes the reflection in the x_1x_2-plane,

$$[R_{12}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Then

$$[R_{12} \circ R_{23}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos\pi & 0 & \sin\pi \\ 0 & 1 & 0 \\ \sin\pi & 0 & \cos\pi \end{bmatrix}$$

This is the matrix of the rotation through angle π about the x_2-axis.

D4 (a) Let A be the matrix of a rotation through angle θ, so that $A = [R_\theta]$. Then we want

$$[R_{2\pi}] = I = A^3 = [R_\theta]^3 = [R_\theta][R_\theta][R_\theta] = [R_{3\theta}]$$

Hence, we want $3\theta = 2\pi$, or $\theta = 2\pi/3$. Thus, if $A = [R_{2\pi/3}]$, we get $A^3 = I$.

(b) Let A be the matrix of the rotation through angle $2\pi/5$. Then $A^5 = I$.

D5 The projection property states that $\text{proj}_{\vec{n}} \circ \text{proj}_{\vec{n}} = \text{proj}_{\vec{n}}$. Therefore

$$[\text{proj}_{\vec{n}}][\text{proj}_{\vec{n}}] = [\text{proj}_{\vec{n}}]$$

It follows that

$$\begin{aligned} [\text{refl}_{\vec{n}}][\text{refl}_{\vec{n}}] &= (I - 2[\text{proj}_{\vec{n}}])(I - 2[\text{proj}_{\vec{n}}]) \\ &= I^2 - 2I[\text{proj}_{\vec{n}}] - 2[\text{proj}_{\vec{n}}]I + 4[\text{proj}_{\vec{n}}][\text{proj}_{\vec{n}}] \\ &= I - 4[\text{proj}_{\vec{n}}] + 4[\text{proj}_{\vec{n}}] \\ &= I \end{aligned}$$

3.4 Special Subspaces for Systems and Mappings: Rank Theorem

Practice Problems

A1 By definition of the matrix of a linear mapping, $\vec{y}$ is in the range of L if and only if the system $[L]\vec{x} = \vec{y}$ is consistent.

(a) Row reducing the augmented matrix for $[L]\vec{x} = \vec{y}$ gives

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 1 & 2 & 1 & 6 \\ 2 & 5 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 11/4 \\ 0 & 1 & 0 & 7/4 \\ 0 & 0 & 1 & -1/4 \\ 0 & 0 & 0 & -13 \end{array}\right]$$

The system is inconsistent so $\begin{bmatrix} 3 \\ 1 \\ 6 \\ 1 \end{bmatrix}$ is not in the range of L.

(b) Row reducing the augmented matrix for $[L]\vec{x} = \vec{y}$ gives

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 3 & -5 \\ 1 & 2 & 1 & 1 \\ 2 & 5 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system is consistent with solution $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$. Thus, $L(1, 1, -2) = \begin{bmatrix} 3 \\ -5 \\ 1 \\ 5 \end{bmatrix}$.

A2 (a) Every vector in the range of L has the form

$$\begin{bmatrix} 2x_1 \\ -x_2 + 2x_3 \end{bmatrix} = 2x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-x_2 + 2x_3) \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Therefore, every vector in the range of L can be written as a linear combination of these two vectors. Let $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. Then, we have shown that $\text{Range}(L) = \text{Span}\, S$. Additionally, S is clearly linearly independent (the vectors are standard basis vectors). Hence, S is a basis for the range of L.

To find a basis for the nullspace of L we need to do a similar procedure. We first need to find the general form of a vector in the nullspace and then write it as a linear combination of vectors. To find the general form, we let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be any vector in the nullspace. Then, by definition of the nullspace and of L we get

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = L(\vec{x}) = \begin{bmatrix} 2x_1 \\ -x_2 + 2x_3 \end{bmatrix}$$

Hence, we must have $2x_1 = 0$ and $-x_2 + 2x_3 = 0$. So, $x_1 = 0$ and $x_2 = 2x_3$. Therefore, every vector in the nullspace of L has the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Therefore, the set $T = \left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$ spans the nullspace and is clearly linearly independent (it only contains one non-zero vector). Thus, it is a basis for Null(L).

(b) Every vector in the range of L has the form

$$\begin{bmatrix} x_4 \\ x_3 \\ 0 \\ x_2 \\ x_1 + x_2 - x_3 \end{bmatrix} = x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + x_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the set $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ spans the range of L.

To show it is linearly independent, we use the definition of linear independence. Consider

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + t_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + t_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} t_4 \\ t_3 \\ 0 \\ t_2 \\ t_1 + t_2 - t_3 \end{bmatrix}$$

This gives the system of equations $t_4 = 0$, $t_3 = 0$, $t_2 = 0$, and $t_1 + t_2 - t_3 = 0$, which clearly has the unique solution of $t_1 = t_2 = t_3 = t_4 = 0$. Hence, S is also linearly independent and so it is a basis for the range of L.

Let $\vec{x} \in$ Null(M). Then,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = M(\vec{x}) = \begin{bmatrix} x_4 \\ x_3 \\ 0 \\ x_2 \\ x_1 + x_2 - x_3 \end{bmatrix}$$

As we just showed above this has the unique solution $x_1 = x_2 = x_3 = x_4 = 0$, so the only vector in the nullspace of M is $\vec{0}$. Thus, a basis for Null(M) is the empty set since Null(M) = $\{\vec{0}\}$.

A3 Since the range of L consists of all vectors that are multiples of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, the columns of the matrix must be multiples of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Since the nullspace consists of all vectors that are multiples of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, the rows of the matrix must satisfy the first entry plus the second entry equals 0 (in particular, the rows must be orthogonal to $\begin{bmatrix}1\\1\end{bmatrix}$). So the second column of the matrix must be -1 times the first column. Hence, the matrix of L is any multiple of $\begin{bmatrix}1 & -1\\2 & -2\\3 & -3\end{bmatrix}$.

A4 Since the range of L consists of all vectors that are multiples of $\begin{bmatrix}1\\1\\1\end{bmatrix}$, the columns of the matrix must be multiples of $\begin{bmatrix}1\\1\\1\end{bmatrix}$. Since the nullspace consists of all vectors that are multiples of $\begin{bmatrix}1\\-2\end{bmatrix}$, the rows of the matrix must satisfy the first entry plus -2 times the second entry equals 0 (in particular, the rows must be orthogonal to the $\begin{bmatrix}1\\-2\end{bmatrix}$). So the first column of the matrix must be 2 times the second column. Hence, the matrix of L is any multiple of $\begin{bmatrix}2 & 1\\2 & 1\\2 & 1\end{bmatrix}$.

A5 (a) Since A has $n = 4$ columns, the number of variables in the system $A\vec{x} = \vec{0}$ is 4. The rank of the matrix equals the number of leadings ones in the RREF of A. Although the matrix is not in RREF, it is easy to see that performing the row operation required to bring it into RREF will not change the number of leadings ones to the rank of A is 2. By the Rank Theorem, the dimension of the solution space is $n - \operatorname{rank}(A) = 4 - 2 = 2$.

(b) Since A has $n = 5$ columns, the number of variables in the system $A\vec{x} = \vec{0}$ is 5. The rank of A is 3. So, by the Rank Theorem, the dimension of the solution space is $n - \operatorname{rank}(A) = 5 - 3 = 2$.

(c) Since A has $n = 5$ columns, the number of variables in the system $A\vec{x} = \vec{0}$ is 5. The rank of A is 2. So, by the Rank Theorem, the dimension of the solution space is $n - \operatorname{rank}(A) = 5 - 2 = 3$.

(d) Since A has $n = 6$ columns, the number of variables in the system $A\vec{x} = \vec{0}$ is 6. The rank of A is 3. So, by the Rank Theorem, the dimension of the solution space is $n - \operatorname{rank}(A) = 6 - 3 = 3$.

A6 To find a basis for the rowspace, columnspace, and nullspace of a matrix A, we start by row reducing A to its RREF R. Then, the non-zero rows of R form a basis for the rowspace of A, and the columns in A which correspond to the leading ones in R form a basis for the columnspace of A. To find a basis for the nullspace of A we find the solution space to $A\vec{x} = \vec{0}$. Of course, to do this we row reduce the coefficient matrix A to R. In particular, we solve $R\vec{x} = \vec{0}$.

(a) Row reducing the matrix to RREF gives

$$\begin{bmatrix}1 & 2 & 8\\1 & 1 & 5\\1 & 0 & -2\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Hence, a basis for the rowspace is $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$. Since every column of the RREF has a leading one, a basis for the columnspace is $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}8\\5\\-2\end{bmatrix}\right\}$. Finally, it is clear that the only solution to $R\vec{x} = \vec{0}$ is $\vec{0}$. Thus, the nullspace of the matrix only contains the zero vector. So, a basis for the nullspace is the empty set.

We have $\operatorname{rank}(A) + \operatorname{nullity}(A) = 3 + 0 = 3$, the number of columns of A as predicted by the Rank Theorem.

(b) Row reducing the matrix to RREF gives

$$\begin{bmatrix} 1 & 1 & -3 & 1 \\ 2 & 3 & -8 & 4 \\ 0 & 1 & -2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence, a basis for the rowspace is $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. The first, second, and fourth columns of the RREF form contain leading ones, so a basis for the columnspace is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \right\}$. Writing $R\vec{x} = \vec{0}$ in equation form gives

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 - 2x_3 &= 0 \\ x_4 &= 0 \end{aligned}$$

Thus, $x_3 = t \in \mathbb{R}$ is a free variable, and we get $x_1 = t$, $x_2 = 2t$, and $x_4 = 0$. Hence, the solution space is

$$\vec{x} = \begin{bmatrix} t \\ 2t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

Hence, a basis for the nullspace is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Then, $\operatorname{rank}(A) + \operatorname{nullity}(A) = 3 + 1 = 4$, the number of columns of A, as predicted by the Rank Theorem.

(c) Row reducing the matrix to RREF gives

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & 1 & 7 & 1 \\ 2 & 4 & 0 & 6 & 1 \\ 3 & 6 & 1 & 13 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for the rowspace is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. The first, third, and fifth columns of the RREF contain leading ones, so a basis for the columnspace is $\left\{\begin{bmatrix}1\\1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\2\end{bmatrix}\right\}$. Writing $R\vec{x} = \vec{0}$ in equation form gives

$$\begin{aligned} x_1 + 2x_2 + 3x_4 &= 0 \\ x_3 + 4x_4 &= 0 \\ x_5 &= 0 \end{aligned}$$

Thus, $x_2 = s \in \mathbb{R}$ and $x_4 = t \in \mathbb{R}$ are free variables, and we get $x_1 = -2s - 3t$, $x_3 = -4t$, and $x_5 = 0$. Hence, the solution space is

$$\vec{x} = \begin{bmatrix}-2s-3t\\s\\-4t\\t\\0\end{bmatrix} = s\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix} + t\begin{bmatrix}-3\\0\\-4\\1\\0\end{bmatrix}$$

Hence, a basis for the nullspace is $\left\{\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\-4\\1\\0\end{bmatrix}\right\}$.

Then, $\operatorname{rank}(A) + \operatorname{nullity}(A) = 3 + 2 = 5$, the number of columns of A as predicted by the Rank Theorem.

A7 (a) $\operatorname{proj}_{\vec{v}}$ maps a vector in $\mathbb{R}^3$ to a vector in the direction of $\begin{bmatrix}1\\-2\\2\end{bmatrix}$. Hence, its range consists of all scalar multiples of $\begin{bmatrix}1\\-2\\2\end{bmatrix}$. Thus, a basis for the range is $\left\{\begin{bmatrix}1\\-2\\2\end{bmatrix}\right\}$. The mapping maps all vectors orthogonal to $\begin{bmatrix}1\\-2\\2\end{bmatrix}$ to the zero vector. Hence, its nullspace is the plane through the origin with normal vector $\vec{n} = \begin{bmatrix}1\\-2\\2\end{bmatrix}$. Any two linearly independent vectors orthogonal to $\vec{n}$ form a basis for this plane. So, one basis for the nullspace is $\left\{\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\-1\end{bmatrix}\right\}$.

(b) $\operatorname{perp}_{\vec{v}}$ maps a vector in $\mathbb{R}^3$ to a vector in the plane orthogonal to $\begin{bmatrix}3\\1\\2\end{bmatrix}$. Hence, its range is the plane through the origin with normal $\vec{n} = \begin{bmatrix}3\\1\\2\end{bmatrix}$. Any two linearly independent vectors orthogonal to $\vec{n}$ form a basis for this plane. So, one basis for the range is $\left\{\begin{bmatrix}1\\-3\\0\end{bmatrix}, \begin{bmatrix}0\\-2\\1\end{bmatrix}\right\}$. The mapping maps all scalar multiples of $\begin{bmatrix}3\\1\\2\end{bmatrix}$ to the zero vector, so a basis for the nullspace is $\left\{\begin{bmatrix}3\\1\\2\end{bmatrix}\right\}$.

(c) $\text{refl}_{\vec{v}}$ maps a vector corresponding to a point in $\mathbb{R}^3$ to its mirror image on the opposite side of the plane through the origin with normal vector $\vec{n} = \begin{bmatrix}0\\1\\0\end{bmatrix}$. Since every point in $\mathbb{R}^3$ is the image under reflection of some point, the range of $\text{refl}_{\vec{v}}$ is $\mathbb{R}^3$. Hence, any basis for $\mathbb{R}^3$ is also a basis for the range of the reflection. For example, we could take the standard basis. Since the only point that is reflected to the origin is the origin itself, the nullspace of the mapping is $\{\vec{0}\}$. Hence, a basis for the nullspace is the empty set.

A8 (a) Each vector in the rowspace of A has five components. Hence, $n = 5$; the rowspace is a subspace of $\mathbb{R}^5$.

(b) $\left\{\begin{bmatrix}1\\0\\2\\0\\3\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\1\end{bmatrix}\right\}$ is a basis for the rowspace of A. Since row reducing a matrix preserves linear combinations, the rowspace of R equals the rowspace of A. Therefore, since this set is a basis for the rowspace of R it is a basis for the rowspace of A.

(c) Each vector in the columnspace of A has four components. Hence, $m = 4$; the columnspace is a subspace of $\mathbb{R}^4$.

(d) The first, second, and fourth columns of R form a linearly independent set, and hence a basis for the columnspace of R. Therefore, the first, second, and fourth columns of A also form a linearly independent set, and hence a basis for the columnspace of A. A basis for the columnspace of A is therefore $\left\{\begin{bmatrix}1\\2\\1\\1\end{bmatrix}, \begin{bmatrix}1\\3\\1\\2\end{bmatrix}, \begin{bmatrix}1\\2\\3\\-1\end{bmatrix}\right\}$.

(e) Let $x_3 = s \in \mathbb{R}$ and $x_5 = t \in \mathbb{R}$, then the general solution of $A\vec{x} = \vec{0}$ is

$$\vec{x} = \begin{bmatrix}-2s-3t\\s-t\\s\\-t\\t\end{bmatrix} = s\begin{bmatrix}-2\\1\\1\\0\\0\end{bmatrix} + t\begin{bmatrix}-3\\-1\\0\\-1\\1\end{bmatrix}$$

So a spanning set for the solution space is $\left\{\begin{bmatrix}-2\\1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\-1\\0\\-1\\1\end{bmatrix}\right\}$.

(f) The set $\left\{\begin{bmatrix}-2\\1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\-1\\0\\-1\\1\end{bmatrix}\right\}$ is also linearly independent since neither vector is a scalar multiple of the other, so it a basis.

(g) The rank of A is 3 and a basis for the solution space has two vectors in it, so the dimension of the solution space is 2. We have $3 + 2 = 5$ is the number of variables in the system.

Homework Problems

B1 (a) $L(2,1,1) = \begin{bmatrix} 5 \\ 5 \\ 4 \\ 4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ -3 \\ 2 \\ 1 \end{bmatrix}$ is not in the range of L.

B2 (a) A basis for Range(L) is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right\}$. A basis for Null($L$) is the empty set since Null(L) = $\{\vec{0}\}$.

(b) The range of M is $\mathbb{R}^3$, so take any basis for $\mathbb{R}^3$. A basis for Null(M) is $\left\{ \begin{bmatrix} -1 \\ -2 \\ -1 \\ 1 \end{bmatrix} \right\}$.

B3 The matrix of L is any multiple of $\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$.

B4 The matrix of L is any multiple of $\begin{bmatrix} 0 & 0 \\ 3 & 3 \\ 5 & 5 \end{bmatrix}$.

B5 (a) Number of variables is 5. Rank of A is 3. Dimension of the solution space is 2.

(b) Number of variables is 4. Rank of A is 3. Dimension of the solution space is 1.

(c) Number of variables is 5. Rank of A is 4. Dimension of the solution space is 1.

(d) Number of variables is 6. Rank of A is 3. Dimension of the solution space is 3.

B6 (a) A basis for the rowspace is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. A basis for the columnspace is $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$. A basis for the nullspace is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Then, rank(A) + nullity(A) = 2 + 1 = 3, the number of columns of A as predicted by the Rank Theorem.

(b) A basis for the rowspace is $\left\{\begin{bmatrix}1\\0\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\0\\-2\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right\}$. A basis for the columnspace is $\left\{\begin{bmatrix}0\\1\\2\end{bmatrix}, \begin{bmatrix}1\\2\\4\end{bmatrix}, \begin{bmatrix}0\\1\\3\end{bmatrix}\right\}$. A basis for the nullspace is $\left\{\begin{bmatrix}-2\\2\\-1\\1\end{bmatrix}\right\}$.

Then, rank(A) + nullity(A) = 3 + 1 = 4, the number of columns of A as predicted by the Rank Theorem.

(c) A basis for the rowspace is $\left\{\begin{bmatrix}1\\0\\0\\1/2\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\-2\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\5/2\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right\}$. A basis for the columnspace is $\left\{\begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\1\\3\end{bmatrix}, \begin{bmatrix}1\\4\\3\\8\end{bmatrix}\right\}$.

A basis for the nullspace is $\left\{\begin{bmatrix}-1/2\\2\\-5/2\\1\\0\end{bmatrix}\right\}$.

Then, rank(A) + nullity(A) = 4 + 1 = 5, the number of columns of A as predicted by the Rank Theorem.

B7 (a) A basis for the nullspace is $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-2\\1\end{bmatrix}\right\}$ (or any pair of linearly independent vectors orthogonal to $\begin{bmatrix}1\\0\\-1\end{bmatrix}$); a basis for the range is $\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$.

(b) For the nullspace a basis is $\left\{\begin{bmatrix}-1\\1\\3\end{bmatrix}\right\}$; for the range $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\3\\-1\end{bmatrix}\right\}$.

(c) A basis for the nullspace is the empty set; the range is $\mathbb{R}^3$, so take any basis for $\mathbb{R}^3$.

B8 (a) $n = 7$

(b) $\left\{\begin{bmatrix}1\\2\\0\\0\\3\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\-1\\0\\-2\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\-2\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\1\\4\end{bmatrix}\right\}$

(c) $m = 5$

(d) $\left\{\begin{bmatrix}1\\1\\1\\1\\3\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\\2\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\\2\end{bmatrix}\right\}$

(e) $\vec{x} = r\begin{bmatrix}-2\\1\\0\\0\\0\\0\\0\end{bmatrix} + s\begin{bmatrix}-3\\0\\1\\2\\1\\0\\0\end{bmatrix} + t\begin{bmatrix}1\\0\\2\\-1\\0\\-4\\1\end{bmatrix}$, $r, s, t \in \mathbb{R}$. Hence, a spanning set is $\left\{\begin{bmatrix}-2\\1\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\2\\-1\\0\\-4\\1\end{bmatrix}\right\}$.

(f) $\left\{\begin{bmatrix}-2\\1\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\1\\2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\2\\-1\\0\\-4\\1\end{bmatrix}\right\}$ is also linearly independent, so it a basis.

(g) The rank of A is 4, so the dimension of the solution space is 3.

Conceptual Problems

D1 Since Range(L) = Col($[L]$) and Null(L) = Null($[L]$), we have that by the Rank Theorem

$$\begin{aligned} n &= \operatorname{rank}([L]) + \operatorname{nullity}([L]) = \dim(\operatorname{Col}([L])) + \dim(\operatorname{Null}([L])) \\ &= \dim(\operatorname{Range}(L)) + \dim(\operatorname{Null}(L)) \end{aligned}$$

D2 (a) First observe that if $\vec{x} \in \mathbb{R}^n$, then $(M \circ L)(\vec{x}) = M(L(\vec{x})) = M(\vec{y})$ for $\vec{y} = L(\vec{x}) = \mathbb{R}^m$. Hence every vector in the range of $M \circ L$ is in the range of M. Thus, the range of $M \circ L$ is a subset of the range of M. Since $(M \circ L)(\vec{0}) = \vec{0}$, we have that $\vec{0} \in$ Range($M \circ L$). Let $\vec{y}_1, \vec{y}_2 \in$ Range($M \circ L$). Then there exists $\vec{x}_1, \vec{x}_2 \in \mathbb{R}^n$ such that $(M \circ L)(\vec{x}_1) = \vec{y}_1$ and $(M \circ L)(\vec{x}_2) = \vec{y}_2$. Hence

$$\vec{y}_1 + \vec{y}_2 = (M \circ L)(\vec{x}_1) + (M \circ L)(\vec{x}_2) = (M \circ L)(\vec{x}_1 + \vec{x}_2)$$

with $\vec{x}_1 + \vec{x}_2 \in \mathbb{R}^n$. Thus, $\vec{y}_1 + \vec{y}_2 \in$ Range($M \circ L$). Similarly, for any $t \in \mathbb{R}$ we have

$$t\vec{y}_1 = t(M \circ L)(\vec{x}_1) = (M \circ L)(t\vec{x}_1)$$

with $t\vec{x}_1 \in \mathbb{R}^n$, so $t\vec{y}_1 \in$ Range($M \circ L$). Thus, $M \circ L$ is a subspace of Range(M).

(b) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ and $M : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $L(x_1, x_2) = (x_1, 0)$ and $M(x_1, x_2) = (x_1, x_2)$. Then, clearly Range(M) = $\mathbb{R}^2$, but Range($M \circ L$) = Span$\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$.

(c) Let $\vec{x} \in$ Null(L). Then, $(M \circ L)(\vec{x}) = M(L(\vec{x})) = M(\vec{0}) = \vec{0}$, so $\vec{x} \in$ Null($M \circ L$). Thus, the nullspace of L is a subset of the nullspace of $M \circ L$. Since $L(\vec{0}) = \vec{0}$, we have $\vec{0} \in$ Null(L). Let $\vec{x}, \vec{y} \in$ Null(L). Then

$$L(\vec{x} + \vec{y}) = L(\vec{x}) + L(\vec{y}) = \vec{0} + \vec{0} = \vec{0}$$

so $\vec{x} + \vec{y} \in$ Null(L). Similarly, for any $t \in \mathbb{R}$ we have

$$L(t\vec{x}) = tL(\vec{x}) = t\vec{0} = \vec{0}$$

so $t\vec{x} \in$ Null(L). Therefore Null(L) is a subspace of Null($M \circ L$).

D3 (a) Since rank$(A) = 4$, we have $\dim(\text{Col}(A)) = 4$. Then, by the Rank Theorem, nullity$(A) = 7 - \text{rank}(A) = 7 - 4 = 3$.

(b) The nullspace is as large as possible when the nullspace is the entire domain. Therefore, the largest possible dimension for the nullspace if 4. The largest possible rank of a matrix is the minimum of the number of rows and columns. Thus, the largest possible rank is 4.

(c) Since $\dim(\text{Row}(A)) = \text{rank}(A)$, we get by the Rank Theorem that the dimension of the rowspace is $5 - \text{nullity}(A) = 5 - 3 = 2$.

D4 Let $\vec{b}$ be in the columnspace of A. Then there exist an $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{b}$. But then

$$A\vec{b} = A(A\vec{x}) = A^2\vec{x} = O_{n,n}\vec{x} = \vec{0}$$

Hence, $\vec{b}$ is in the nullspace of A.

3.5 Inverse Matrices and Inverse Mappings

Practice Problems

A1 To determine if a matrix A is invertible, write the matrix $[A \mid I]$ and row reduce.

(a) $\left[\begin{array}{cc|cc} 3 & -4 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 5/23 & 4/23 \\ 0 & 1 & -2/23 & 3/23 \end{array}\right]$

Thus, the matrix is invertible, and its inverse is $\dfrac{1}{23}\begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}$.

(b) $\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$.

Thus, the matrix is invertible, and its inverse is $\begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$.

(c) $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 & 1 & 0 \\ 3 & 1 & 7 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & -1 & 1 \end{array}\right]$

Since the reduced row echelon form of C is not I, it follows that C is not invertible.

(d) $\left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$

Thus, the matrix is invertible, and its inverse is $\begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

(e) $\left[\begin{array}{cccc|cccc} 1 & 1 & 3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 7 & 1 & 0 & 0 & 1 & 0 \\ 0 & 6 & 3 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 6 & 10 & -5/2 & -7/2 \\ 0 & 1 & 0 & 0 & 1 & 2 & -1/2 & -1/2 \\ 0 & 0 & 1 & 0 & -2 & -3 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -3 & 0 & 1 \end{array}\right]$

Thus, the matrix is invertible, and its inverse is $\begin{bmatrix} 6 & 10 & -5/2 & -7/2 \\ 1 & 2 & -1/2 & -1/2 \\ -2 & -3 & 1 & 1 \\ 0 & -3 & 0 & 1 \end{bmatrix}$.

(f) $\left[\begin{array}{ccccc|ccccc} 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & -2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$

Thus, the matrix is invertible, and its inverse is $\begin{bmatrix} 1 & 0 & -1 & 1 & -2 \\ 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

A2 From question A1 (b) we have that $B^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$.

(a) The solution of $B\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is

$$\vec{x} = B^{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

(b) The solution of $B\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is

$$\vec{x} = B^{-1}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$$

(c) The solution of $B\vec{x} = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$ is

$$\vec{x} = B^{-1}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right) = B^{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + B^{-1}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

A3 (a) Row reducing $[A \mid I]$ and $[B \mid I]$ gives

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -3 & 2 \end{array}\right]$$
$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -5 & 2 \\ 0 & 1 & 3 & -1 \end{array}\right]$$

Hence, $A^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$.

(b) We have $(AB) = \begin{bmatrix} 5 & 9 \\ 9 & 16 \end{bmatrix}$. Row reducing $[AB \mid I]$ gives

$$\left[\begin{array}{cc|cc} 5 & 9 & 1 & 0 \\ 9 & 16 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -16 & 9 \\ 0 & 1 & 9 & -5 \end{array}\right]$$

Thus, $(AB)^{-1} = \begin{bmatrix} -16 & 9 \\ 9 & -5 \end{bmatrix}$. We also have

$$B^{-1}A^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -16 & 9 \\ 9 & -5 \end{bmatrix} = (AB)^{-1}$$

(c) Row reducing $[3A \mid I]$ gives

$$\left[\begin{array}{cc|cc} 6 & 3 & 1 & 0 \\ 9 & 6 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 2/3 & -1/3 \\ 0 & 1 & -1 & 2/3 \end{array}\right]$$

Hence, $(3A)^{-1} = \begin{bmatrix} 2/3 & -1/3 \\ -1 & 2/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \frac{1}{3}A^{-1}$.

(d) Row reducing $\left[A^T \mid I\right]$ gives

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

Hence, $(A^T)^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$. We have

$$A^T(A^T)^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

as required.

A4 (a) The mapping $R_{\pi/6}$ rotates each vector in the plane about the origin counterclockwise through the angle $\frac{\pi}{6}$. The inverse of $R_{\pi/6}$ maps each image vector back to its original coordinates, so the inverse is a counterclockwise rotation through angle $-\frac{\pi}{6}$. Hence,

$$[R_{\pi/6}]^{-1} = [R_{-\pi/6}] = \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$$

(b) The matrix represents a shear in the x_1 direction by an amount -3. This linear transformation moves each vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ parallel to the x_1-axis by an amount $-3x_2$ to the new position $\begin{bmatrix} x_1 - 3x_2 \\ x_2 \end{bmatrix}$. The inverse transformation maps each image vector from the new position by an amount $3x_2$ back to the original position. Thus the inverse is a shear in the direction x_1 by amount 3. The matrix of the inverse is $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$.

(c) The matrix represents a stretch in the x_1 direction by a factor of 5. Thus, the inverse of this transformation must "shrink" the x_1 coordinate back to its original value. Hence, the inverse is a stretch in the x_1 direction by a factor of $\frac{1}{5}$. The matrix of the inverse is $\begin{bmatrix} 1/5 & 0 \\ 0 & 1 \end{bmatrix}$.

(d) The matrix represents a reflection of a vector in $\mathbb{R}^3$ over the x_1x_3-plane. Thus, the inverse of this transformation must reflect the vector back over the x_1x_3-plane. Therefore, the transformation is its own inverse. Hence, the matrix of the inverse is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

A5 (a) We know that the matrix of a shear S by a factor of 2 in the x_2 direction is $[S] = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$. By either row reducing $[S \mid I]$ or using geometrical arguments as in the question A4, we find that $[S^{-1}] = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$.

(b) A reflection R in the line $x_1 - x_2 = 0$ maps a vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in the plane to its mirror image $\begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$ in the line $x_1 - x_2 = 0$. Thus, $R(\vec{e}_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $R(\vec{e}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, hence $[R] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The inverse of the mapping is also a reflection in the line $x_1 - x_2 = 0$, so $[R^{-1}] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = [R]$.

(c) Composition of mappings is represented by products of matrices, so

$$[(R \circ S)^{-1}] = ([R][S])^{-1} = [S]^{-1}[R]^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$
$$[(S \circ R)^{-1}] = ([S][R])^{-1} = [R]^{-1}[S]^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$$

A6 Let $\vec{v}, \vec{y} \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Then there exists $\vec{u}, \vec{x} \in \mathbb{R}^n$ such that $\vec{x} = M(\vec{y})$ and $\vec{u} = M(\vec{v})$. Then $L(\vec{x}) = \vec{y}$ and $L(\vec{u}) = \vec{v}$. Since L is linear $L(t\vec{x} + \vec{u}) = tL(\vec{x}) + L(\vec{u}) = t\vec{y} + \vec{v}$. It follows that

$$M(t\vec{y} + \vec{v}) = t\vec{x} + \vec{u} = tM(\vec{y}) + M(\vec{v})$$

so M is linear.

Homework Problems

B1 (a) $\begin{bmatrix} 7/15 & -4/15 \\ 2/15 & 1/15 \end{bmatrix}$

(b) It is not invertible.

(c) $\begin{bmatrix} 11/9 & -2/3 & 10/9 \\ -1/9 & 1/3 & -5/9 \\ -4/9 & 1/3 & -2/9 \end{bmatrix}$

(d) It is not invertible.

(e) $\begin{bmatrix} -2/3 & -1/3 & 8/3 \\ -1/3 & 1/3 & 1/3 \\ 2/3 & 1/3 & -5/3 \end{bmatrix}$

(f) $\begin{bmatrix} 10/3 & 8/3 & -1/3 & -5/3 \\ 1/3 & 2/3 & -1/3 & 1/3 \\ -1/3 & 1/3 & 1/3 & -1/3 \\ -1 & -1 & 0 & 1 \end{bmatrix}$

(g) $\begin{bmatrix} 5 & -2 & -10 & 11/3 \\ -3 & 1 & 6 & -2 \\ 1 & -1 & -1 & 1/3 \\ 1 & 0 & -2 & 2/3 \end{bmatrix}$

(h) $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}$

(i) $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$

(j) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & 1 \end{bmatrix}$

B2 (a) $A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ -1/2 & 3/2 & 1 \\ 1/2 & -1/2 & -1 \end{bmatrix}$

(b) (i) $\vec{x} = \begin{bmatrix} 2 \\ 5/2 \\ 1/2 \end{bmatrix}$, (ii) $\vec{x} = \begin{bmatrix} 6 \\ 15/2 \\ 3/2 \end{bmatrix}$, (iii) $\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$

B3 (a) $A^{-1} = \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}$

(b) $AB = \begin{bmatrix} 17 & 39 \\ 7 & 16 \end{bmatrix}$, $(AB)^{-1} = \begin{bmatrix} -16 & 39 \\ 7 & -17 \end{bmatrix}$

(c) $(5A)^{-1} = \begin{bmatrix} -2/5 & 1 \\ 1/5 & -2/5 \end{bmatrix}$

(d) $(A^T)^{-1} = \begin{bmatrix} -2 & 1 \\ 5 & -2 \end{bmatrix}$

B4 (a) $[R_{\pi/4}]^{-1} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$

(d) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

B5 (a) $[S] = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $[S^{-1}] = \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix}$

(b) $[R] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, $[R^{-1}] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

(c) $[(R \circ S)^{-1}] = \begin{bmatrix} 0 & -1 \\ -1/3 & 0 \end{bmatrix}$, $[(S \circ R)^{-1}] = \begin{bmatrix} 0 & -1/3 \\ -1 & 0 \end{bmatrix}$

B6 (a) $[R_{\pi/2}^{-1}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$, $[S^{-1}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $[(R_{\pi/2} \circ S)^{-1}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix}$

(b) $[R^{-1}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $[S^{-1}] = \begin{bmatrix} -0.4 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $[(R \circ S)^{-1}] = \begin{bmatrix} 1 & 0 & -0.4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Computer Problems

C1 (a) $\begin{bmatrix} 0.26315861440198551 & 0.49258011208722540 & 0.16149453057662802 & -0.23038787380832820 \\ 0.21479273169845791 & -0.15523431868049322 & -0.022592485249228511 & 0.034067093077257845 \\ 0.0082272363399542546 & -0.12187604630787456 & -0.12710459750906119 & 0.17567170823165517 \\ -0.10019869774999662 & -0.45503652011314232 & 0.067154043885201963 & 0.45015271726511108 \end{bmatrix}$

(b) $\begin{bmatrix} 1.2300000000000002 & 3.1099999999999994 & 1.0100000000000002 & -2.2204460492503130 \times 10^{-16} \\ 2.0099999999999993 & -2.5600000000000000 & 3.0299999999999993 & 0.040000000000000035 \\ 1.1100000000000007 & 0.029999999999999360 & -5.1099999999999994 & 2.5600000000000009 \\ 2.1399999999999996 & -1.8999999999999996 & 4.0499999999999989 & 1.8799999999999996 \end{bmatrix}$

It is not quite equal to A due to computer round-off error.

Conceptual Problems

D1 Using properties of transpose and inverse we get

$$((AB)^T)^{-1} = (B^T A^T)^{-1} = (A^T)^{-1}(B^T)^{-1} = (A^{-1})^T (B^{-1})^T$$

D2 (a) $I = A^3 = AA^2$, so $A^{-1} = A^2$.

(b) $B(B^4 + B^2 + I) = I$, so $B^{-1} = B^4 + B^2 + I$.

D3 We must not assume that A^{-1} and B^{-1} exist, so we cannot use $(AB)^{-1} = B^{-1}A^{-1}$. We know that $(AB)^{-1}$ exists. Therefore, $I = (AB)^{-1}AB = ((AB)^{-1}A)B$ and so, by Theorem 2, B is invertible and $B^{-1} = (AB)^{-1}A$. Similarly, we have $I = AB(AB)^{-1} = A(B(AB)^{-1})$, hence A is invertible.

D4 (a) We need to find $B = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix}$ such that

$$I = AB = A\begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix} = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 \end{bmatrix}$$

So, we need to solve the equations $A\vec{b}_1 = \vec{e}_1$ and $A\vec{b}_2 = \vec{e}_2$. Solving these we find that $\vec{b}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ and $\vec{b}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$. Thus, a right inverse of A is $B = \begin{bmatrix} -1 & 2 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}$.

(b) Observe that $(CA)_{i3} = c_{i1}a_{13} + c_{i2}a_{23} = c_{i1}(0) + c_{i2}(0) = 0$. Hence, the last column of CA is $\vec{0}$. Thus, $CA \neq I$.

D5 (1) $\Rightarrow$ (2): If A is invertible, then $\operatorname{rank}(A) = n$ by the Invertible Matrix Theorem. Hence, $\operatorname{nullity}(A) = 0$ by Theorem 3.4.7. Thus, $\operatorname{Null}(A) = \{\vec{0}\}$.
(2) $\Rightarrow$ (3): If $\operatorname{Null}(A) = \{\vec{0}\}$, then $\operatorname{nullity}(A) = 0$ and so $\operatorname{rank}(A) = n$. Hence, by Theorem 3.4.5, we get $n = \operatorname{rank}(A) = \dim(\operatorname{Row}(A))$. In particular, the n rows of A form a basis for $\mathbb{R}^n$, and hence are linearly independent.
(3) $\Rightarrow$ (4): The columns of A^T are the rows of A. So, if the rows of A are linearly independent, then the columns of A^T are linearly independent, so A^T is invertible by the Invertible Matrix Theorem.
(4) $\Rightarrow$ (1): We have shown above that if A is invertible, then A^T is invertible. Thus, if A^T is invertible we have that $(A^T)^T = A$ is invertible.

3.6 Elementary Matrices

Practice Problems

A1 To find the elementary matrix E, we perform the specified elementary row operation on the 3×3 identity matrix.

(a) We have $E = \begin{bmatrix} 1 & -5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and

$$EA = \begin{bmatrix} 1 & -5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 4 \\ 4 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 6 & -13 & -17 \\ -1 & 3 & 4 \\ 4 & 2 & 0 \end{bmatrix}$$

which equals the matrix obtained from A by performing the elementary row operation.

(b) We have $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 4 \\ 4 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 0 \\ -1 & 3 & 4 \end{bmatrix}$$

which equals the matrix obtained from A by performing the elementary row operation.

(c) We have $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 4 \\ 4 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 4 \\ -4 & -2 & 0 \end{bmatrix}$$

which equals the matrix obtained from A by performing the elementary row operation.

(d) We have $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 4 \\ 4 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ -6 & 18 & 24 \\ 4 & 2 & 0 \end{bmatrix}$$

which equals the matrix obtained from A by performing the elementary row operation.

(e) We have $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$ and

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 4 \\ 4 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 4 \\ 8 & 10 & 12 \end{bmatrix}$$

which equals the matrix obtained from A by performing the elementary row operation.

A2 To find the elementary matrix E, we perform the specified elementary row operation on the 4×4 identity matrix.

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

A3 (a) It is elementary. The corresponding elementary row operation is $R_3 + (-4)R_2$.

(b) It is not elementary. Both row 1 and row 3 have been multiplied by -1.

(c) It is not elementary. We have multiplied row 1 by 3 and then added row 3 to row 1.

(d) It is elementary. The corresponding elementary row operation is $R_1 \updownarrow R_3$.

(e) It is not elementary. All three rows have been swapped.

(f) It is elementary. A corresponding elementary row operation is $1R_1$.

A4 To find a sequence of elementary matrices such that $E_k \cdots E_1 A = I$, we row reduce A to I keeping track of the elementary row operations used. Note that since we obtain one elementary matrix for each elementary row operation used, it is wise to try to minimize the number of elementary row operations required.

(a) We have

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \begin{matrix} \\ \\ R_2 \updownarrow R_3 \end{matrix} \sim \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{matrix} \\ \\ \frac{1}{2}R_3 \end{matrix} \sim$$
$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 - 4R_3 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 - 3R_2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The first elementary row operation performed was $R_2 \updownarrow R_3$. So, E_1 is the 3×3 elementary matrix associated with this row operation. Hence, $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. The second row operation performed was $\frac{1}{2}R_3$, so $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$. The third elementary row operation performed was $R_1 - 4R_3$, so $E_3 = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and the fourth elementary row operation performed was $R_1 - 3R_2$ so $E_4 = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Hence,

$$A^{-1} = E_4E_3E_2E_1 = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 0 & 1 \\ 0 & 1/2 & 0 \end{bmatrix}$$

Since $A^{-1} = E_4E_3E_2E_1$ we have that $A = (E_4E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-2}E_3^{-1}E_4^{-1}$. Hence,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) We have

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 2 & 4 & 5 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 - 2R_3 \\ R_2 - 3R_3 \\ \\ \end{matrix} \sim$$
$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 - 2R_2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Observe that we performed the second and third operation "at the same time". Technically speaking, we actually need to perform one and then the other, however, which order we do them in does not matter. We will say that we did $R_2 - 3R_3$ first and then $R_1 - 2R_3$. Therefore, the elementary row operations corresponding to these row operations are: $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $E_4 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Hence,

$$A^{-1} = E_4E_3E_2E_1 = \begin{bmatrix} -7 & -2 & 4 \\ 6 & 1 & -3 \\ -2 & 0 & 1 \end{bmatrix}$$

and

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) We have

$$\begin{bmatrix} 1 & 0 & -1 \\ -2 & 0 & -2 \\ -4 & 1 & 4 \end{bmatrix} \begin{matrix} \\ R_2 + 2R_1 \\ R_3 + 4R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & -4 \\ 0 & 1 & 0 \end{bmatrix} \; -\tfrac{1}{4}R_2 \sim$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{matrix} \\ \\ R_2 \updownarrow R_3 \end{matrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 + R_3 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, and $E_5 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Then

$$A^{-1} = E_5E_4E_3E_2E_1 = \begin{bmatrix} 1/2 & -1/4 & 0 \\ 4 & 0 & 1 \\ -1/2 & -1/4 & 0 \end{bmatrix}$$

and

$$\begin{aligned} A &= E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

(d) We have

$$\begin{bmatrix} 1 & -2 & 4 & 1 \\ -1 & 3 & -4 & -1 \\ 0 & 1 & 2 & 0 \\ -2 & 4 & -8 & -1 \end{bmatrix} \begin{matrix} \\ R_2 + R_1 \\ \\ R_4 + 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & -2 & 4 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 + 2R_2 \\ \\ R_3 - R_2 \\ \\ \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ \frac{1}{2}R_3 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 4 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 - 4R_3 \\ \\ \\ \\ \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ R_1 - R_4 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, we get $E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $E_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $E_6 = \begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, and $E_7 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Then

$$A^{-1} = E_7E_6E_5E_4E_3E_2E_1 = \begin{bmatrix} 3 & 4 & -2 & -1 \\ 1 & 1 & 0 & 0 \\ -1/2 & -1/2 & 1/2 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{aligned} A =& E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}E_7^{-1} \\ =& \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Homework Problems

B1

(a) $E = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, EA = \begin{bmatrix} 10 & 1 & 19 \\ 2 & 0 & 5 \\ 1 & -3 & -2 \end{bmatrix}$ (b) $E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, EA = \begin{bmatrix} 1 & -3 & -2 \\ 2 & 0 & 5 \\ 2 & 1 & -1 \end{bmatrix}$

(c) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, EA = \begin{bmatrix} 2 & 1 & -1 \\ -6 & 0 & -15 \\ 1 & -3 & -2 \end{bmatrix}$ (d) $E = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, EA = \begin{bmatrix} 4 & 2 & -2 \\ 2 & 0 & 5 \\ 1 & -3 & -2 \end{bmatrix}$

(e) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, EA = \begin{bmatrix} 2 & 1 & -1 \\ 2 & 0 & 5 \\ -3 & -5 & 0 \end{bmatrix}$ (f) $E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, EA = \begin{bmatrix} 2 & 0 & 5 \\ 2 & 1 & -1 \\ 1 & -3 & -2 \end{bmatrix}$

B2

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (f) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

B3

(a) It is not elementary. All three rows have been swapped.

(b) It is elementary. The corresponding elementary row operation is $R_3 + 3R_2$.

(c) It is not elementary. Row 2 and 3 have been swapped and row 1 has been multiplied by -1.

(d) It is not elementary. You cannot obtain this matrix from I with elementary row operations.

(e) It is elementary. The corresponding elementary row operation is $2R_1$.

(f) It is elementary. The corresponding elementary row operation is $R_2 - R_1$.

B4

(a) $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}, E_4 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_5 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} -4 & -2 & 5/2 \\ 2 & 1 & -1 \\ -1 & 0 & 1/2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1/3 & 0 \\ 5/3 & -4/3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

(c) $E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} 0 & -1 & -3 \\ -1 & 2 & 8 \\ 1 & -1 & -4 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(d) $E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix}, E_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, E_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix},$

$E_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, E_7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}, E_8 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, E_9 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1/2 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \\ 7/2 & -1/2 & 5/2 & 1 \\ 3/2 & -1/2 & 3/2 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Conceptual Problems

D1 (a) Row reducing A to I, we find that $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$. We see that $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is the standard matrix of a reflection R in the line $x_1 = x_2$, $\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ is the standard matrix of a stretch S by factor -2 in the x_2-axis, and $\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$ is a shear H by factor -4 in the x_1-direction. Thus, $L = R \circ S \circ H$.

(b) Since the standard matrix of an invertible linear operator is invertible, it can be written as a product of elementary matrices. Moreover, each elementary matrix corresponds to either a shear, a stretch, or a reflection. Hence, since multiplying the matrices of two mappings corresponds to a composition of the mappings, we have that we can write L as a composition of the shears, stretches, and reflections.

D2 For "add t times row 1 to row 2" we have that the corresponding elementary matrix is $E = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$. Then,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{matrix} \\ R_2 + tR_1 \end{matrix} \sim \begin{bmatrix} a_{11} & a_{12} \\ a_{21} + ta_{11} & a_{22} \end{bmatrix}$$

while

$$EA = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} + ta_{11} & a_{22} \end{bmatrix}$$

For "multiply row 2 by a factor of $t \neq 0$" we have that the corresponding elementary matrix is $E = \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix}$. Then,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{matrix} \\ tR_2 \end{matrix} \sim \begin{bmatrix} a_{11} & a_{12} \\ ta_{21} & ta_{22} \end{bmatrix}$$

while

$$EA = \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ ta_{21} & ta_{22} \end{bmatrix}$$

D3 (a) Row reducing A to I we find that $E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$.

(b) We find that $E_1\vec{b} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, and $\vec{x} = E_2E_1\vec{b} = E_2\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$.

(c) Row reducing $\left[\begin{array}{c|c} A & \vec{b} \end{array}\right]$, we see that we perform the same two row operation on $\vec{b}$ and hence get the same answer.

3.7 *LU*-Decomposition

Practice Problems

A1 (a) Row reducing we get

$$\begin{bmatrix} -2 & -1 & 5 \\ -4 & 0 & -2 \\ 2 & 1 & 3 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{matrix} \sim \begin{bmatrix} -2 & -1 & 5 \\ 0 & 2 & -12 \\ 0 & 0 & 8 \end{bmatrix} \quad \Rightarrow \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & * & 1 \end{bmatrix}$$

We now have the matrix in a row echelon form, so no further row operations are required. Thus, we have

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & -1 & 5 \\ 0 & 2 & -12 \\ 0 & 0 & 8 \end{bmatrix}$$

(b) Row reducing we get

$$\begin{bmatrix} 1 & -2 & 4 \\ 3 & -2 & 4 \\ 2 & 2 & -5 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & -2 & 4 \\ 0 & 4 & -8 \\ 0 & 6 & -13 \end{bmatrix} \quad \Rightarrow \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & * & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 4 \\ 0 & 4 & -8 \\ 0 & 6 & -13 \end{bmatrix} \begin{matrix} \\ \\ R_3 - \frac{3}{2}R_2 \end{matrix} \sim \begin{bmatrix} 1 & -2 & 4 \\ 0 & 4 & -8 \\ 0 & 0 & -1 \end{bmatrix} \quad \Rightarrow \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 3/2 & 1 \end{bmatrix}$$

Hence,

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 3/2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -2 & 4 \\ 0 & 4 & -8 \\ 0 & 0 & -1 \end{bmatrix}$$

(c) Row reducing we get

$$\begin{bmatrix} 2 & -4 & 5 \\ 2 & 5 & 2 \\ 2 & -1 & 5 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix} \sim \begin{bmatrix} 2 & -4 & 5 \\ 0 & 9 & -3 \\ 0 & 3 & 0 \end{bmatrix} \quad \Rightarrow \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & * & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 & 5 \\ 0 & 9 & -3 \\ 0 & 3 & 0 \end{bmatrix} \begin{matrix} \\ \\ R_3 - \frac{1}{3}R_2 \end{matrix} \sim \begin{bmatrix} 2 & -4 & 5 \\ 0 & 9 & -3 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1/3 & 1 \end{bmatrix}$$

Hence,

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 & 5 \\ 0 & 9 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

(d) Row reducing we get

$$\begin{bmatrix} 1 & 5 & 3 & 4 \\ -2 & -6 & -1 & 3 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_2 + 2R_1 \sim \begin{bmatrix} 1 & 5 & 3 & 4 \\ 0 & 4 & 5 & 11 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & * & 1 & 0 \\ 0 & * & * & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 3 & 4 \\ 0 & 4 & 5 & 11 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 - \tfrac{1}{2}R_2 \sim \begin{bmatrix} 1 & 5 & 3 & 4 \\ 0 & 4 & 5 & 11 \\ 0 & 0 & -7/2 & -13/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 1/2 & 1 & 0 \\ 0 & 0 & * & 1 \end{bmatrix}$$

We now have the matrix in a row echelon form, so no further row operations are required. Thus, we have

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 1/2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 & 3 & 4 \\ 0 & 4 & 5 & 11 \\ 0 & 0 & -7/2 & -13/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(e) Row reducing we get

$$\begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & -3 & -2 & 1 \\ 3 & -3 & 2 & -1 \\ 0 & 4 & -3 & 0 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 3R_1 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & -3 & -2 & 1 \\ 0 & 3 & -1 & -4 \\ 0 & 4 & -3 & 0 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & * & 1 & 0 \\ 0 & * & * & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & -3 & -2 & 1 \\ 0 & 3 & -1 & -4 \\ 0 & 4 & -3 & 0 \end{bmatrix} \begin{matrix} \\ \\ R_3 + R_2 \\ R_4 + \frac{4}{3}R_2 \end{matrix} \sim \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & -3 & -2 & 1 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & -\frac{17}{3} & \frac{4}{3} \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 \\ 0 & -\frac{4}{3} & * & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & -3 & -2 & 1 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & -\frac{17}{3} & \frac{4}{3} \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 - \frac{17}{9}R_3 \end{matrix} \sim \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & -3 & -2 & 1 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 7 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 \\ 0 & -\frac{4}{3} & \frac{17}{9} & 1 \end{bmatrix}$$

Hence,

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 \\ 0 & -4/3 & 17/9 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & -3 & -2 & 1 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

(f) Row reducing we get

$$\begin{bmatrix} -2 & -1 & 2 & 0 \\ 4 & 3 & -2 & 2 \\ 3 & 3 & 4 & 3 \\ 2 & -1 & 2 & -4 \end{bmatrix} \begin{matrix} \\ R_2 + 2R_1 \\ R_3 + \frac{3}{2}R_1 \\ R_4 + R_1 \end{matrix} \sim \begin{bmatrix} -2 & -1 & 2 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 3/2 & 7 & 3 \\ 0 & -2 & 4 & -4 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -3/2 & * & 1 & 0 \\ -1 & * & * & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -1 & 2 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 3/2 & 7 & 3 \\ 0 & -2 & 4 & -4 \end{bmatrix} \begin{matrix} \\ \\ R_3 - \frac{3}{2}R_2 \\ R_4 + 2R_2 \end{matrix} \sim \begin{bmatrix} -2 & -1 & 2 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 8 & 0 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -3/2 & 3/2 & 1 & 0 \\ -1 & -2 & * & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -1 & 2 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 8 & 0 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 - 2R_3 \end{matrix} \sim \begin{bmatrix} -2 & -1 & 2 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -3/2 & 3/2 & 1 & 0 \\ -1 & -2 & 2 & 1 \end{bmatrix}$$

Hence,

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -3/2 & 3/2 & 1 & 0 \\ -1 & -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & -1 & 2 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A2 (a) Row reducing A we get

$$\begin{bmatrix} 1 & 0 & 3 \\ -2 & 1 & -3 \\ -1 & 4 & 5 \end{bmatrix} \begin{matrix} \\ R_2 + 2R_1 \\ R_3 + R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 4 & 8 \end{bmatrix} \qquad \Rightarrow \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & * & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 4 & 8 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 4R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & -4 \end{bmatrix} \qquad \Rightarrow \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix}$$

Hence,

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & -4 \end{bmatrix}$$

To solve $A\vec{x}_1 = \vec{b}_1$ we let $\vec{y} = U\vec{x}_1$ and solve $L\vec{y} = \vec{b}_1$. This gives

$$\begin{aligned} y_1 &= 3 \\ -2y_1 + y_2 &= -4 \\ -y_1 + 4y_2 + y_3 &= -3 \end{aligned}$$

We solve this by forward substitution. We have $y_1 = 3$, $y_2 = -4 + 2y_1 = -4 + 2(3) = 2$, and $y_3 = -3 + y_1 - 4y_2 = -3 + 3 - 4(2) = -8$. Then we solve $U\vec{x}_1 = \begin{bmatrix} 4 \\ 4 \\ -8 \end{bmatrix}$ by back substitution. We have

$$\begin{aligned} x_1 + 3x_3 &= 3 \\ x_2 + 3x_3 &= 2 \\ -4x_3 &= -8 \end{aligned}$$

which gives $x_3 = 2$, $x_2 = 2 - 3x_3 = 2 - 3(2) = -4$, and $x_1 = 3 - 3x_3 = 3 - 3(2) = -3$. Hence the solution is $\vec{x}_1 = \begin{bmatrix} -3 \\ -4 \\ 2 \end{bmatrix}$.

To solve $A\vec{x}_2 = \vec{b}_2$ we let $\vec{y} = U\vec{x}_2$ and solve $L\vec{y} = \vec{b}_2$. This gives

$$\begin{aligned} y_1 &= 2 \\ -2y_1 + y_2 &= -5 \\ -y_1 + 4y_2 + y_3 &= -2 \end{aligned}$$

We solve this by forward substitution. We have $y_1 = 2$, $y_2 = -5 + 2y_1 = -5 + 2(2) = -1$, and $y_3 = -2 + y_1 - 4y_2 = -2 + 2 - 4(-1) = 4$. Then we solve $U\vec{x}_2 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ by back substitution. We have

$$\begin{aligned} x_1 + 3x_3 &= 2 \\ x_2 + 3x_3 &= -1 \\ -4x_3 &= 4 \end{aligned}$$

which gives $x_3 = -1$, $x_2 = -1 - 3x_3 = -1 - 3(-1) = 2$, and $x_1 = 2 - 3x_3 = 2 - 3(-1) = 5$. Hence the solution is $\vec{x}_2 = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$.

(b) Row reducing A we get

$$\begin{bmatrix} 1 & 0 & -2 \\ -1 & -4 & 4 \\ 3 & -4 & -1 \end{bmatrix} \begin{matrix} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & -4 & 2 \\ 0 & -4 & 5 \end{bmatrix} \quad \Rightarrow \quad L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & * & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & -4 & 2 \\ 0 & -4 & 5 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & -4 & 2 \\ 0 & 0 & 3 \end{bmatrix} \quad \Rightarrow \quad L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}$$

Hence,

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & -4 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

To solve $A\vec{x}_1 = \vec{b}_1$ we let $\vec{y} = U\vec{x}_1$ and solve $L\vec{y} = \vec{b}_1$. This gives

$$\begin{aligned} y_1 &= -1 \\ -y_1 + y_2 &= -7 \\ 3y_1 + y_2 + y_3 &= -5 \end{aligned}$$

We solve this by forward substitution. We have $y_1 = -1$, $y_2 = -7 + y_1 = -7 + (-1) = -8$, and $y_3 = -5 - 3y_1 - y_2 = -5 - 3(-1) - (-8) = 6$. Then we solve $U\vec{x}_1 = \begin{bmatrix} -1 \\ -8 \\ 6 \end{bmatrix}$ by back substitution. We have

$$\begin{aligned} x_1 - 2x_3 &= -1 \\ -4x_2 + 2x_3 &= -8 \\ 3x_3 &= 6 \end{aligned}$$

which gives $x_3 = 2$ and $-4x_2 = -8 - 2x_3 = -8 - 2(2) = -12$, so $x_2 = 3$. Then $x_1 = -1 + 2x_3 = -1 + 2(2) = 3$. Hence the solution is $\vec{x}_1 = \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}$.

To solve $A\vec{x}_2 = \vec{b}_2$ we let $\vec{y} = U\vec{x}_2$ and solve $L\vec{y} = \vec{b}_2$. This gives

$$\begin{aligned} y_1 &= 2 \\ -y_1 + y_2 &= 0 \\ 3y_1 + y_2 + y_3 &= -1 \end{aligned}$$

We solve this by forward substitution. We have $y_1 = 2$, $y_2 = y_1 = 2$, and $y_3 = -1-3y_1-y_2 = -1-3(2)-2 = -9$. Then we solve $U\vec{x}_2 = \begin{bmatrix} 2 \\ 2 \\ -9 \end{bmatrix}$ by back substitution. We have

$$\begin{aligned} x_1 - 2x_3 &= 2 \\ -4x_2 + 2x_3 &= 2 \\ 3x_3 &= -9 \end{aligned}$$

which gives $x_3 = -3$ and $-4x_2 = 2-2x_3 = 2-2(-3) = 8$, so $x_2 = -2$. Then $x_1 = 2+2x_3 = 2+2(-3) = -4$. Hence the solution is $\vec{x}_2 = \begin{bmatrix} -4 \\ -2 \\ -3 \end{bmatrix}$.

(c) Row reducing A we get

$$\begin{bmatrix} 1 & 0 & 1 \\ -3 & 2 & -1 \\ -3 & 4 & 2 \end{bmatrix} \begin{matrix} \\ R_2 + 3R_1 \\ R_3 + 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 4 & 5 \end{bmatrix} \quad \Rightarrow \quad L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -3 & * & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 4 & 5 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 2R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -3 & 2 & 1 \end{bmatrix}$$

Hence,

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

To solve $A\vec{x}_1 = \vec{b}_1$ we let $\vec{y} = U\vec{x}_1$ and solve $L\vec{y} = \vec{b}_1$. This gives

$$\begin{aligned} y_1 &= 3 \\ -3y_1 + y_2 &= -5 \\ -3y_1 + 2y_2 + y_3 &= -1 \end{aligned}$$

We solve this by forward substitution. We have $y_1 = 3$, $y_2 = -5 + 3y_1 = -5 + 3(3) = 4$, and $y_3 = -1 + 3y_1 - 2y_2 = -1 + 3(3) - 2(4) = 0$. Then we solve $U\vec{x}_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$ by back substitution. We have

$$\begin{aligned} x_1 + x_3 &= 3 \\ 2x_2 + 2x_3 &= 4 \\ x_3 &= 0 \end{aligned}$$

which gives $x_3 = 0$ and $2x_2 = 4-2x_3 = 4$, so $x_2 = 2$. Then $x_1 = 3-x_3 = 3$. Hence the solution is $\vec{x}_1 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$.

To solve $A\vec{x}_2 = \vec{b}_2$ we let $\vec{y} = U\vec{x}_2$ and solve $L\vec{y} = \vec{b}_2$. This gives

$$\begin{aligned} y_1 &= -4 \\ -3y_1 + y_2 &= 4 \\ -3y_1 + 2y_2 + y_3 &= -5 \end{aligned}$$

We solve this by forward substitution. We have $y_1 = -4$, $y_2 = 4 + 3y_1 = 4 + 3(-4) = -8$, and $y_3 = -5 + 3y_1 - 2y_2 = -5 + 3(-4) - 2(-8) = -1$. Then we solve $U\vec{x}_2 = \begin{bmatrix} -4 \\ -8 \\ -1 \end{bmatrix}$ by back substitution. We have

$$\begin{aligned} x_1 + x_3 &= -4 \\ 2x_2 + 2x_3 &= -8 \\ x_3 &= -1 \end{aligned}$$

which gives $x_3 = -1$ and $2x_2 = -8-2x_3 = -8-2(-1) = -6$, so $x_2 = -3$. Then $x_1 = -4-x_3 = -4-(-1) = -3$. Hence the solution is $\vec{x}_2 = \begin{bmatrix} -3 \\ -3 \\ -1 \end{bmatrix}$.

(d) Row reducing A we get

$$\begin{bmatrix} -1 & 2 & -3 & 0 \\ 0 & -1 & 3 & 1 \\ 3 & -8 & 3 & 2 \\ 1 & -2 & 3 & 1 \end{bmatrix} \begin{matrix} \\ \\ R_3 + 3R_1 \\ R_4 + R_1 \end{matrix} \sim \begin{bmatrix} -1 & 2 & -3 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & -2 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & * & 1 & 0 \\ -1 & * & * & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & -3 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & -2 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 2R_2 \\ \\ \end{matrix} \sim \begin{bmatrix} -1 & 2 & -3 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & -12 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 2 & 1 & 0 \\ -1 & 0 & * & 1 \end{bmatrix}$$

Hence,

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 2 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & -3 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & -12 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To solve $A\vec{x}_1 = \vec{b}_1$ we let $\vec{y} = U\vec{x}_1$ and solve $L\vec{y} = \vec{b}_1$. This gives

$$\begin{aligned} y_1 &= -6 \\ y_2 &= 7 \\ -3y_1 + 2y_2 + y_3 &= -4 \\ -y_1 + y_4 &= 5 \end{aligned}$$

We solve this by forward substitution. We have $y_1 = -6$, $y_2 = 7$, $y_3 = -4 + 3y_1 - 2y_2 = -36$, and $y_4 = 5 + y_1 = -1$. Then we solve $U\vec{x}_1 = \begin{bmatrix} -6 \\ 7 \\ -36 \\ -1 \end{bmatrix}$ by back substitution. We have

$$\begin{aligned} -x_1 + 2x_2 - 3x_3 &= -6 \\ -x_2 + 3x_3 + x_4 &= 7 \\ -12x_3 &= -36 \\ x_4 &= -1 \end{aligned}$$

which gives $x_4 = -1$, $x_3 = 3$, $x_2 = -(7 - 3x_3 - x_4) = 1$, and $x_1 = -(-6 - 2x_2 + 3x_3) = -1$. Hence the solution is $\vec{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 3 \\ -1 \end{bmatrix}$.

To solve $A\vec{x}_2 = \vec{b}_2$ we let $\vec{y} = U\vec{x}_2$ and solve $L\vec{y} = \vec{b}_2$. This gives

$$\begin{aligned} y_1 &= 5 \\ y_2 &= -3 \\ -3y_1 + 2y_2 + y_3 &= 3 \\ -y_1 + y_4 &= -5 \end{aligned}$$

We solve this by forward substitution. We have $y_1 = 5$, $y_2 = -3$, $y_3 = 3 + 3y_1 - 2y_2 = 24$, and $y_4 = -5 + y_1 = 0$. Then we solve $U\vec{x}_2 = \begin{bmatrix} 5 \\ -3 \\ 24 \\ 0 \end{bmatrix}$ by back substitution. We have

$$\begin{aligned} -x_1 + 2x_2 - 3x_3 &= 5 \\ -x_2 + 3x_3 + x_4 &= -3 \\ -12x_3 &= 24 \\ x_4 &= 0 \end{aligned}$$

which gives $x_4 = 0$, $x_3 = -2$, $x_2 = -(-3 - 3x_3 - x_4) = -3$, and $x_1 = -(5 - 2x_2 + 3x_3) = -5$. Hence the solution is $\vec{x}_2 = \begin{bmatrix} -5 \\ -3 \\ -2 \\ 0 \end{bmatrix}$.

Homework Problems

B1 (a) $\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -2 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 5/4 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & -3/2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ 4 & 3 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 7 & 1 & 0 \\ -3/2 & 5/2 & 1/2 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 & 3 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

B2 (a) $LU = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & -6 \\ 0 & 0 & -3 \end{bmatrix}$; $\vec{x}_1 = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 4 \\ -5 \\ -3 \end{bmatrix}$

(b) $LU = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -5/2 & 2 & 1 \end{bmatrix}\begin{bmatrix} -2 & -4 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $\vec{x}_1 = \begin{bmatrix} -5 \\ -1 \\ -8 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$

(c) $LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -4 & -7/5 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 4 \\ 0 & 5 & 5 \\ 0 & 0 & 15 \end{bmatrix}$; $\vec{x}_1 = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$

(d) $LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -2 & -1 & 1 & 0 \\ 4 & 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & -2 & 0 \\ 0 & -3 & 1 & -2 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 5 \end{bmatrix}$; $\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 9/5 \\ -1 \\ 7/5 \\ 1/5 \end{bmatrix}$.

Conceptual Problems

D1 (a) Let E be a lower triangular elementary matrix. Then E can be obtained from the identity matrix by a row operation of the form $R_i + tR_j$ where $i > j$ or by the row operation cR_i where $c \neq 0$. In the first case, we can bring E back to the identity matrix with the row operation $R_i - tR_j$. Hence, this is the row operation associated with E^{-1} and so E^{-1} is lower triangular. In the second case, the row operation corresponding to E^{-1} is $\frac{1}{c}R_i$, so E is lower triangular in this case as well.

(b) Assume that L and M are both $n \times n$ lower triangular matrices. Then whenever $i < j$ we have $(L)_{ij} = 0$ and $(M)_{ij} = 0$. Hence, if $i < j$, we have

$$\begin{aligned}(LM)_{ij} &= (L)_{i1}(M)_{1j} + \cdots + (L)_{i(j-1)}(M)_{j(j+1)} + (L)_{ij}(M)_{jj} + \cdots + (L)_{in}(M)_{nj}\\ &= (L)_{i1}(0) + \cdots + (L)_{i(j-1)}(0) + (0)(M)_{jj} + (0)(M)_{(j+1)j} + \cdots + (0)(M)_{nj}\\ &= 0\end{aligned}$$

Thus, LM is lower triangular.

Chapter 3 Quiz

Problems

E1 (a) Since the number of columns of A equals the number of rows of B the product AB is defined and

$$AB = \begin{bmatrix} 2 & -5 & -3 \\ -3 & 4 & -7 \end{bmatrix}\begin{bmatrix} 2 & -1 & 4 \\ 3 & 0 & 2 \\ 1 & -1 & 5 \end{bmatrix} = \begin{bmatrix} -14 & 1 & -17 \\ -1 & 10 & -39 \end{bmatrix}$$

(b) The number of columns of B does not equal the number of rows of A, so BA is not defined.

(c) A has 3 columns and 2 rows, so A^T has 2 columns and 3 rows. Thus, the number of columns of B equals the number of rows of A^T, so BA^T is defined. We have

$$BA^T = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 0 & 2 \\ 1 & -1 & 5 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -5 & 4 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} -3 & -38 \\ 0 & -23 \\ -8 & -42 \end{bmatrix}$$

E2 (a) We have

$$f_A(\vec{u}) = \begin{bmatrix} -3 & 0 & 4 \\ 2 & -4 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -11 \\ 0 \end{bmatrix}$$

$$f_A(\vec{v}) = \begin{bmatrix} -3 & 0 & 4 \\ 2 & -4 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -16 \\ 17 \end{bmatrix}$$

(b) We have

$$A\begin{bmatrix} 4 & 1 \\ -2 & 1 \\ -1 & -2 \end{bmatrix} = A\begin{bmatrix} \vec{v} & \vec{u} \end{bmatrix} = \begin{bmatrix} A\vec{v} & A\vec{u} \end{bmatrix} = \begin{bmatrix} -16 & -11 \\ 17 & 0 \end{bmatrix}$$

E3 (a) We have

$$[R] = [R_{\pi/3}] = \begin{bmatrix} \cos\pi/3 & -\sin\pi/3 & 0 \\ \sin\pi/3 & \cos\pi/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) A normal to the plane $-x_1 - x_2 + 2x_3 = 0$ is $\vec{n} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$. The columns of $[M]$ are found by calculating the images of the standard basis vectors under the reflection.

$$\text{refl}_{\vec{n}}\, \vec{e}_1 = \vec{e}_1 - 2\frac{\vec{e}_1 \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 2\frac{-1}{6}\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

$$\text{refl}_{\vec{n}}\, \vec{e}_2 = \vec{e}_2 - 2\frac{\vec{e}_2 \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 2\frac{-1}{6}\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$\text{refl}_{\vec{n}}\, \vec{e}_3 = \vec{e}_3 - 2\frac{\vec{e}_3 \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 2\frac{2}{6}\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$$

Thus,

$$[M] = [\text{refl}_{(-1,-1,2)}] = \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

(c) A composition of mappings is represented by the product of the matrices. Hence,

$$[R \circ M] = [R][M] = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 2+\sqrt{3} & -1-2\sqrt{3} & 2-2\sqrt{3} \\ 2\sqrt{3}-1 & -\sqrt{3}+2 & 2\sqrt{3}+2 \\ 4 & 4 & -2 \end{bmatrix}$$

E4 The reduced row echelon form of the augmented matrix $\left[A \mid \vec{b}\right]$ is $\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 0 & 5 \\ 0 & 1 & -1 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right]$. We see that x_3 and x_4 are free variables, so let $x_3 = s \in \mathbb{R}$ and $x_4 = t \in \mathbb{R}$. Then, the general solution is

$$\vec{x} = \begin{bmatrix} 5 \\ 6 \\ 0 \\ 0 \\ 7 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

Replacing $\vec{b}$ with $\vec{0}$, we see that the solution space of $A\vec{x} = \vec{0}$ is

$$\vec{x} = s\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

In particular, the solution set is obtained from the solution space of $A\vec{x} = \vec{0}$ by translating by the vector $\begin{bmatrix}5\\6\\0\\0\\7\end{bmatrix}$.

E5 (a) The range of f_B is equal to the columnspace of B, so $\vec{v}$ is in the range of f_B if and only if $\vec{v}$ is in the columnspace of B. $\vec{u}$ and $\vec{v}$ are in the columnspace if and only if the systems $B\vec{x} = \vec{u}$ and $B\vec{x} = \vec{v}$ are consistent. Since the coefficient matrix is the same for the two systems, we can row reduce the doubly augmented matrix $[B \mid \vec{u} \mid \vec{v}]$. We get

$$\left[\begin{array}{ccc|c|c} 1 & 2 & 0 & 4 & -5 \\ -1 & -1 & -1 & -3 & 6 \\ 1 & 3 & 0 & 5 & -7 \\ 0 & 2 & -1 & 3 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

We see that the system $[B \mid \vec{u}]$ is inconsistent so $\vec{u}$ is not in the columnspace of B. However, the system $[B \mid \vec{v}]$ is consistent so $\vec{v}$ is in the columspace of B and hence in the range of f_B.

(b) From the reduced row echelon form of $[B \mid \vec{v}]$ we get that $\vec{x} = \begin{bmatrix}-1\\-2\\-3\end{bmatrix}$ satisfies $f_B(\vec{x}) = \vec{v}$.

(c) Use the fact that $B\vec{y}$ is a linear combination of columns of B.

$$\begin{bmatrix}2\\-1\\3\\2\end{bmatrix} = 0\begin{bmatrix}1\\-1\\1\\0\end{bmatrix} + 1\begin{bmatrix}2\\-1\\3\\2\end{bmatrix} + 0\begin{bmatrix}0\\-1\\0\\-1\end{bmatrix}$$

So, $\vec{y} = \begin{bmatrix}0\\1\\0\end{bmatrix}$.

E6 Observe that the given row echelon form of A is not the reduced row echelon form. Therefore, we must first finish row reducing this to reduced row echelon form. We get

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 & 5 \\ 0 & 2 & -2 & 1 & 8 \\ 3 & 3 & 0 & 4 & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for the rowspace of A is the set of non-zero rows from the reduced row echelon form. Hence, a basis for Row(A) is $\left\{\begin{bmatrix}1\\0\\1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\0\\3\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\2\end{bmatrix}\right\}$. The columns from A which correspond to columns with leading ones in the reduced row echelon form of A form a basis for the columnspace of A. Hence, a basis for the columnspace of A

is $\left\{\begin{bmatrix}1\\2\\0\\3\end{bmatrix}, \begin{bmatrix}0\\1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\2\\1\\4\end{bmatrix}\right\}$. To find a basis for the nullspace we find the general solution to $A\vec{x} = \vec{0}$. Let $x_3 = s \in \mathbb{R}$ and $x_5 = t \in \mathbb{R}$. Then the solution space is

$$\vec{x} = \begin{bmatrix}-s+t\\s-3t\\s\\-2t\\t\end{bmatrix} = s\begin{bmatrix}-1\\1\\1\\0\\0\end{bmatrix} + t\begin{bmatrix}1\\-3\\0\\-2\\1\end{bmatrix}$$

Therefore, a basis for Null(A) is $\left\{\begin{bmatrix}-1\\1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\-3\\0\\-2\\1\end{bmatrix}\right\}$.

E7 To find the inverse we row reduce $[A \mid I]$. We get

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0\\ 0 & 2 & 0 & 1 & 0 & 0 & 1 & 0\\ 1 & 0 & 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2/3 & 0 & 0 & 1/3\\ 0 & 1 & 0 & 0 & 1/6 & 0 & 1/2 & -1/6\\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & -1/3 & 0 & 0 & 1/3 \end{array}\right]$$

Hence, $A^{-1} = \begin{bmatrix} 2/3 & 0 & 0 & 1/3\\ 1/6 & 0 & 1/2 & -1/6\\ 0 & 1 & 0 & 0\\ -1/3 & 0 & 0 & 1/3 \end{bmatrix}$.

E8 To find the inverse we row reduce $[A \mid I]$. We get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & p & 1 & 0 & 0\\ 1 & 1 & 0 & 0 & 1 & 0\\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & p & 1 & 0 & 0\\ 0 & 1 & -p & -1 & 1 & 0\\ 0 & 0 & 1-p & -1 & -1 & 1 \end{array}\right]$$

A matrix is invertible if and only if its reduced row echelon form is I. We can get a leading one in the third column only when $1 - p \neq 0$. Hence, the above matrix is invertible only for $p \neq 1$. Assume $p \neq 1$, to get the inverse, we must continue row reducing.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & p & 1 & 0 & 0\\ 0 & 1 & -p & -1 & 1 & 0\\ 0 & 0 & 1-p & -1 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{1-p} & \frac{p}{1-p} & \frac{-p}{1-p}\\ 0 & 1 & 0 & \frac{-1}{1-p} & \frac{1-2p}{1-p} & \frac{p}{1-p}\\ 0 & 0 & 1 & \frac{-1}{1-p} & \frac{-1}{1-p} & \frac{1}{1-p} \end{array}\right]$$

Thus, when $p \neq 1$ we get that the inverse is $\dfrac{1}{1-p}\begin{bmatrix} 1 & p & -p\\ -1 & 1-2p & p\\ -1 & -1 & 1 \end{bmatrix}$.

E9 By definition, the range of L is a subset of $\mathbb{R}^m$. We have $L(\vec{0}) = \vec{0}$, so $\vec{0} \in$ Range(L). If $\vec{x}, \vec{y} \in$ Range(L), then there exists $\vec{u}, \vec{v} \in \mathbb{R}^n$ such that $L(\vec{u}) = \vec{x}$ and $L(\vec{v}) = \vec{y}$. Hence $L(\vec{u} + \vec{v}) = L(\vec{u}) + L(\vec{v}) = \vec{x} + \vec{y}$, so $\vec{x} + \vec{y} \in$ Range(L). Similarly, $L(t\vec{u}) = tL(\vec{u}) = t\vec{x}$, so $t\vec{x} \in$ Range(L). Thus, L is a subspace of $\mathbb{R}^m$.

E10 To prove the set is linearly independent, we use the definition. Consider $c_1L(\vec{v}_1) + \cdots + c_kL(\vec{v}_k) = \vec{0}$. Since L is linear we get $L(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) = \vec{0}$. Thus $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k \in \operatorname{Null}(L)$. But, $\operatorname{Null}(L) = \{\vec{0}\}$, so $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$. But, this implies that $c_1 = \cdots = c_k = 0$ since $\{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent. Therefore, $\{L(\vec{v}_1), \dots, L(\vec{v}_k)\}$ is linearly independent.

E11 (a) To find a sequence of elementary matrices such that $E_k \cdots E_1 A = I$, we row reduce A to I keeping track of the elementary row operations used. We have

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & -3 \\ 0 & 0 & 4 \end{bmatrix} \begin{matrix} \\ \frac{1}{2}R_2 \\ \frac{1}{4}R_3 \end{matrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 + 2R_3 \\ R_2 + \frac{3}{2}R_3 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, we have $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and $E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) We have

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3/2 \\ 0 & 0 & 1 \end{bmatrix}$$

E12 (a) Let $K = I_3$. Then $KM = MK$ for all 3×3 matrices M.

(b) For KM to be defined K must have 3 columns. For MK to be defined, K must have 4 rows. Hence, if there exists such a K, it must be a 4×3 matrix. However, this implies that KM has size 4×4 and that MK has size 3×3. Thus, KM cannot equal MK for any matrix K.

(c) The range is a subspace of $\mathbb{R}^3$ and hence cannot be spanned by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ because this vector is not in $\mathbb{R}^3$.

(d) Since the range of L consists of all vectors that are multiples of $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, the columns of L must be multiples of $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. Since the nullspace of L consists of all vectors that are multiples of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, we must have the rows are multiples of $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Hence, the matrix of L is any multiple of $\begin{bmatrix} 1 & -2/3 \\ 1 & -2/3 \\ 2 & -4/3 \end{bmatrix}$.

(e) The rank of this linear mapping is 3 and the nullity is 1. But then $\operatorname{rank}(L) + \operatorname{nullity}(L) = 3 + 1 = 4 \neq \dim \mathbb{R}^3$. Hence, this contradicts the Rank Theorem, so there can be no such mapping L.

(f) This contradicts Theorem 3.5.2, so there can be no such matrix.

Chapter 3 Further Problems

Problems

F1 Certainly $A(pI+qA)=pA+qA^2=(pI+qA)A$, so matrices of this form do commute with A. Suppose that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

so that

$$\begin{bmatrix} 3a & 2a+b \\ 3c & 2c+d \end{bmatrix} = \begin{bmatrix} 3a+2c & 3b+2d \\ c & d \end{bmatrix}$$

By comparing corresponding entries, we see immediately that for this equation to be satisfied, it is necessary and sufficient that $c=0$ and $a=b+d$. Note that b and d may be chosen arbitrarily. Thus the general form of a matrix that commutes with A is $\begin{bmatrix} b+d & b \\ 0 & d \end{bmatrix}$. We want to write this in the form

$$pI+qA = \begin{bmatrix} p+3q & 2q \\ 0 & p+q \end{bmatrix}$$

We equate corresponding entries and solve. Thus, let $b=2q$, $d=p+q$, and we see that the general matrix that commutes with A is

$$\begin{bmatrix} b+d & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} 2q+p+q & 2q \\ 0 & p+q \end{bmatrix} = pI+qA$$

as claimed.

F2 Let $P,Q \in C(A)$, $t \in \mathbb{R}$, so that $PA=AP$ and $QA=AQ$. Then

$$(P+Q)A = PA+QA = AP+AQ = A(P+Q)$$

So, $C(A)$ is closed under addition. Also,

$$(tP)A = t(PA) = t(AP) = A(tP)$$

Hence, $C(A)$ is closed under scalar multiplication. Finally,

$$(PQ)A = P(QA) = P(AQ) = (PA)Q = (AP)Q = A(PQ)$$

Therefore, $PQ \in C(A)$, and $C(A)$ is closed under matrix multiplication.

F3 Let $A = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix}$. Then

$$A^2 = \begin{bmatrix} 0 & 0 & a_{12}a_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We claim that any $n \times n$ matrix such that all entries on and below the main diagonal are zero is nilpotent. The general condition describing this kind of matrix is

$$a_{ij} = 0 \quad \text{whenever } j < i+1$$

We now prove that such matrices are nilpotent. We have

$$(A^2)_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj}$$

Since $a_{ik} = 0$ unless $k \geq i + 1$, this sum can be rewritten

$$(A^2)_{ij} = \sum_{k=i+1}^{n} a_{ik}a_{kj}$$

But, $a_{kj} = 0$ unless $j \geq k + 1 \geq i + 2$, so $(A^2)_{ij} = 0$ unless $j \geq i + 2$. Using induction, we can show that $(A^m)_{ij} = 0$ unless $j \geq i + m$. Since $i \geq 1$ and $k \leq n$, it is impossible to satisfy $k \geq j + n$, so $(A^n)_{ij} = 0$ for all $i, j = 1, \dots, n$. Thus, A is nilpotent.

F4 (a) refl_θ maps $(1, 0)$ to $(\cos 2\theta, \sin 2\theta)$. To see this, think of rotating $(1, 0)$ to lie along the line making angle θ with the x_1-axis, then rotating a further angle θ so that the final image is as far from the line as the original starting point.

refl_θ maps $(0, 1)$ to $\left(\cos(-\frac{\pi}{2} + 2\theta), \sin(-\frac{\pi}{2} + 2\theta)\right)$. To see this, note that $(0, 1)$ starts at angle $\pi/2$; to obtain its image under the reflection in the line, rotate it backwards by twice the angle $\frac{\pi}{2} - \theta$.

By standard trigonometric identities,

$$\cos(-\frac{\pi}{2} + 2\theta) = \sin 2\theta, \qquad \sin(-\frac{\pi}{2} + 2\theta) = -\cos 2\theta$$

Hence,

$$[\text{refl}_\theta] = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

(b) It follows that

$$\begin{aligned}
[\text{refl}_\alpha \circ \text{refl}_\theta] &= \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix} \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos 2\alpha \cos 2\theta + \sin 2\alpha \sin 2\theta & \cos 2\alpha \sin 2\theta - \sin 2\alpha \cos 2\theta \\ \sin 2\alpha \cos 2\theta - \cos 2\alpha \sin 2\theta & \sin 2\alpha \sin 2\theta + \cos 2\alpha \cos 2\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos 2(\alpha - \theta) & -\sin 2(\alpha - \theta) \\ \sin 2(\alpha - \theta) & \cos 2(\alpha - \theta) \end{bmatrix}
\end{aligned}$$

Thus the composition of these reflections is a rotation through angle $2(\alpha - \theta)$.

F5 (a) For every $\vec{x} \in \mathbb{R}^2$ we have $\|L(\vec{x})\| = \|\vec{x}\|$, so $L(\vec{x}) \cdot L(\vec{x}) = \|\vec{x}\|^2$ for every $\vec{x} \in \mathbb{R}^2$ and $\|L(\vec{x} + \vec{y})\|^2 = \|\vec{x} + \vec{y}\|^2$ for every $\vec{x} + \vec{y} \in \mathbb{R}^2$. Hence,

$$\begin{aligned}
(L(\vec{x} + \vec{y})) \cdot (L(\vec{x} + \vec{y})) &= (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) \\
(L(\vec{x}) + L(\vec{y})) \cdot (L(\vec{x}) + L(\vec{y})) &= \vec{x} \cdot \vec{x} + 2\vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y} \\
L(\vec{x}) \cdot L(\vec{x}) + 2L(\vec{x}) \cdot L(\vec{y}) + L(\vec{y}) \cdot L(\vec{y}) &= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2 \\
\|\vec{x}\|^2 + 2L(\vec{x}) \cdot L(\vec{y}) + \|\vec{y}\|^2 &= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2 \\
L(\vec{x}) \cdot L(\vec{y}) &= \vec{x} \cdot \vec{y}
\end{aligned}$$

as required.

(b) The columns of $[L]$ are $L(\vec{e}_1)$ and $L(\vec{e}_2)$. We know that

$$L(\vec{e}_1) \cdot L(\vec{e}_2) = \vec{e}_1 \cdot \vec{e}_2 = 0$$

so the columns of $[L]$ are orthogonal. Moreover, $\|L(\vec{e}_i)\| = \|\vec{e}_i\| = 1$ for $i = 1, 2$, so the columns have length 1. We may always write a vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ of length 1 as $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ for some θ since we have $1 = \|\vec{v}\|^2 = v_1^2 + v_2^2$. If we take this to be the first column of L, then by orthogonality the second column must be either $\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ or $\begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$. In the first case, the matrix of the isometry L is

$$[L] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

and the isometry L is a rotation through angle θ. In the second case,

$$[L] = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

and L is a reflection in the line that makes angle $\theta/2$ with the x_1-axis, by Problem F4.

F6 (a) Add the two equations: $(A + B)X + (A + B)Y = C + D$. Since $(A + B)^{-1}$ exists,

$$X + Y = (A + B)^{-1}(C + D)$$

Similarly, by subtraction, $(A - B)X - (A - B)Y = C - D$, so

$$X - Y = (A - B)^{-1}(C - D)$$

Hence, by addition

$$\begin{aligned} X &= \frac{1}{2}\Big((A + B)^{-1}(C + D) + (A - B)^{-1}(C - D)\Big) \\ &= \frac{1}{2}\Big((A + B)^{-1} + (A - B)^{-1}\Big)C + \frac{1}{2}\Big((A + B)^{-1} - (A - B)^{-1}\Big)D \end{aligned}$$

Similarly,

$$Y = \frac{1}{2}\Big((A + B)^{-1} - (A - B)^{-1}\Big)C + \frac{1}{2}\Big((A + B)^{-1} + (A - B)^{-1}\Big)D$$

Thus we have shown that the required X and Y exist.

(b) Using block multiplication, we can write the original equations as

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} C \\ D \end{bmatrix}$$

We have seen that this system has a unique solution for arbitrary C and D. Consider only the first column of the matrix $\begin{bmatrix} C \\ D \end{bmatrix}$ and the first columns of the matrix $\begin{bmatrix} X \\ Y \end{bmatrix}$. The system

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix} \vec{x} = \vec{c}$$

must have a unique solution, so the coefficient matrix is invertible. From the solution for X and Y, we see that

$$\begin{bmatrix} A & B \\ B & A \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} (A + B)^{-1} + (A - B)^{-1} & (A + B)^{-1} - (A - B)^{-1} \\ (A + B)^{-1} - (A - B)^{-1} & (A + B)^{-1} + (A - B)^{-1} \end{bmatrix}$$

CHAPTER 4 Vector Spaces

4.1 Spaces of Polynomials

Practice Problems

A1 (a) $(2 - 2x + 3x^2 + 4x^3) + (-3 - 4x + x^2 + 2x^3) = -1 - 6x + 4x^2 + 6x^3$

(b) $(-3)(1 - 2x + 2x^2 + x^3 + 4x^4) = -3 + 6x - 6x^2 - 3x^3 - 12x^4$

(c) $(2 + 3x + x^2 - 2x^3) - 3(1 - 2x + 4x^2 + 5x^3) = -1 + 9x - 11x^2 - 17x^3$

(d) $(2 + 3x + 4x^2) - (5 + x - 2x^2) = -3 + 2x + 6x^2$

(e) $-2(-5 + x + x^2) + 3(-1 - x^2) = 7 - 2x - 5x^2$

(f) $2\left(\frac{2}{3} - \frac{1}{3}x + 2x^2\right) + \frac{1}{3}\left(3 - 2x + x^2\right) = \frac{7}{3} - \frac{4}{3}x + \frac{13}{3}x^2$

(g) $\sqrt{2}(1 + x + x^2) + \pi(-1 + x^2) = \sqrt{2} - \pi + \sqrt{2}x + (\sqrt{2} + \pi)x^2$

A2 (a) Clearly $0 \in \operatorname{Span}\mathcal{B}$ as $0 = 0(1 + x^2 + x^3) + 0(2 + x + x^3) + 0(-1 + x + 2x^2 + x^3)$.

(b) $2 + 4x + 3x^2 + 4x^3$ is in the span of $\mathcal{B}$ if and only if there exists real numbers t_1, t_2, and t_3, such that

$$\begin{aligned} 2 + 4x + 3x^2 + 4x^3 &= t_1(1 + x^2 + x^3) + t_2(2 + x + x^3) + t_3(-1 + x + 2x^2 + x^3) \\ &= (t_1 + 2t_2 - t_3) + (t_2 + t_3)x + (t_1 + 2t_3)x^2 + (t_1 + t_2 + t_3)x^3 \end{aligned}$$

Since two polynomials are equal if and only if the coefficients of like powers of x are equal, this gives the system of linear equations

$$\begin{aligned} t_1 + 2t_2 - t_3 &= 2 \\ t_2 + t_3 &= 4 \\ t_1 + 2t_3 &= 3 \\ t_1 + t_2 + t_3 &= 4 \end{aligned}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 1 & 4 \\ 1 & 0 & 2 & 3 \\ 1 & 1 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 5 & 9 \\ 0 & 0 & 0 & 3/5 \end{array}\right]$$

The system is inconsistent, so $2 + 4x + 3x^2 + 4x^3$ is not in the span.

(c) Repeating our work in part (a) we get a system of linear equations

$$\begin{aligned} t_1 + 2t_2 - t_3 &= 0 \\ t_2 + t_3 &= -1 \\ t_1 + 2t_3 &= 2 \\ t_1 + t_2 + t_3 &= 1 \end{aligned}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system is consistent with solution $\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$. Thus, $-x + 2x^2 + x^3$ is in the span and

$$-x + 2x^2 + x^3 = 2(1 + x^2 + x^3) + (-1)(2 + x + x^3) + 0(-1 + x + 2x^2 + x^3)$$

(d) Repeating our work in part (a) we get a system of linear equations

$$\begin{aligned} t_1 + 2t_2 - t_3 &= -4 \\ t_2 + t_3 &= -1 \\ t_1 + 2t_3 &= 3 \\ t_1 + t_2 + t_3 &= 0 \end{aligned}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -4 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 2 & 3 \\ 1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system is consistent with solution $\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Thus, $-4 - x + 3x^2$ is in the span and

$$-4 - x + 3x^2 = 1(1 + x^2 + x^3) + (-2)(2 + x + x^3) + 1(-1 + x + 2x^2 + x^3)$$

(e) Repeating our work in part (a) we get a system of linear equations

$$\begin{aligned} t_1 + 2t_2 - t_3 &= -1 \\ t_2 + t_3 &= 7 \\ t_1 + 2t_3 &= 5 \\ t_1 + t_2 + t_3 &= 4 \end{aligned}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & 1 & 1 & 7 \\ 1 & 0 & 2 & 5 \\ 1 & 1 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system is consistent with solution $\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 4 \end{bmatrix}$. Thus, $-1 - 7x + 5x^2 + 4x^3$ is in the span and

$$-1 - 7x + 5x^2 + 4x^3 = (-3)(1 + x^2 + x^3) + 3(2 + x + x^3) + 4(-1 + x + 2x^2 + x^3)$$

(f) Repeating our work in part (a) we get a system of linear equations

$$\begin{aligned} t_1 + 2t_2 - t_3 &= 2 \\ t_2 + t_3 &= 1 \\ t_1 + 2t_3 &= 0 \\ t_1 + t_2 + t_3 &= 5 \end{aligned}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

The system is inconsistent, so $2 + x + 5x^3$ is not in the span.

A3 (a) Consider

$$\begin{aligned} 0 &= t_1(1 + 2x + x^2 - x^3) + t_2(5x + x^2) + t_3(1 - 3x + 2x^2 + x^3) \\ &= (t_1 + t_3) + (2t_1 + 5t_2 - 3t_3)x + (t_1 + t_2 + 2t_3)x^2 + (-t_1 + t_3)x^3 \end{aligned}$$

Comparing coefficients of powers of x we get the homogeneous system of equations

$$\begin{aligned} t_1 + t_3 &= 0 \\ 2t_1 + 5t_2 - 3t_3 &= 0 \\ t_1 + t_2 + 2t_3 &= 0 \\ -t_1 + t_3 &= 0 \end{aligned}$$

Row reducing the coefficient matrix of the system gives

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 5 & -3 \\ 1 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the only solution is $t_1 = t_2 = t_3 = 0$, so the set is linearly independent.

(b) Consider

$$\begin{aligned}0 &= t_1(1+x+x^2)+t_2x+t_3(x^2+x^3)+t_4(3+2x+2x^2-x^3)\\ &= (t_1+3t_4)+(t_1+t_2+2t_4)x+(t_1+t_3+2t_4)x^2+(t_3-t_4)x^3\end{aligned}$$

Comparing coefficients of powers of x we get the homogeneous system of equations

$$\begin{aligned}t_1+3t_4&=0\\ t_1+t_2+2t_4&=0\\ t_1+t_3+2t_4&=0\\ t_3-t_4&=0\end{aligned}$$

Row reducing the coefficient matrix of the system gives

$$\begin{bmatrix}1&0&0&3\\1&1&0&2\\1&0&1&2\\0&0&1&-1\end{bmatrix}\sim\begin{bmatrix}1&0&0&3\\0&1&0&-1\\0&0&1&-1\\0&0&0&0\end{bmatrix}$$

t_4 is a free variable so the system has infinitely many solutions, so the the set is linearly dependent. The general solution of the homogeneous system is $\begin{bmatrix}t_1\\t_2\\t_3\\t_4\end{bmatrix}=t\begin{bmatrix}-3\\1\\1\\1\end{bmatrix}$, so we have

$$0=(-3t)(1+x+x^2)+tx+t(x^2+x^3)+t(3+2x+2x^2-x^3),\quad t\in\mathbb{R}$$

(c) Consider

$$\begin{aligned}0 &= t_1(3+x+x^2)+t_2(4+x-x^2)+t_3(1+2x+x^2+2x^3)+t_4(-1+5x^2+x^3)\\ &= (3t_1+4t_2+t_3-t_4)+(t_1+t_2+2t_3)x+(t_1-t_2+t_3+5t_4)x^2+(2t_3+t_4)x^3\end{aligned}$$

Comparing coefficients of powers of x we get the homogeneous system of equations

$$\begin{aligned}3t_1+4t_2+t_3-t_4&=0\\ t_1+t_2+2t_3&=0\\ t_1-t_2+t_3+5t_4&=0\\ 2t_3+t_4&=0\end{aligned}$$

Row reducing the coefficient matrix of the system gives

$$\begin{bmatrix}3&4&1&-1\\1&1&2&0\\1&-1&1&5\\0&0&2&1\end{bmatrix}\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

Hence, the only solution is $t_1=t_2=t_3=t_4=0$, so the set is linearly independent.

(d) Consider

$$\begin{aligned}0 &= t_1(1 + x + x^3 + x^4) + t_2(2 + x - x^2 + x^3 + x^4) + t_3(x + x^2 + x^3 + x^4)\\ &= (t_1 + 2t_2) + (t_1 + t_2 + t_3)x + (-t_2 + t_3)x^2 + (t_1 + t_2 + t_3)x^3 + (t_1 + t_2 + t_3)x^4\end{aligned}$$

Comparing coefficients of powers of x we get the homogeneous system of equations

$$\begin{aligned}t_1 + 2t_2 &= 0\\ t_1 + t_2 + t_3 &= 0\\ -t_2 + t_3 &= 0\\ t_1 + t_2 + t_3 &= 0\\ t_1 + t_2 + t_3 &= 0\end{aligned}$$

Row reducing the coefficient matrix of the system gives

$$\begin{bmatrix}1 & 2 & 0\\ 1 & 1 & 1\\ 0 & -1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 2\\ 0 & 1 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

t_3 is a free variable so the system has infinitely many solutions, so the the set is linearly dependent. The general solution of the homogeneous system is $\begin{bmatrix}t_1\\ t_2\\ t_3\end{bmatrix} = t\begin{bmatrix}-2\\ 1\\ 1\end{bmatrix}$, so we have

$$0 = (-2t)(1 + x + x^3 + x^4) + t(2 + x - x^2 + x^3 + x^4) + t(x + x^2 + x^3 + x^4), \quad t \in \mathbb{R}$$

A4 Consider

$$\begin{aligned}a_1 + a_2x + a_3x^2 &= t_1 1 + t_2(x - 1) + t_3(x - 1)^2\\ &= t_1 1 + t_2(x - 1) + t_3(x^2 - 2x + 1)\\ &= (t_1 - t_2 + t_3) + (t_2 - 2t_3)x + t_3x^2\end{aligned}$$

The corresponding augmented matrix is $\left[\begin{array}{ccc|c}1 & -1 & 1 & a_1\\ 0 & 1 & -2 & a_2\\ 0 & 0 & 1 & a_3\end{array}\right]$.

Since there is a leading one in each row the system is consistent for all polynomials $a_1 + a_2x + a_3x^2$. Thus, $\mathcal{B}$ spans P_2. Moreover, since there is a leading one in each column there is a unique solution and so $\mathcal{B}$ is also linearly independent. Therefore, it is a basis for P_2.

Homework Problems

B1 (a) $2 + 6x - 2x^2$

(b) $-4 - 2x - 2x^2 - 6x^3 + 2x^4$

(c) $3x$

(d) $3 + 5x - 2x^2 + 5x^3$

(e) 0

(f) $\frac{7}{4} + \frac{11}{6}x + \frac{3}{4}x^2$

(g) 0

B2 (a) 1 is not in the span.

(b) $5x + 2x^2 + 3x^3 = 3(1 + x) + 2(x + x^2) + (-3)(1 - x^3)$

(c) $3 + x^2 - 4x^3 = (-1)(1 + x) + 1(x + x^2) + 4(1 - x^3)$

(d) $1 + x^3$ is not in the span.

B3 (a) The set is linearly independent.

(b) The set is linearly independent.

(c) The set is linearly dependent. We have

$$0 = (-t)(1 + x + x^3) + t(x + x^3 + x^5) + t(1 - x^5), \quad t \in \mathbb{R}$$

(d) The set is linearly independent.

(e) The set is linearly dependent. We have

$$\begin{aligned} 0 =&(s - t)(1 + 2x + x^2 - x^3) + (-s - 2t)(2 + 3x - x^2 + x^3 + x^4) \\ &+ s(1 + x - 2x^2 + 2x^3 + x^4) + t(1 + 2x + x^2 + x^3 - 3x^4) \\ &+ t(4 + 6x - 2x^2 + 5x^4), \quad s, t \in \mathbb{R} \end{aligned}$$

B4 Consider

$$\begin{aligned} a_1 + a_2x + a_3x^2 + a_4x^3 &= t_1 1 + t_2(x - 2) + t_3(x - 2)^2 + t_4(x - 2)^3 \\ &= (t_1 - 2t_2 + 4t_3 - 8t_4) + (t_2 - 4t_3 + 12t_4)x + (t_3 - 6t_4)x^2 + t_4x^3 \end{aligned}$$

The corresponding augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & -4 & 4 & 8 & a_1 \\ 0 & 1 & -4 & 12 & a_2 \\ 0 & 0 & 1 & -6 & a_3 \\ 0 & 0 & 0 & 1 & a_4 \end{array}\right]$$

Since there is a leading one in each row the system is consistent for all polynomials $a_1 + a_2x + a_3x^2 + a_4s^3$. Thus, $\mathcal{B}$ spans P_3. Moreover, since there is a leading one in each column there is a unique solution and so $\mathcal{B}$ is also linearly independent. Therefore, it is a basis for P_3.

Conceptual Problems

D1 (a) If there does not exists such a polynomial $q(x)$, then the equation

$$t_1p_1(x) + \cdots + t_kp_k(x) = q(x)$$

is consistent for all n-th degree polynomials $q(x)$. This equation corresponds to a system of $n+1$ equations (one for each power of x in the polynomials) in k unknowns $(t_1, \ldots, t_k)$. By Theorem 2.2.2, this implies that $n + 1 \leq k$ which contradicts the fact that $k < n + 1$. Hence, there must exists a polynomial $q(x)$ which is not in the span of $\mathcal{B}$.

(b) Consider $t_1p_1(x) + \cdots + t_kp_k(x) = 0$. As in part (a), this corresponds to a system of $n + 1$ equations in k unknowns. By Theorem 2.2.2, the system has at least $k - (n + 1) > 0$ parameters. Hence, there is a non-trivial solution, so the set is linearly dependent.

4.2 Vector Spaces

Practice Problems

A1 (a) Call the set S. Since the condition of the set is a linear equation, we suspect that S is a subspace. By definition S is a subset of $\mathbb{R}^4$ and $\vec{0} \in S$ since $0 + 2(0) = 0$. Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$ be vectors in S. Then, they satisfy the condition of S, so we have $x_1 + 2x_2 = 0$ and $y_1 + 2y_2 = 0$. We need to show that $\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 \end{bmatrix}$ is in S. We have

$$(x_1 + y_1) + 2(x_2 + y_2) = x_1 + 2x_2 + y_1 + 2y_2 = 0 + 0 = 0$$

so $\vec{x} + \vec{y}$ satisfies the condition of the set and hence $\vec{x} + \vec{y} \in S$. Therefore, S is closed under addition. Let $t \in \mathbb{R}$. Then $t\vec{x} = \begin{bmatrix} tx_1 \\ tx_2 \\ tx_3 \\ tx_4 \end{bmatrix}$ and $tx_1 + 2(tx_2) = t(x_1 + 2x_2) = t(0) = 0$, so $t\vec{x} \in S$. Hence S is also closed under scalar multiplication. Thus, S is a subspace of $\mathbb{R}^4$.

(b) Call the set S. Since the condition of the set is a linear equation, we suspect that S is a subspace. By definition S is a subset of $M(2, 2)$ and $\vec{0} \in S$ since $0 + 2(0) = 0$. Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ be vectors in S. Then, they satisfy the condition of S, so we have $a_1 + 2a_2 = 0$ and $b_1 + 2b_2 = 0$. We need to show that $A + B = \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ a_3 + b_3 & a_4 + b_4 \end{bmatrix}$ is in S. We have

$$(a_1 + b_1) + 2(a_2 + b_2) = a_1 + 2a_2 + b_1 + 2b_2 = 0 + 0 = 0$$

so $A + B$ satisfies the condition of the set and hence $A + B \in S$. Therefore, S is closed under addition. Let $t \in \mathbb{R}$. Then $tA = \begin{bmatrix} ta_1 & ta_2 \\ ta_3 & ta_4 \end{bmatrix}$ and $ta_1 + 2(ta_2) = t(a_1 + 2a_2) = t(0) = 0$, so $tA \in S$. Hence S is also closed under scalar multiplication. Thus, S is a subspace of $M(2, 2)$.

(c) Call the set S. Since the condition of the set is a linear equation, we suspect that S is a subspace. By definition S is a subset of P_3 and $\vec{0} \in S$ since $0 + 2(0) = 0$. Let $p = a_0 + a_1x + a_2x^2 + a_3x^3$ and $q = b_0 + b_1x + b_2x^2 + b_3x^3$ be vectors in S. Then, they satisfy the condition of S, so we have $a_0 + 2a_1 = 0$ and $b_0 + 2b_1 = 0$. We need to show that $p + q = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3$ is in S. We have

$$(a_0 + b_0) + 2(a_1 + b_1) = a_0 + 2a_1 + b_0 + 2b_1 = 0 + 0 = 0$$

so $p + q$ satisfies the condition of the set and hence $p + q \in S$. Therefore, S is closed under addition. Let $t \in \mathbb{R}$. Then $tp = ta_0 + ta_1x + ta_2x^2 + ta_3x^3$ and $ta_0 + 2(ta_1) = t(a_0 + 2a_1) = t(0) = 0$, so $tp \in S$. Hence S is also closed under scalar multiplication. Thus, S is a subspace of P_3.

(d) At first glance we may be tempted to think that this is a subspace. In fact, if one is not careful, one could think they have a proof that it is a subspace. The important thing to remember in this problem is that the scalars in a vector space can be any real number. So, the condition that the entries of the matrices in the set are integers should be problematic. Indeed, observe that $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is in the set, but the scalar multiple $\sqrt{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$ is not in the set since the entries are not integers. Thus, the set is not closed under scalar multiplication and hence is not a subspace.

(e) Since the condition of the set involves multiplication of entries we expect that it is not a subspace. Observe that $\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ are both in the set since $2(3) - 1(6) = 0$ and $1(4) - 2(2) = 0$. But, their sum $\begin{bmatrix} 3 & 3 \\ 8 & 7 \end{bmatrix}$ is not in the set since $3(7) - 3(8) \neq 0$.

(f) Call the set S. The condition of the set is linear, so we suspect that this is a subspace. By definition S is a subset of $M(2,2)$ and $\vec{0} \in S$ since $0 = 0$. Let $A = \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 \\ 0 & 0 \end{bmatrix}$ be vectors in S. Then, they satisfy the condition of S, so we have $a_1 = a_2$ and $b_1 = b_2$. We need to show that $A+B = \begin{bmatrix} a_1 + b_1 & a_2 + b_2 \\ 0 & 0 \end{bmatrix}$ is in S. We have $a_1 + b_1 = a_2 + b_2$ so $A + B$ satisfies the condition of the set and hence $A + B \in S$. Therefore, S is closed under addition. Let $t \in \mathbb{R}$. Then $tA = \begin{bmatrix} ta_1 & ta_2 \\ 0 & 0 \end{bmatrix}$ and $ta_1 = ta_2$, so $tA \in S$. Hence S is also closed under scalar multiplication. Thus, S is a subspace of $M(2,2)$.

A2 Note that we are given that each of the following sets are subsets of $M(n,n)$.

(a) The $n \times n$ zero matrix is a diagonal matrix, so the set is non-empty. Let $A = \operatorname{diag}(a_{11}, \dots, a_{nn})$ and $B = \operatorname{diag}(b_{11}, \dots, b_{nn})$, then $A + B$ is the diagonal matrix $\operatorname{diag}(a_{11} + b_{11}, \dots, a_{nn} + b_{nn})$ and for any $t \in \mathbb{R}$, tA is the diagonal matrix $\operatorname{diag}(ta_{11}, \dots, ta_{nn})$. Hence, the set is closed under addition and scalar multiplication and hence is a subspace of $M(n,n)$.

(b) Observe that $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ are both in row echelon form, but $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is not in row echelon form. Thus, the set is not closed under addition and hence is not a subspace.

(c) The $n \times n$ zero matrix $O_{n,n}$ satisfies $O_{n,n}^T = O_{n,n}$, so the zero matrix is in the set. Let A and B be symmetric matrices. Then $A^T = A$ and $B^T = B$. Using properties of the transpose gives $(A + B)^T = A^T + B^T = A + B$ and $(tA)^T = tA^T = tA$, so $(A + B)$ and (tA) are symmetric matrices. Thus, the set is closed under addition and scalar multiplication and hence is a subspace of $M(n,n)$.

(d) The $n \times n$ zero matrix is an upper triangular matrix, so the set is non-empty. Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ 0 & \ddots & \vdots \\ 0 & 0 & a_{nn} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ 0 & \ddots & \vdots \\ 0 & 0 & b_{nn} \end{bmatrix}$ be upper triangular matrices. Then, for any $t \in \mathbb{R}$ we have $A+B = \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ 0 & \ddots & \vdots \\ 0 & 0 & a_{nn}+b_{nn} \end{bmatrix}$ and $tA = \begin{bmatrix} ta_{11} & \cdots & ta_{1n} \\ 0 & \ddots & \vdots \\ 0 & 0 & ta_{nn} \end{bmatrix}$ are both upper triangular, so the set is closed under addition and scalar multiplication and hence is a subspace of $M(n, n)$.

A3 Note that we are given that each of the following sets are subsets of P_5.

(a) The zero polynomial is an even polynomial, so the set is non-empty. Let $p(x)$ and $q(x)$ be even polynomials. Then $p(-x) = p(x)$ and $q(-x) = q(x)$. Then, for any $t \in \mathbb{R}$ we have $(p+q)(-x) = p(-x) + q(-x) = p(x) + q(x) = (p+q)(x)$ and $(tp)(-x) = tp(-x) = tp(x) = (tp)(x)$, so $p+q$ and tp are even polynomials. Thus, the set is closed under addition and scalar multiplication and hence is a subspace of P_5.

(b) Since P_3 is a vector space it contains the zero polynomial. Thus, $(1+x^2)(0) = 0$ is in the set so it also contains the zero polynomial and hence is non-empty. Let $q_1(x)$ and $q_2(x)$ be polynomials in the set. Then there exists polynomials $p_1(x)$ and $p_2(x)$ in P_3 such that $q_1(x) = (1+x^2)p_1(x)$ and $q_2(x) = (1+x^2)p_2(x)$. We have

$$(q_1+q_2)(x) = q_1(x) + q_2(x) = (1+x^2)p_1(x) + (1+x^2)p_2(x) = (1+x^2)(p_1(x)+p_2(x))$$

But, P_3 is a vector space and so is closed under addition. Hence, $p_1(x) + p_2(x) \in P_3$ and so $q_1 + q_2$ is in the set. Therefore the set is closed under addition. Similarly, since P_3 is closed under scalar multiplication, for any $t \in \mathbb{R}$ we get $tq_1(x) = t(1+x^2)p_1(x) = (1+x^2)(tp_1(x))$ is in the set. So, the set is also closed under scalar multiplication and hence is a subspace of P_5.

(c) The zero polynomial is in the set since its satisfies the conditions of the set. Let $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ and $q(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4$ be polynomials in the set. Then $a_0 = a_4$, $a_1 = a_3$, $b_0 = b_4$, and $b_1 = b_3$. Then, $p(x) + q(x) = (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + (a_3+b_3)x^3 + (a_4+b_4)x^4$ satisfies $a_0 + b_0 = a_4 + b_4$ and $a_1 + b_1 = a_3 + b_3$ so it is in the set. Therefore, the set is closed under addition. Similarly, for any $t \in \mathbb{R}$, $tp(x) = ta_0 + ta_1x + ta_2x^2 + ta_3x^3 + ta_4x^4$ is in the set since $ta_0 = ta_4$ and $ta_1 = ta_3$. Therefore, the set is also closed under scalar multiplication and hence is a subspace of P_5.

(d) The zero polynomial does not satisfy the condition of the set, so it cannot be a subspace.

(e) The set is equal to $\mathrm{Span}\{1, x, x^2\}$ and hence is a subspace of P_5 by Theorem 4.2.2.

A4 Note that we are given that each of the following sets are subsets of $\mathcal{F}$.

(a) Call the set S. The zero vector in $\mathcal{F}$ is the function which maps all x to 0. Thus, it certainly maps 3 to 0 and hence is in S. Let $f, g \in S$. Then $f(3) = 0$ and $g(3) = 0$. Therefore, we get $(f+g)(3) = f(3) + g(3) = 0 + 0 = 0$ and $(tf)(3) = tf(3) = t(0) = 0$ for any $t \in \mathbb{R}$. Therefore, S is closed under addition and scalar multiplication and hence is a subspace of $\mathcal{F}$.

(b) The zero vector in $\mathcal{F}$ does not satisfy the condition of the set, so it cannot be a subspace.

(c) Call the set S. The zero vector in $\mathcal{F}$ is even so it is in S. Let $f, g \in S$. Then $f(-x) = f(x)$ and $g(-x) = g(x)$. Thus, we get $(f+g)(-x) = f(-x)+g(-x) = f(x)+g(x) = (f+g)(x)$ and $(tf)(-x) = tf(-x) = tf(x) = (tf)(x)$ for any $t \in \mathbb{R}$. Therefore, S is closed under addition and scalar multiplication and hence is a subspace of $\mathcal{F}$.

(d) Observe that $f(x) = x^2 + 1$ is a function in the set since $f(x) \geq 0$ for all $x \in \mathbb{R}$. But then $(-1)f(x) = -x^2 - 1 < 0$ for all $x \in \mathbb{R}$ and hence is not in the set. Consequently, the set is not closed under scalar multiplication and hence is not a subspace.

A5 Let the set be $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ and assume that $\mathbf{v}_i$ is the zero vector. Then we have

$$\mathbf{0} = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_{i-1} + \mathbf{v}_i + 0\mathbf{v}_{i+1} + \cdots + 0\mathbf{v}_k$$

Hence, by definition, $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly dependent.

Homework Problems

B1 (a) It is a subspace.

(b) It is a subspace.

(c) It is a subspace.

(d) It is a subspace.

(e) It is not a subspace.

(f) It is not a subspace.

B2 (a) It is a subspace.

(b) It is not a subspace.

(c) It is a subspace.

(d) It is not a subspace.

(e) It is a subspace.

B3 (a) It is a subspace.

(b) It is not a subspace.

(c) It is not a subspace.

(d) It is a subspace.

(e) It is a subspace.

B4 (a) It is a subspace.

(b) It is not a subspace.

(c) It is not a subspace.

(d) It is not a subspace.

Conceptual Problems

D1 (a) We have

$$\begin{aligned}(-1)\mathbf{x} &= \mathbf{0} + (-1)\mathbf{x} \quad \text{by V3}\\ &= (-\mathbf{x}) + \mathbf{x} + (-1)\mathbf{x} \quad \text{by V4}\\ &= (-\mathbf{x}) + 1\mathbf{x} + (-1)\mathbf{x} \quad \text{by V10}\\ &= (-\mathbf{x}) + (1 + (-1))\mathbf{x} \quad \text{by V8}\\ &= (-\mathbf{x}) + 0\mathbf{x} \quad \text{by addition in } \mathbb{R}\\ &= (-\mathbf{x}) + \mathbf{0} \quad \text{by Theorem 4.2.1 part (1)}\\ &= (-\mathbf{x}) \quad \text{by V3}\end{aligned}$$

(b) Assume $\mathbf{0}_1$ and $\mathbf{0}_2$ are two zero vectors in $\mathbb{V}$. Then, we have

$$\begin{aligned}\mathbf{0}_1 &= \mathbf{0}_1 + \mathbf{0}_2 \quad \text{by V3 since } \mathbf{0}_2 \text{ is a zero vector}\\ &= \mathbf{0}_2 \quad \text{by V3 since } \mathbf{0}_1 \text{ is a zero vector}\end{aligned}$$

(c) If $t = 0$, then the result follows by Theorem 4.2.1 part (1). If $t \neq 0$, then we have

$$\begin{aligned}t\mathbf{0} &= t\mathbf{0} + \mathbf{0} \quad \text{by V3}\\ &= t\left(\frac{1}{t}\mathbf{0} - \frac{1}{t}\mathbf{0}\right) + \mathbf{0} \quad \text{by V4 and Theorem 4.2.1. part (2)}\\ &= \frac{t}{t}\mathbf{0} - \frac{t}{t}\mathbf{0} + \mathbf{0} \quad \text{by V9}\\ &= \mathbf{0} - \mathbf{0} + \mathbf{0} \quad \text{by V10}\\ &= \mathbf{0} \quad \text{by V4 and Theorem 4.2.1 part (2)}\end{aligned}$$

D2 We need to show all that all ten axioms of a vector space hold. Let $\mathbf{x} = (a, b) \in \mathbb{V}$, $\mathbf{y} = (c, d) \in \mathbb{V}$, $\mathbf{z} = (e, f) \in \mathbb{V}$ and $s, t \in \mathbb{R}$.

V1. $\mathbf{x} \oplus \mathbf{y} = (ad + bc, bd)$ and $bd > 0$ since $b > 0$ and $d > 0$, hence $\mathbf{x} \oplus \mathbf{y} \in \mathbb{V}$.

V2. $(\mathbf{x} \oplus \mathbf{y}) \oplus \mathbf{z} = (ad + bc, bd) \oplus (e, f) = (adf + bcf + bde, bdf) = (a, b) \oplus (cf + de, df) = \mathbf{x} \oplus (\mathbf{y} \oplus \mathbf{z})$.

V3. $(0, 1) \in \mathbb{V}$ and $(a, b) \oplus (0, 1) = (a(1) + b(0), b(1)) = (a, b)$. Hence $\mathbf{0} = (0, 1)$ is the zero element.

V4. The additive inverse of $\vec{x}$ is $(-ab^{-2}, b^{-1})$ which is in $\mathbb{V}$ since $b^{-1} > 0$ and

$$(a, b) \oplus (-ab^{-2}, b^{-1}) = (a(b^{-1}) + (b)(-ab^{-2}), b(b^{-1})) = (\frac{a}{b} - \frac{a}{b}, 1) = (0, 1) = \mathbf{0}$$

V5. $\mathbf{x} \oplus \mathbf{y} = (ad + bc, bd) = \mathbf{y} \oplus \mathbf{x}$.

V6. $t \odot \mathbf{x} = (tab^{t-1}, b^t) \in \mathbb{V}$ as $b^k > 0$ since $b > 0$.

V7.

$$\begin{aligned}s \odot (t \odot \mathbf{x}) &= s \odot (tab^{t-1}, b^t) = (s(tab^{t-1})(b^t)^{s-1}, (b^t)^s)\\ &= \left((st)ab^{st-1}, b^{st}\right) = (st) \odot \mathbf{x}\end{aligned}$$

V8.

$$\begin{aligned}(s+t)\odot \mathbf{x} &= \big((s+t)ab^{s+t-1}, b^{s+t}\big) = \big((sab^{s-1})(b^t) + (b^s)(tab^{s-1}), b^s b^t\big)\\ &= (sab^{s-1}, b^s) \oplus (tab^{t-1}, b^t) = [s\odot \mathbf{x}] \oplus [t \odot \mathbf{x}]\end{aligned}$$

V9.

$$\begin{aligned}t\odot(\mathbf{x}\oplus\mathbf{y}) &= t\odot(ad+bc, bd) = (t(ad+bc)(bd)^{t-1}, (bd)^t)\\ &= (tad(bd)^{t-1} + tbc(bd)^{t-1}, b^t d^t) = (tcd^{t-1}b^t + tab^{t-1}d^t, b^t d^t)\\ &= (tab^{t-1}, b^t) \oplus (tcd^{t-1}, d^t) = [t\odot(a,b)] \oplus [t\odot(c,d)] = [t\odot\mathbf{x}] \oplus [t\odot\mathbf{y}]\end{aligned}$$

V10. $1\odot \mathbf{x} = (1ab^{1-1}, b^1) = \mathbf{x}$.

Hence, $\mathbb{V}$ is a vector space.

D3 We need to show all that all ten axioms of a vector space hold. Let $x, y, z \in \mathbb{V}$ and $s, t \in \mathbb{R}$. Then, $x > 0$, $y > 0$, and $z > 0$.

V1: Since $x, y > 0$ we have $x \oplus y = xy > 0$ so $x \oplus y \in \mathbb{V}$.

V2: $x \oplus (y \oplus z) = x \oplus yz = x(yz) = (xy)z = (xy) \oplus z = (x \oplus y) \oplus z$.

V3: Observe that $1 \in \mathbb{V}$ and $x \oplus 1 = x(1) = x$ for all $x \in \mathbb{V}$. Thus, the zero vector of $\mathbb{V}$ is $\mathbf{0} = 1$.

V4: We have $x \oplus \frac{1}{x} = x\left(\frac{1}{x}\right) = 1 = \mathbf{0}$, so the additive inverse of x is $\frac{1}{x}$. Moreover, $\frac{1}{x} \in \mathbb{V}$ since $\frac{1}{x} > 0$ whenever $x > 0$.

V5: $x \oplus y = xy = yx = y \oplus x$

V6: $t \odot x = x^t \in \mathbb{V}$ since $x^t > 0$ whenever $x > 0$.

V7: $s \odot (t \odot x) = s \odot (x^t) = (x^t)^s = x^{ts} = (ts) \odot x$.

V8: $(s+t) \odot x = x^{s+t} = x^s x^t = x^s \oplus x^t = [s \odot x] \oplus [t \odot x]$.

V9: $t(x \oplus y) = t(xy) = (xy)^t = x^t y^t = x^t \oplus y^t = [t \odot x] \oplus [t \odot y]$.

V10: $1x = x^1 = x$

Hence, $\mathbb{V}$ is a vector space.

D4 We need to show all that all ten axioms of a vector space hold. Let $L, M, N \in \mathbb{L}$ and $s, t \in \mathbb{R}$. Then, L, M, and N are linear mappings from $\mathbb{V}$ to $\mathbb{R}^n$.

V1: By Theorem 3.2.4 $L + M$ is a linear mapping from $\mathbb{R}^n \to \mathbb{R}^n$, hence $L + M \in \mathbb{L}$.

V2: For any $\vec{x} \in \mathbb{R}^n$ we have

$$\begin{aligned}(L + (M+N))(\vec{x}) &= L(\vec{x}) + (M+N)(\vec{x}) = L(\vec{x}) + \big(M(\vec{x}) + N(\vec{x})\big)\\ &= \big(L(\vec{x}) + M(\vec{x})\big) + N(\vec{x}) = (L+M)(\vec{x}) + N(\vec{x})\\ &= ((L+M)+N)(\vec{x})\end{aligned}$$

Thus, $L + (M+N) = (L+M)+N$.

V3: Let $Z : \mathbb{R}^n \to \mathbb{R}^n$ be the linear mapping defined by $Z(\vec{x}) = \vec{0}$ for all $\vec{x} \in \mathbb{R}^n$. Then, for any $\vec{x} \in \mathbb{R}^n$ we have

$$(L + Z)(\vec{x}) = L(\vec{x}) + Z(\vec{x}) = L(\vec{x}) + \vec{0} = L(\vec{x})$$

Thus, $L + Z = L$. Since $Z(\vec{x})$ is linear, we have $Z(\vec{x}) \in \mathbb{L}$. Hence, $Z(\vec{x})$ is the zero vector of $\mathbb{L}$.

V4: For any $L \in \mathbb{L}$, define $(-L)$ by $(-L)(\vec{x}) = (-1)L(\vec{x})$. Then, for any $\vec{x} \in \mathbb{R}^n$ we have

$$(L + (-L))(\vec{x}) = L(\vec{x}) + (-L)(\vec{x}) = L(\vec{x}) - L(\vec{x}) = \vec{0} = Z(\vec{x})$$

Thus, $L + (-L) = Z$. Moreover, it is easy to verify that $(-L)$ is linear, so $(-L) \in \mathbb{L}$.

The other vector space axioms are proven in a similar manner.

D5 (a) We need to show all that all ten axioms of a vector space hold. Let $(\mathbf{u}_1, \mathbf{v}_1), (\mathbf{u}_2, \mathbf{v}_2), (\mathbf{u}_3, \mathbf{v}_3) \in \mathbb{U} \times \mathbb{V}$ and $s, t \in \mathbb{R}$. Then $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbb{U}$ and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{V}$.

V1: Observe that $\mathbf{u}_1 + \mathbf{u}_2 \in \mathbb{U}$ and $\mathbf{v}_1 + \mathbf{v}_2 \in \mathbb{V}$ as $\mathbb{U}$ and $\mathbb{V}$ are both vector spaces. Thus,

$$(\mathbf{u}_1, \mathbf{v}_1) \oplus (\mathbf{u}_2, \mathbf{v}_2) = (\mathbf{u}_1 + \mathbf{u}_2, \mathbf{v}_1 + \mathbf{v}_2) \in \mathbb{V}$$

V2:

$$\begin{aligned}
(\mathbf{u}_1, \mathbf{v}_1) \oplus [(\mathbf{u}_2, \mathbf{v}_2) \oplus (\mathbf{u}_3, \mathbf{v}_3)] &= (\mathbf{u}_1, \mathbf{v}_1) \oplus (\mathbf{u}_2 + \mathbf{u}_3, \mathbf{v}_2 + \mathbf{v}_3) \\
&= (\mathbf{u}_1 + (\mathbf{u}_2 + \mathbf{u}_3), \mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3)) \\
&= ((\mathbf{u}_1 + \mathbf{u}_2) + \mathbf{u}_3, (\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3) \\
&= (\mathbf{u}_1 + \mathbf{u}_2, \mathbf{v}_1 + \mathbf{v}_2) \oplus (\mathbf{u}_3, \mathbf{v}_3) \\
&= [(\mathbf{u}_1, \mathbf{v}_1) \oplus (\mathbf{u}_2, \mathbf{v}_2)] \oplus (\mathbf{u}_3, \mathbf{v}_3)
\end{aligned}$$

V3: $(\mathbf{u}_1, \mathbf{v}_1) \oplus (\mathbf{0}_{\mathbb{U}}, \mathbf{0}_{\mathbb{V}}) = (\mathbf{u}_1 + \mathbf{0}_{\mathbb{U}}, \mathbf{v}_1 + \mathbf{0}_{\mathbb{V}}) = (\mathbf{u}_1, \mathbf{v}_1)$. Thus, $(\mathbf{0}_{\mathbb{U}}, \mathbf{0}_{\mathbb{V}}) \in \mathbb{U} \times \mathbb{V}$ is the zero vector of $\mathbb{U} \times \mathbb{V}$.

The other vector space axioms are proven in a similar manner.

(b) By definition, $\mathbb{U} \times \{\mathbf{0}_{\mathbb{V}}\}$ is a subset of $\mathbb{U} \times \mathbb{V}$. Also, $(\mathbf{0}_{\mathbb{U}}, \mathbf{0}_{\mathbb{V}}) \in \mathbb{U} \times \mathbb{V}$, so it is non-empty. For any $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{U}$ and $t \in \mathbb{R}$ we have

$$(\mathbf{u}_1, \mathbf{0}_{\mathbb{V}}) \oplus (\mathbf{u}_2, \mathbf{0}_{\mathbb{V}}) = (\mathbf{u}_1 + \mathbf{u}_2, \mathbf{0}_{\mathbb{V}}) \in \mathbb{U} \times \mathbb{V}$$

and

$$t \odot (\mathbf{u}_1, \mathbf{0}_{\mathbb{V}}) = (t\mathbf{u}_1, \mathbf{0}_{\mathbb{V}}) \in \mathbb{U} \times \mathbb{V}$$

Thus, $\mathbb{U} \times \{\mathbf{0}_{\mathbb{V}}\}$ is a subspace of $\mathbb{U} \times \mathbb{V}$.

(c) With this rule for scalar multiplication, it is not a subspace because

$$0(\mathbf{u}_1, \mathbf{v}_1) = (\mathbf{0}_{\mathbb{U}}, \mathbf{v}_1) \neq (\mathbf{0}_{\mathbb{U}}, \mathbf{0}_{\mathbb{V}})$$

and $(\mathbf{0}_{\mathbb{U}}, \mathbf{0}_{\mathbb{V}})$ is the zero vector in $\mathbb{U} \times \mathbb{V}$.

4.3 Bases and Dimensions

Practice Problems

A1 (a) Call the set $\mathcal{B}$. To show that it is a basis for $\mathbb{R}^3$ we need to show that Span $\mathcal{B} = \mathbb{R}^3$ and that $\mathcal{B}$ is linearly independent. To prove that Span $\mathcal{B} = \mathbb{R}^3$, we need to show that every vector $\vec{x} \in \mathbb{R}^3$ can be written as a linear combination of the vectors in $\mathcal{B}$. Consider

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + t_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} t_1 + t_2 + 2t_3 \\ t_1 - t_2 + t_3 \\ 2t_1 - t_2 + t_3 \end{bmatrix}$$

Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 2 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Observe that the rank of the coefficient matrix equals the number of rows, so by Theorem 2.2.2, the system is consistent for every $\vec{x} \in \mathbb{R}^3$. Hence, Span $\mathcal{B} = \mathbb{R}^3$. Moreover, since the rank of the coefficient matrix equals the number of columns, there are no parameters in the general solution. Therefore, we have a unique solution when we take $\vec{x} = \vec{0}$, so $\mathcal{B}$ is also linearly independent. Hence, it is a basis for $\mathbb{R}^3$.

(b) Since it only has two vectors in $\mathbb{R}^3$ it cannot span $\mathbb{R}^3$ by Theorem 4.3.4 (2) and hence it cannot be a basis.

(c) Since it has four vectors in $\mathbb{R}^3$ it is linearly dependent by Theorem 4.3.4 (1) and hence cannot be a basis.

(d) Call the set $\mathcal{B}$. To show that it is a basis for $\mathbb{R}^3$ we need to show that Span $\mathcal{B} = \mathbb{R}^3$ and that $\mathcal{B}$ is linearly independent. To check whether $\mathcal{B}$ is linearly independent, we consider

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = t_1 \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} + t_2 \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -t_1 + 2t_2 + t_3 \\ 3t_1 + 4t_2 + 4t_3 \\ 5t_1 + 2t_3 \end{bmatrix}$$

Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix} -1 & 2 & 1 \\ 3 & 4 & 4 \\ 5 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 1 \\ 0 & 10 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the system homogeneous system has infinitely many solutions and consequently the set is linearly dependent. Therefore, it is not a basis.

(e) Call the set $\mathcal{B}$. To show that it is a basis for $\mathbb{R}^3$ we need to show that Span $\mathcal{B} = \mathbb{R}^3$ and that $\mathcal{B}$ is linearly independent. To prove that $\mathcal{B}$ is linearly independent, we consider

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} t_1 + t_2 + 3t_3 \\ -t_1 + 2t_2 \\ t_1 - t_2 + t_3 \end{bmatrix}$$

Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix} 1 & 1 & 3 \\ -1 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the system has a unique solution, so $\mathcal{B}$ is linearly independent. Then, since the dimension of $\mathbb{R}^3$ is 3 and $\mathcal{B}$ is a set of three linearly independent vectors in $\mathbb{R}^3$, we can apply Theorem 4.3.4 (3) to get that $\mathcal{B}$ is also a spanning set for $\mathbb{R}^3$. Thus, $\mathcal{B}$ is a basis for $\mathbb{R}^3$.

A2 Consider

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} + t_2 \begin{bmatrix} 2 \\ 3 \\ 1 \\ 3 \end{bmatrix} + t_3 \begin{bmatrix} 3 \\ 3 \\ 4 \\ 11 \end{bmatrix} + t_4 \begin{bmatrix} 2 \\ 3 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} t_1 + 2t_2 + 3t_3 + 2t_4 \\ t_1 + 3t_2 + 3t_3 + 3t_4 \\ t_1 + t_2 + 4t_3 + t_4 \\ 2t_1 + 3t_2 + 11t_3 + 4t_4 \end{bmatrix}$$

Row reducing the coefficient matrix of the corresponding homogeneous system gives

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 3 & 3 & 3 \\ 1 & 1 & 4 & 1 \\ 2 & 3 & 11 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, the system has a unique solution so the set is linearly independent. By Theorem 4.3.4 (3), a set of four linearly independent vectors in $\mathbb{R}^4$ also spans $\mathbb{R}^4$. Thus, the set is a basis for $\mathbb{R}^4$.

A3 (a) Since we have a spanning set, we only need to remove linearly dependent vectors until we have a linearly independent set. Consider

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + t_3 \begin{bmatrix} 2 \\ 0 \\ 10 \end{bmatrix} + t_4 \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} t_1 + 2t_3 + t_4 \\ -2t_1 + t_2 + t_4 \\ t_1 + 2t_2 + 10t_3 + 7t_4 \end{bmatrix}$$

Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ -2 & 1 & 0 & 1 \\ 1 & 2 & 10 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus we see that the first two columns of the reduced row echelon form make a linearly independent set and that the third and fourth columns can be written as linear combinations of the first two columns. Hence, this is also true about the original matrix. In particular, this tells us that two times the first vector in $\mathcal{B}$ plus four times the second vector in $\mathcal{B}$ equals the third vector, and the first vector in $\mathcal{B}$ plus three times the second vector in $\mathcal{B}$ equals the fourth vector. Thus, one basis of Span $\mathcal{B}$ is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$. Hence, the dimension is 2.

(b) Since we have a spanning set, we only need to remove linearly dependent vectors until we have a linearly independent set. Consider

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = t_1\begin{bmatrix}1\\3\\2\end{bmatrix} + t_2\begin{bmatrix}-2\\-6\\-4\end{bmatrix} + t_3\begin{bmatrix}-1\\-1\\2\end{bmatrix} + t_4\begin{bmatrix}0\\4\\8\end{bmatrix} + t_5\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}t_1 - 2t_2 - t_3\\3t_1 - 6t_2 - t_3 + 4t_4 + t_5\\2t_1 - 4t_2 + 2t_3 + 8t_4 + t_5\end{bmatrix}$$

Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix}1 & -2 & -1 & 0 & 0\\3 & -6 & -1 & 4 & 1\\2 & -4 & 2 & 8 & 1\end{bmatrix} \sim \begin{bmatrix}1 & -2 & 0 & 2 & 0\\0 & 0 & 1 & 2 & 0\\0 & 0 & 0 & 0 & 1\end{bmatrix}$$

Therefore, the first, third, and fifth vectors in $\mathcal{B}$ for a basis for Span $\mathcal{B}$. Hence, the dimension is 3.

A4 (a) Since we have a spanning set, we only need to remove linearly dependent vectors until we have a linearly independent set. Consider

$$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = t_1\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix} + t_2\begin{bmatrix}0 & 1\\3 & -1\end{bmatrix} + t_3\begin{bmatrix}1 & -1\\2 & -3\end{bmatrix} + t_4\begin{bmatrix}2 & 1\\4 & -3\end{bmatrix}$$
$$= \begin{bmatrix}t_1 + t_3 + 2t_4\\t_1 + t_2 - t_3 + t_4\\-t_1 + 3t_2 + 2t_3 + 4t_4\\t_1 - t_2 - 3t_3 - 3t_4\end{bmatrix}$$

Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix}1 & 0 & 1 & 2\\1 & 1 & -1 & 1\\-1 & 3 & 2 & 4\\1 & -1 & -3 & -3\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & 1\\0 & 0 & 1 & 1\\0 & 0 & 0 & 0\end{bmatrix}$$

Therefore, the first three vectors in $\mathcal{B}$ form a basis for Span $\mathcal{B}$. Hence, the dimension is 3.

(b) Consider

$$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = t_1\begin{bmatrix}1 & -1\\-1 & -1\end{bmatrix} + t_2\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} + t_3\begin{bmatrix}0 & 1\\2 & 1\end{bmatrix} + t_4\begin{bmatrix}3 & 0\\1 & -2\end{bmatrix} + t_5\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}$$
$$= \begin{bmatrix}t_1 + t_2 + 3t_4\\-t_1 + t_2 + t_3 + t_5\\-t_1 + t_2 + 2t_3 + t_4\\-t_1 - t_2 + t_3 - 2t_4 + t_5\end{bmatrix}$$

Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix}1 & 1 & 0 & 3 & 0\\-1 & 1 & 1 & 0 & 1\\-1 & 1 & 2 & 1 & 0\\-1 & -1 & 1 & -2 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0 & 2 & 0\\0 & 1 & 0 & 1 & 0\\0 & 0 & 1 & 1 & 0\\0 & 0 & 0 & 0 & 1\end{bmatrix}$$

Therefore, a basis is $\left\{\begin{bmatrix}1 & -1\\-1 & -1\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}, \begin{bmatrix}0 & 1\\2 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}\right\}$. Hence, the dimension is 4. In particular Span $\mathcal{B} = M(2,2)$.

(c) Consider

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + t_3 \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + t_4 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} t_1 + t_4 \\ t_2 + t_3 + t_4 \\ t_2 + t_4 \\ t_1 - t_3 \end{bmatrix}$$

Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, $\mathcal{B}$ is a linearly independent set and hence is a basis for Span $\mathcal{B}$. Hence, the dimension is 4.

A5 (a) The set $\mathcal{B}$ is clearly linearly independent since the highest power of x in each vector is different. Thus, $\mathcal{B}$ is a basis for Span $\mathcal{B}$ and so the dimension is 3.

(b) Consider

$$0 = t_1(1 + x) + t_2(1 - x) + t_3(1 + x^3) + t_4(1 - x^3) = (t_1 + t_2 + t_3 + t_4) + (t_1 - t_2)x + (t_3 - t_4)x^3$$

Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the first three vectors form a basis for Span $\mathcal{B}$ and so the dimension is 3.

(c) Consider

$$\begin{aligned} 0 =& t_1(1 + x + x^2) + t_2(1 - x^3) + t_3(1 - 2x + 2x^2 - x^3) + t_4(1 - x^2 + 2x^3) \\ &+ t_5(x^2 + x^3) \\ =&(t_1 + t_2 + t_3 + t_4) + (t_1 - 2t_3)x + (t_1 + 2t_3 - t_4 + t_5)x^2 \\ &+ (-t_2 - t_3 + 2t_4 + t_5)x^3 \end{aligned}$$

Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & -2 & 0 & 0 \\ 1 & 0 & 2 & -1 & 1 \\ 0 & -1 & -1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 4/7 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2/7 \\ 0 & 0 & 0 & 1 & 1/7 \end{bmatrix}$$

Therefore, the first four vectors form a basis for Span $\mathcal{B}$ and hence the dimension is 4.

A6 Alternate correct answers are possible.

(a) Since a plane is two dimensional, to find a basis for the plane we just need to find two linearly independent vectors in the plane. Observe that the vectors $\begin{bmatrix}1\\2\\0\end{bmatrix}$ and $\begin{bmatrix}1\\0\\2\end{bmatrix}$ satisfy the equation of the plane and hence are in the plane. Thus, $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}\right\}$ is a basis for the plane.

(b) To extend our basis for the plane in part (a) to a basis for $\mathbb{R}^3$ we need to pick one vector which is not a linear combination of the basis vectors for the plane. The basis vectors span the plane, so any vector not in the plane will be linearly independent with the basis vectors. Clearly the normal vector $\vec{n} = \begin{bmatrix}2\\-1\\-1\end{bmatrix}$ is not in the plane, so $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}2\\-1\\-1\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^3$.

A7 Alternate correct answers are possible.

(a) Since a hyperplane in $\mathbb{R}^4$ is three dimensional, we just need to pick three linearly independent vectors which satisfy the equation of the hyperplane. Hence, $\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\end{bmatrix}\right\}$ is a basis for the hyperplane.

(b) To extend our basis for the hyperplane in part (a) to a basis for $\mathbb{R}^4$, we just need to pick one vector which does not lie in the hyperplane. Observe that $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ does not satisfy the equation of the hyperplane, and hence $\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^4$.

A8 The general procedure for finding a basis for a vector space is to find the general form of a vector in the vector space and write it as a linear combination of vectors. This obtains a spanning set for the vector space. Then we just need to remove linearly dependent vectors until we have a basis.

(a) Since $a = -c$, every polynomial in $\mathbb{S}$ has the form

$$a + bx + cx^2 = a + bx - ax^2 = bx + a(1 - x^2)$$

Thus, $\mathcal{B} = \{x, 1 - x^2\}$ spans $\mathbb{S}$. Moreover, the set is clearly linearly independent and hence is a basis. Hence, the dimension of $\mathbb{S}$ is 2.

(b) Every matrix in $\mathbb{S}$ has the form

$$\begin{bmatrix}a & b\\0 & c\end{bmatrix} = a\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + b\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + c\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

Hence, $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ spans $\mathbb{S}$ and is clearly linearly independent, so it is a basis for $\mathbb{S}$. Thus, the dimension of $\mathbb{S}$ is 3.

(c) If $\vec{s} \in \mathbb{S}$, then by the condition of $\mathbb{S}$ we have $x_1 + x_2 + x_3 = 0$. Thus, $x_3 = -x_1 - x_2$, so every vector in $\mathbb{S}$ has the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ -x_1 - x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Hence, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ spans $\mathbb{S}$ and is clearly linearly independent, so it is a basis for $\mathbb{S}$. Thus, the dimension of $\mathbb{S}$ is 2.

(d) Every polynomial $p(x)$ in $\mathbb{S}$ has 2 as a root and thus has $(x - 2)$ as a factor. Since $p(x)$ is also of degree at most 2, by factoring we get that every polynomial in $\mathbb{S}$ has the form

$$(x - 2)(ax + b) = a(x^2 - 2x) + b(x - 2)$$

Hence, $\mathcal{B} = \left\{ x^2 - 2x, x - 2 \right\}$ spans $\mathbb{S}$ and is clearly linearly independent, so it is a basis for $\mathbb{S}$. Thus, the dimension of $\mathbb{S}$ is 2.

(e) Every matrix in $\mathbb{S}$ has the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ -a & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ spans $\mathbb{S}$ and is clearly linearly independent, so it is a basis for $\mathbb{S}$. Thus, the dimension of $\mathbb{S}$ is 3.

Homework Problems

B1 (a) It is not a basis.

(b) Since it has four vectors in $\mathbb{R}^3$ it is linearly dependent and hence cannot be a basis.

(c) It is a basis.

(d) Since it only has two vectors in $\mathbb{R}^3$ it cannot span $\mathbb{R}^3$ and hence it cannot be a basis.

(e) Since the first and last vector are identical, the set is linearly dependent and hence cannot be a basis.

B2 Show that it is a linearly independent spanning set.

B3 (a) Since it only has two vectors in P_2 it cannot span P_2 and hence it cannot be a basis.

(b) It is a basis for P_2.

(c) Since it has four vectors in P_2 it is linearly dependent and hence cannot be a basis.

(d) It is not a basis for P_2.

B4 (a) One possible basis is $\left\{\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}2\\3\\-2\end{bmatrix}\right\}$. Hence, the dimension is 2.

(b) One possible basis is $\left\{\begin{bmatrix}1\\1\\-1\end{bmatrix},\begin{bmatrix}1\\-1\\2\end{bmatrix},\begin{bmatrix}3\\-3\\-2\end{bmatrix}\right\}$. Hence, the dimension is 3.

B5 (a) One possible basis is $\left\{\begin{bmatrix}1&1\\2&1\end{bmatrix},\begin{bmatrix}1&-1\\1&0\end{bmatrix}\right\}$. Hence, the dimension is 2.

(b) One possible basis is $\left\{\begin{bmatrix}1&2\\-1&1\end{bmatrix},\begin{bmatrix}1&2\\2&1\end{bmatrix},\begin{bmatrix}0&1\\0&-2\end{bmatrix}\right\}$. Hence, the dimension is 3.

B6 (a) The dimension is 2.

(b) The dimension is 3.

B7 Alternate correct answers are possible.

(a) $\left\{\begin{bmatrix}4\\0\\-1\end{bmatrix},\begin{bmatrix}3\\-1\\0\end{bmatrix}\right\}$.

(b) $\left\{\begin{bmatrix}4\\0\\-1\end{bmatrix},\begin{bmatrix}3\\-1\\0\end{bmatrix},\begin{bmatrix}1\\3\\4\end{bmatrix}\right\}$.

B8 Alternate correct answers are possible.

(a) $\left\{\begin{bmatrix}1\\-1\\0\\0\end{bmatrix},\begin{bmatrix}0\\2\\-1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\-2\end{bmatrix}\right\}$.

(b) $\left\{\begin{bmatrix}1\\-1\\0\\0\end{bmatrix},\begin{bmatrix}0\\2\\-1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\-2\end{bmatrix},\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right\}$.

B9 (a) One possible basis is $\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$. Hence, the dimension is 2.

(b) One possible basis is $\{1+x^2, x+x^3, x^2+x^4\}$. Hence, the dimension is 3.

(c) One possible basis is $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$. Hence, the dimension is 2.

(d) One possible basis is $\left\{\begin{bmatrix}1&1\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$. Hence, the dimension is 2.

(e) One possible basis is $\left\{1, x^2, x^4\right\}$. Hence, the dimension is 3.

Conceptual Problems

D1 (a) Suppose that $\dim \mathbb{V} = n$. By Theorem 4.3.4, a basis for $\mathbb{V}$ must contain n linearly independent vectors. However, if a set contains more than n vectors, then it is linearly dependent. Therefore, to be a basis, a set must be a linearly independent set with as many vectors as possible - a "maximal linearly independent set" in $\mathbb{V}$.

(b) Suppose that $\dim \mathbb{V} = n$. By Theorem 4.3.4, a basis for $\mathbb{V}$ must contain n vectors which span $\mathbb{V}$. However, if a set contains less that n vectors, then it cannot span all of $\mathbb{V}$. Therefore, to be a basis, we must have a spanning set with as few vectors as possible - a "minimal spanning set" in $\mathbb{V}$.

D2 If $\mathbb{S}$ is an n-dimensional subspace of $\mathbb{V}$, then $\mathbb{S}$ has a basis $\mathcal{B}$ containing n vectors. But then, $\mathcal{B}$ is a linearly independent set of n vectors in $\mathbb{V}$ and hence is a basis for $\mathbb{V}$. Thus, $\mathbb{S} = \operatorname{Span} \mathcal{B} = \mathbb{V}$.

D3 (a) First we show that $\{\mathbf{v}_1, \mathbf{v}_2 + t\mathbf{v}_1\}$ is a spanning set for $\mathbb{V}$. Let $\mathbf{w} \in \mathbb{V}$. Then since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{V}$ we can write $\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2$. We now want to write $\mathbf{w}$ as a linear combination of $\mathbf{v}_1$ and $(\mathbf{v}_2 + t\mathbf{v}_1)$. Observe that

$$\begin{aligned}\mathbf{w} &= a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + ta_2\mathbf{v}_1 - ta_2\mathbf{v}_1\\ &= a_1\mathbf{v}_1 - ta_2\mathbf{v}_1 + a_2\mathbf{v}_2 + ta_2\mathbf{v}_1 = (a_1 - ta_2)\mathbf{v}_1 + a_2(\mathbf{v}_2 + t\mathbf{v}_1)\end{aligned}$$

Thus, $\{\mathbf{v}_1, \mathbf{v}_2 + t\mathbf{v}_1\}$ is a spanning set for $\mathbb{V}$. We then immediately get that $\{\mathbf{v}_1, \mathbf{v}_2 + t\mathbf{v}_1\}$ is a basis for $\mathbb{V}$ since $\dim \mathbb{V} = 2$.

(b) Let $\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 \in \mathbb{V}$. We can write this as

$$\mathbf{w} = (a_1 - ta_3)\mathbf{v}_1 + (a_2 - sa_3)\mathbf{v}_2 + a_3(\mathbf{v}_3 + t\mathbf{v}_1 + s\mathbf{v}_2)$$

Thus, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 + t\mathbf{v}_1 + s\mathbf{v}_2\}$ is a spanning set for $\mathbb{V}$. We then immediately get that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 + t\mathbf{v}_1 + s\mathbf{v}_2\}$ is a basis for $\mathbb{V}$ since $\dim \mathbb{V} = 3$.

D4 We see that $0(a, b) = (0, 1)$ is the zero vector and so cannot be in our basis. Thus, we consider the vector $(1, 1)$. Scalar multiples of $(1, 1)$ give $k(1, 1) = (k(1)(1)^{k-1}, 1^k) = (k, 1)$. This clearly does not span $\mathbb{V}$, so we need to pick another vector which will change the second coordinate. We pick $(0, 2)$. Since $(0, 2)$ is not a multiple of $(1, 1)$, we have that $\mathcal{B} = \{(1, 1), (0, 2)\}$ is linearly independent. Additionally, for any vector $\mathbf{x} = (a, b) \in \mathbb{V}$ we have

$$\begin{aligned}\frac{a}{b}(1, 1) + \log_2 b(0, 2) &= (\frac{a}{b}, 1) + (0, 2^{\log_2 b})\\ &= (\frac{a}{b}, 1) + (0, b)\\ &= (\frac{a}{b}b + (1)(0), (1)b)\\ &= (a, b)\end{aligned}$$

Hence $\mathcal{B}$ also spans $\mathbb{V}$ and thus is a basis. Therefore, $\dim \mathbb{V} = 2$.

4.4 Coordinates with Respect to a Basis

Practice Problems

A1 (a) Since the two vectors are not scalar multiples of each other, the set $\mathcal{B}$ is linearly independent and hence forms a basis for Span $\mathcal{B}$. The coordinates of $\mathbf{x}$ and $\mathbf{y}$ with respect to $\mathcal{B}$ are determined by finding s_1, s_2, t_1, and t_2 such that

$$s_1\begin{bmatrix}1\\0\\1\\1\end{bmatrix}+s_2\begin{bmatrix}-1\\1\\-1\\0\end{bmatrix}=\begin{bmatrix}5\\-2\\5\\3\end{bmatrix}\quad\text{and}\quad t_1\begin{bmatrix}1\\0\\1\\1\end{bmatrix}+t_2\begin{bmatrix}-1\\1\\-1\\0\end{bmatrix}=\begin{bmatrix}-1\\3\\-1\\2\end{bmatrix}$$

Since both systems have the same coefficient matrix, we row reduce the doubly augmented matrix to get

$$\left[\begin{array}{cc|c|c}1&-1&5&-1\\0&1&-2&3\\1&-1&5&-1\\1&0&3&2\end{array}\right]\sim\left[\begin{array}{cc|c|c}1&0&3&2\\0&1&-2&3\\0&0&0&0\\0&0&0&0\end{array}\right]$$

Thus, we see that $s_1=3$, $s_2=-2$, $t_1=2$, and $t_2=3$. Therefore, $[\mathbf{x}]_{\mathcal{B}}=\begin{bmatrix}3\\-2\end{bmatrix}$ and $[\mathbf{y}]_{\mathcal{B}}=\begin{bmatrix}2\\3\end{bmatrix}$.

(b) Consider

$$\begin{aligned}0&=t_1(1+x+x^2)+t_2(1+3x+2x^2)+t_3(4+x^2)\\&=(t_1+t_2+4t_3)+(t_1+3t_2)x+(t_1+2t_2+t_3)x^2\end{aligned}$$

Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix}1&1&4\\1&3&0\\1&2&1\end{bmatrix}\sim\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Hence, the solution is unique so $\mathcal{B}$ is linearly independent and hence a basis for Span $\mathcal{B}$. Next, we need to find s_1, s_2, s_3, t_1, t_2, and t_3 such that

$$s_1(1+x+x^2)+s_2(1+3x+2x^2)+s_3(4+x^2)=-2+8x+5x^2$$

and

$$t_1(1+x+x^2)+t_2(1+3x+2x^2)+t_3(4+x^2)=-4+6x+6x^2$$

Row reducing the corresponding doubly augmented matrix gives

$$\left[\begin{array}{ccc|c|c}1&1&4&-2&-4\\1&3&0&8&8\\1&2&1&5&4\end{array}\right]\sim\left[\begin{array}{ccc|c|c}1&0&0&5&2\\0&1&0&1&2\\0&0&1&-2&-2\end{array}\right]$$

Thus, $[\mathbf{x}]_{\mathcal{B}}=\begin{bmatrix}5\\1\\-2\end{bmatrix}$ and $[\mathbf{y}]_{\mathcal{B}}=\begin{bmatrix}2\\2\\-2\end{bmatrix}$.

(c) Consider

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = t_1 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + t_3 \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} t_1 + 2t_3 & t_1 + t_2 \\ t_1 + t_2 & t_2 - t_3 \end{bmatrix}$$

Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the solution is unique so $\mathcal{B}$ is linearly independent and hence a basis for Span $\mathcal{B}$. Next, we need to find s_1, s_2, s_3, t_1, t_2, and t_3 such that

$$s_1 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + s_2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + s_3 \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

and

$$t_1 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} + t_3 \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 1 & 4 \end{bmatrix}$$

Row reducing the corresponding doubly augmented matrix gives

$$\left[\begin{array}{ccc|c|c} 1 & 0 & 2 & 0 & -4 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & -2 & -2 \\ 0 & 1 & 0 & 3 & 3 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$ and $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$.

(d) Since the two vectors are not scalar multiples of each other, the set $\mathcal{B}$ is linearly independent and hence forms a basis for Span $\mathcal{B}$. We need to find s_1, s_2, t_1, and t_2 such that

$$s_1 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} + s_2 \begin{bmatrix} 0 & 2 & -1 \\ 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 4 & 0 \end{bmatrix}$$

and

$$t_1 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} + t_2 \begin{bmatrix} 0 & 2 & -1 \\ 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 \\ -2 & -3 & 5 \end{bmatrix}$$

Row reducing the corresponding doubly augmented matrix gives

$$\left[\begin{array}{cc|c|c} 1 & 0 & 1 & 3 \\ 1 & 2 & 3 & -1 \\ 0 & -1 & -1 & 2 \\ 0 & 1 & 1 & -2 \\ 1 & 3 & 4 & -3 \\ 1 & -1 & 0 & 5 \end{array}\right] \sim \left[\begin{array}{cc|c|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

(e) Consider

$$0 = t_1(1 + x^2 + x^4) + t_2(1 + x + 2x^2 + x^3 + x^4) + t_3(x - x^2 + x^3 - 2x^4)$$

Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the solution is unique so $\mathcal{B}$ is linearly independent and hence a basis for Span $\mathcal{B}$. Next, we need to find s_1, s_2, s_3, t_1, t_2, and t_3 such that

$$s_1(1 + x^2 + x^4) + s_2(1 + x + 2x^2 + x^3 + x^4) + s_3(x - x^2 + x^3 - 2x^4) = 2 + x - 5x^2 + x^3 - 6x^4$$

and

$$t_1(1 + x^2 + x^4) + t_2(1 + x + 2x^2 + x^3 + x^4) + t_3(x - x^2 + x^3 - 2x^4) = 1 + x + 4x^2 + x^3 + 3x^4$$

Row reducing the corresponding doubly augmented matrix gives

$$\left[\begin{array}{ccc|c|c} 1 & 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & -5 & 4 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & -2 & -6 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 5 & -1 \\ 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -3 \\ 4 \end{bmatrix}$, $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$

A2 (a) It is easy to verify that the two vectors satisfy the equation of the plane and hence are in the plane. Also, they are clearly linearly independent, so we have a set of two linearly independent vectors in a two dimensional vector space, so the set is a basis.

(b) The vectors $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$ do not satisfy the equation of the plane, so they are not in the plane. $\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$ satisfies the equation of the plane, so it is in the plane. To find its $\mathcal{B}$-coordinates, we need to find t_1 and t_2 such that

$$t_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 2 & 2 & 2 \\ 0 & -1 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

A3 (a) Consider

$$0 = t_1(1 + x^2) + t_2(1 - x + 2x^2) + t_3(-1 - x + x^2)$$

Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, $\mathcal{B}$ is a set of three linearly independent vectors in the three dimensional vector space P_2 and so it is a basis for P_2.

(b) To find the coordinates, we need to solve the three systems

$$\begin{aligned} b_1(1 + x^2) + b_2(1 - x + 2x^2) + b_3(-1 - x + x^2) &= 1 \\ s_1(1 + x^2) + s_2(1 - x + 2x^2) + s_3(-1 - x + x^2) &= 4 - 2x + 7x^2 \\ t_1(1 + x^2) + t_2(1 - x + 2x^2) + t_3(-1 - x + x^2) &= -2 - 2x + 3x^2 \end{aligned}$$

Row reducing the triple augmented matrix gives

$$\left[\begin{array}{ccc|c|c|c} 1 & 1 & -1 & 1 & 4 & -2 \\ 0 & -1 & -1 & 0 & -2 & -2 \\ 1 & 2 & 1 & 0 & 7 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & -1 & 4 & 2 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & 3 \end{array}\right]$$

Hence, $[p(x)]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$, $[q(x)]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$, and $[r(x)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$.

(c) We need to solve

$$t_1(1 + x^2) + t_2(1 - x + 2x^2) + t_3(-1 - x + x^2) = 2 - 4x + 10x^2$$

Row reducing the augmented matrix gives

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 0 & -1 & -1 & -4 \\ 1 & 2 & 1 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \end{array}\right]$$

Hence, $[2 - 4x + 10x^2]_{\mathcal{B}} = \begin{bmatrix} 6 \\ 0 \\ 4 \end{bmatrix}$. We then have

$$\begin{aligned} [4 - 2x + 7x^2]_{\mathcal{B}} + [-2 - 2x + 3x^2]_{\mathcal{B}} &= \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 4 \end{bmatrix} \\ &= [2 - 4x + 10x^2]_{\mathcal{B}} = [(4 - 2) + (-2 - 2)x + (7 + 3)x^2]_{\mathcal{B}} \end{aligned}$$

A4 (a) i. We need to determine if there exists t_1, t_2, and t_3 such that

$$t_1 \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} + t_2 \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} + t_3 \begin{bmatrix} -2 & 2 \\ 4 & 10 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & 7 \end{bmatrix}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 1 & 2 & 1 \\ 1 & -1 & 4 & 2 \\ 3 & 2 & 10 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 3/4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Hence, A is in the span of $\mathcal{B}$.

ii. Observe that the augmented system in (i) has a unique solution, thus if we replace A with the zero matrix, we will still have a unique solution and so $\mathcal{B}$ is linearly independent. Thus, it forms a basis for Span $\mathcal{B}$.

iii. Since $\mathcal{B}$ is a basis and $A \in \operatorname{Span}\mathcal{B}$, we can use our result from part (i) to get $[A]_{\mathcal{B}} = \begin{bmatrix} -1/2 \\ 1/2 \\ 3/4 \end{bmatrix}$.

(b) i. We need to determine if there exits t_1, t_2, and t_3 such that

$$t_1 \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} + t_2 \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix} + t_3 \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 9 & 3 \end{bmatrix}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 3 & -2 & 1 & -1 \\ 2 & 1 & -1 & 9 \\ 3 & 2 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & -5 & 1 & -13 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & -4 \end{array}\right]$$

Thus, the system is inconsistent so A is not in the span.

ii. Observe that if we replace A with the zero matrix in the augmented system in part (i) the system will have a unique solution. Hence, $\mathcal{B}$ is linearly independent and forms a basis for Span $\mathcal{B}$.

A5 (a) To find the change of coordinates matrix Q from $\mathcal{B}$-coordinates to $\mathcal{S}$-coordinates we need to find the coordinates of the vectors in $\mathcal{B}$ with respect to the standard basis $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. We get

$$Q = \begin{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}_{\mathcal{S}} & \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{S}} & \begin{bmatrix} -2 \\ -3 \\ 3 \end{bmatrix}_{\mathcal{S}} \end{bmatrix} = \begin{bmatrix} 3 & 0 & -2 \\ 4 & 1 & -3 \\ 1 & 0 & 3 \end{bmatrix}$$

To find the change of coordinates matrix from $\mathcal{S}$-coordinates to $\mathcal{B}$-coordinates, we need to find the coordinates of the vectors in $\mathcal{S}$ with respect to the basis $\mathcal{B}$. So, we need to solve the systems of equations

$$a_1 \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} -2 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$b_1 \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} -2 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$c_1 \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Row reducing the triple augmented matrix gives

$$\left[\begin{array}{ccc|ccc} 3 & 0 & -2 & 1 & 0 & 0 \\ 4 & 1 & -3 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/11 & 0 & 2/11 \\ 0 & 1 & 0 & -15/11 & 1 & 1/11 \\ 0 & 0 & 1 & -1/11 & 0 & 3/11 \end{array}\right]$$

Thus, the change of coordinates matrix P from $\mathcal{S}$-coordinates to $\mathcal{B}$-coordinates is $P = \begin{bmatrix} 3/11 & 0 & 2/11 \\ -15/11 & 1 & 1/11 \\ -1/11 & 0 & 3/11 \end{bmatrix}$.

(b) To find the change of coordinates matrix Q from $\mathcal{B}$-coordinates to $\mathcal{S}$-coordinates we need to find the coordinates of the vectors in $\mathcal{B}$ with respect to the standard basis $\mathcal{S} = \{1, x, x^2\}$. We get

$$Q = \begin{bmatrix} [1 - 2x + 5x^2]_{\mathcal{S}} & [1 - 2x^2]_{\mathcal{S}} & [x + x^2]_{\mathcal{S}} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \\ 5 & -2 & 1 \end{bmatrix}$$

To find the change of coordinates matrix from $\mathcal{S}$-coordinates to $\mathcal{B}$-coordinates, we need to find the coordinates of the vectors in $\mathcal{S}$ with respect to the basis $\mathcal{B}$. So, we need to solve the systems of equations

$$\begin{aligned} a_1(1 - 2x + 5x^2) + a_2(1 - 2x^2) + a_3(x + x^2) &= 1 \\ b_1(1 - 2x + 5x^2) + b_2(1 - 2x^2) + b_3(x + x^2) &= x \\ c_1(1 - 2x + 5x^2) + c_2(1 - 2x^2) + c_3(x + x^2) &= x^2 \end{aligned}$$

Row reducing the triple augmented matrix gives

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \\ 5 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/9 & -1/9 & 1/9 \\ 0 & 1 & 0 & 7/9 & 1/9 & -1/9 \\ 0 & 0 & 1 & 4/9 & 7/9 & 2/9 \end{array}\right]$$

Thus, the change of coordinates matrix P from $\mathcal{S}$-coordinates to $\mathcal{B}$-coordinates is $P = \begin{bmatrix} 2/9 & -1/9 & 1/9 \\ 7/9 & 1/9 & -1/9 \\ 4/9 & 7/9 & 2/9 \end{bmatrix}$.

(c) To find the change of coordinates matrix Q from $\mathcal{B}$-coordinates to $\mathcal{S}$-coordinates we need to find the coordinates of the vectors in $\mathcal{B}$ with respect to the standard basis $\mathcal{S} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$. We have

$$\begin{aligned} \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} &= 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & -4 \\ 0 & -1 \end{bmatrix} &= 0\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-4)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} &= 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Hence,

$$Q = \begin{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}_{\mathcal{S}} & \begin{bmatrix} 0 & -4 \\ 0 & -1 \end{bmatrix}_{\mathcal{S}} & \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}_{\mathcal{S}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ -1 & -4 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

To find the change of coordinates matrix from $\mathcal{S}$-coordinates to $\mathcal{B}$-coordinates, we need to find the coordinates of the vectors in $\mathcal{S}$ with respect to the basis $\mathcal{B}$. So, we need to solve the systems of equations

$$a_1\begin{bmatrix}1 & -1\\0 & -1\end{bmatrix}+a_2\begin{bmatrix}0 & -4\\0 & -1\end{bmatrix}+a_3\begin{bmatrix}2 & 1\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$$
$$b_1\begin{bmatrix}1 & -1\\0 & -1\end{bmatrix}+b_2\begin{bmatrix}0 & -4\\0 & -1\end{bmatrix}+b_3\begin{bmatrix}2 & 1\\0 & 1\end{bmatrix}=\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$$
$$c_1\begin{bmatrix}1 & -1\\0 & -1\end{bmatrix}+c_2\begin{bmatrix}0 & -4\\0 & -1\end{bmatrix}+c_3\begin{bmatrix}2 & 1\\0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$$

Row reducing the triple augmented matrix gives

$$\left[\begin{array}{ccc|ccc}1 & 0 & 2 & 1 & 0 & 0\\-1 & -4 & 1 & 0 & 1 & 0\\-1 & -1 & 1 & 0 & 0 & 1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1/3 & 2/9 & -8/9\\0 & 1 & 0 & 0 & -1/3 & 1/3\\0 & 0 & 1 & 1/3 & -1/9 & 4/9\end{array}\right]$$

So, the change of coordinates matrix P from $\mathcal{S}$-coordinates to $\mathcal{B}$-coordinates is $P=\begin{bmatrix}1/3 & 2/9 & -8/9\\0 & -1/3 & 1/3\\1/3 & -1/9 & 4/9\end{bmatrix}$.

Homework Problems

B1 (a) $[\mathbf{x}]_{\mathcal{B}}=\begin{bmatrix}-3\\4\end{bmatrix}$, $[\mathbf{y}]_{\mathcal{B}}=\begin{bmatrix}2\\-1\end{bmatrix}$

(b) $[\mathbf{x}]_{\mathcal{B}}=\begin{bmatrix}-2\\3\\-1\end{bmatrix}$, $[\mathbf{y}]_{\mathcal{B}}=\begin{bmatrix}-2\\1\\2\end{bmatrix}$

(c) $[\mathbf{x}]_{\mathcal{B}}=\begin{bmatrix}5/2\\-3/2\\-1\end{bmatrix}$, $[\mathbf{y}]_{\mathcal{B}}=\begin{bmatrix}-1\\2\\3\end{bmatrix}$

(d) $[\mathbf{x}]_{\mathcal{B}}=\begin{bmatrix}0\\1\\-1\end{bmatrix}$, $[\mathbf{y}]_{\mathcal{B}}=\begin{bmatrix}2\\1\\1/3\end{bmatrix}$

(e) $[\mathbf{x}]_{\mathcal{B}}=\begin{bmatrix}-2\\4\\-3\end{bmatrix}$, $[\mathbf{y}]_{\mathcal{B}}=\begin{bmatrix}-3\\2\\-1\end{bmatrix}$

B2 (a) Show that it is linearly independent and spans the plane.

(b) $\begin{bmatrix}4\\2\\2\end{bmatrix}$ is not in the plane. The $\mathcal{B}$-coordinates of $\mathbf{x}=\begin{bmatrix}1\\2\\2\end{bmatrix}$ and $\mathbf{y}=\begin{bmatrix}5\\4\\1\end{bmatrix}$ are $[\mathbf{x}]_{\mathcal{B}}=\begin{bmatrix}1/3\\2/3\end{bmatrix}$ and $[\mathbf{y}]_{\mathcal{B}}=\begin{bmatrix}5/3\\1/3\end{bmatrix}$.

B3 (a) Show that it is linearly independent and spans $\mathbb{R}^3$.

(b) $\begin{bmatrix}6\\5\\3\end{bmatrix}_{\mathcal{B}}=\begin{bmatrix}1\\1\\2\end{bmatrix}$, $\begin{bmatrix}4\\5\\-4\end{bmatrix}_{\mathcal{B}}=\begin{bmatrix}3\\-1\\1\end{bmatrix}$, $\begin{bmatrix}5\\4\\1\end{bmatrix}_{\mathcal{B}}=\begin{bmatrix}2\\1\\1\end{bmatrix}$

(c) $\begin{bmatrix} 9 \\ 9 \\ -3 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}$. We have

$$\begin{bmatrix} 4 \\ 5 \\ -4 \end{bmatrix}_{\mathcal{B}} + \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ -3 \end{bmatrix}_{\mathcal{B}}$$

B4 (a) Show that it is linearly independent and spans P_2.

(b) $[3 + 4x + 5x^2]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $[4 + 5x - 7x^2]_{\mathcal{B}} = \begin{bmatrix} -4 \\ 8 \\ -3 \end{bmatrix}$, $[1 + x + x^2]_{\mathcal{B}} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$

(c) $[4 + 5x + 6x^2]_{\mathcal{B}} = \begin{bmatrix} 5/2 \\ 3/2 \\ 7/2 \end{bmatrix}$. We have

$$[3 + 4x + 5x^2]_{\mathcal{B}} + [1 + x + x^2]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 3/2 \\ 7/2 \end{bmatrix} = [4 + 5x + 6x^2]_{\mathcal{B}}$$

B5 (a) i. A is not in the span of $\mathcal{B}$.

ii. $\mathcal{B}$ is linearly dependent, so it does not form a basis for Span $\mathcal{B}$.

(b) i. A is in the span of $\mathcal{B}$.

ii. $\mathcal{B}$ is linearly independent, so it forms a basis for Span $\mathcal{B}$.

iii. $[A]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix}$

B6 (a) The change of coordinates matrix Q from $\mathcal{B}$-coordinates to $\mathcal{S}$-coordinates is $\begin{bmatrix} 1 & 1 & 5 \\ 1 & 1 & 1 \\ 0 & -1 & -3 \end{bmatrix}$. The change of coordinates matrix P from $\mathcal{S}$-coordinates to $\mathcal{B}$-coordinates is $P = \begin{bmatrix} 1/2 & 1/2 & 1 \\ -3/4 & 3/4 & -1 \\ 1/4 & -1/4 & 0 \end{bmatrix}$.

(b) The change of coordinates matrix Q from $\mathcal{B}$-coordinates to $\mathcal{S}$-coordinates is $\begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -1 \\ 2 & 1 & -3 \end{bmatrix}$. The change of coordinates matrix P from $\mathcal{S}$-coordinates to $\mathcal{B}$-coordinates is $P = \begin{bmatrix} 1 & -2 & 1 \\ 1 & -1/2 & 1/2 \\ 1 & -3/2 & 1/2 \end{bmatrix}$.

(c) The change of coordinates matrix Q from $\mathcal{B}$-coordinates to $\mathcal{S}$-coordinates is $\begin{bmatrix} 3 & 5 \\ -2 & 3 \end{bmatrix}$. The change of coordinates matrix P from $\mathcal{S}$-coordinates to $\mathcal{B}$-coordinates is $P = \begin{bmatrix} 3/19 & -5/19 \\ 2/19 & 3/19 \end{bmatrix}$.

Conceptual Problems

D1 Yes, $\mathbf{v}_i = \mathbf{w}_i$ for all i. The easiest way to see this is to consider special cases. The $\mathcal{B}$-coordinates of $\mathbf{v}_1$ are given by $[\mathbf{v}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. We are told that $[\mathbf{v}_1]_{\mathcal{B}} = [\mathbf{v}_1]_{C}$, hence $[\mathbf{v}_1]_{C} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. But this means that

$$\mathbf{v}_1 = \mathbf{w}_1 + 0\mathbf{w}_2 + \cdots + 0\mathbf{w}_n = \mathbf{w}_1$$

Similarly, $\mathbf{v}_i = \mathbf{w}_i$ for all i.

D2 Write the new basis in terms of the old, and make these the columns of the matrix: Let the vectors in $\mathcal{B}$ be denoted $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$. Then we have

$$\begin{aligned}
\mathbf{w}_1 &= 0\mathbf{v}_3 + 0\mathbf{v}_2 + 0\mathbf{v}_4 + 1\mathbf{v}_1 \\
\mathbf{w}_2 &= 0\mathbf{v}_3 + 1\mathbf{v}_2 + 0\mathbf{v}_4 + 0\mathbf{v}_1 \\
\mathbf{w}_3 &= 1\mathbf{v}_3 + 0\mathbf{v}_2 + 0\mathbf{v}_4 + 0\mathbf{v}_1 \\
\mathbf{w}_4 &= 0\mathbf{v}_3 + 0\mathbf{v}_2 + 1\mathbf{v}_4 + 0\mathbf{v}_1
\end{aligned}$$

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

D3 (a) Let $\vec{x} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$. Then, we have $\begin{bmatrix} 3 \\ 5 \end{bmatrix}_{\mathcal{B}} = \left[L\left(\begin{bmatrix} 3 \\ 5 \end{bmatrix}\right)\right]_C$.

Observe that $\begin{bmatrix} 3 \\ 5 \end{bmatrix} = 1\begin{bmatrix} 2 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, hence $\begin{bmatrix} 3 \\ 5 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Thus, $\left[L\left(\begin{bmatrix} 3 \\ 5 \end{bmatrix}\right)\right]_C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ which implies that $L\left(\begin{bmatrix} 3 \\ 5 \end{bmatrix}\right) = 1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

(b) We repeat what we did in (a), but in general. We first need to find c_1 and c_2 such that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1\begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2c_1 + c_2 \\ 3c_1 + 2c_2 \end{bmatrix}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{cc|c} 2 & 1 & x_1 \\ 3 & 2 & x_2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2x_1 - x_2 \\ 0 & 1 & -3x_1 + 2x_2 \end{array}\right]$$

Hence,

$$L(\vec{x}) = (2x_1 - x_2)\begin{bmatrix} 2 \\ 1 \end{bmatrix} + (-3x_1 + 2x_2)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \end{bmatrix}$$

D4 Let Q be the change of coordinates matrix from $\mathcal{B}$-coordinates to C-coordinates. By definition of the change of coordinates matrix we have that $[\mathbf{x}]_{\mathcal{B}} = P[\mathbf{x}]_C$ and $[\mathbf{x}]_C = Q[\mathbf{x}]_{\mathcal{B}}$ for any $\mathbf{x} \in \mathbb{V}$. Hence, we get

$$[\mathbf{x}]_{\mathcal{B}} = P[\mathbf{x}]_C = PQ[\mathbf{x}]_{\mathcal{B}}$$

for all $[\mathbf{x}]_{\mathcal{B}} \in \mathbb{R}^n$. Thus, $PQ = I$ by Theorem 3.1.4. Therefore P is invertible and $Q = P^{-1}$ by Theorem 3.5.2.

4.5 General Linear Mappings

Practice Problems

A1 (a) For any $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ in $\mathbb{R}^3$, and $t \in \mathbb{R}$ we have

$$\begin{aligned} L(t\vec{x} + \vec{y}) &= L(tx_1 + y_1, tx_2 + y_2, tx_3 + y_3) \\ &= \begin{bmatrix} tx_1 + y_1 + tx_2 + y_2 \\ tx_1 + y_1 + tx_2 + y_2 + tx_3 + y_3 \end{bmatrix} \\ &= t\begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 + x_3 \end{bmatrix} + \begin{bmatrix} y_1 + y_2 \\ y_1 + y_2 + y_3 \end{bmatrix} \\ &= tL(\vec{x}) + L(\vec{y}) \end{aligned}$$

Hence, L is linear.

(b) For any $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ in $\mathbb{R}^3$, and $t \in \mathbb{R}$ we have

$$\begin{aligned} L(t\vec{x} + \vec{y}) &= L(tx_1 + y_1, tx_2 + y_2, tx_3 + y_3) \\ &= (tx_1 + y_1 + tx_2 + y_2) + (tx_1 + y_1 + tx_2 + y_2 + tx_3 + y_3)x \\ &= t(x_1 + x_2) + t(x_1 + x_2 + x_3)x + (y_1 + y_2) + (y_1 + y_2 + y_3)x \\ &= tL(\vec{x}) + L(\vec{y}) \end{aligned}$$

Hence, L is linear.

(c) For any $A_0 = \begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix}$ and $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ in $M(2,2)$, and $t \in \mathbb{R}$ we have

$$\begin{aligned} \operatorname{tr}(tA_0 + A_1) &= \operatorname{tr}\left(\begin{bmatrix} ta_0 + a_1 & tb_0 + b_1 \\ tc_0 + c_1 & td_0 + d_1 \end{bmatrix}\right) \\ &= ta_0 + a_1 + td_0 + d_1 \\ &= t(a_0 + d_0) + a_1 + d_1 \\ &= t\operatorname{tr}(A_0) + \operatorname{tr}(A_1) \end{aligned}$$

Hence, tr is linear.

(d) For any $p = a_0 + b_0x + c_0x^2 + d_0x^3$ and $q = a_1 + b_1x + c_1x^2 + d_1x^3$ in P_3, and $t \in \mathbb{R}$ we have

$$\begin{aligned} T(tp + q) &= T(ta_0 + a_1 + (tb_0 + b_1)x + (tc_0 + c_1)x^2 + (td_0 + d_1)x^3) \\ &= \begin{bmatrix} ta_0 + a_1 & (tb_0 + b_1) \\ (tc_0 + c_1) & (td_0 + d_1) \end{bmatrix} \\ &= t\begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \\ &= tT(p) + T(q) \end{aligned}$$

Hence, T is linear.

A2 (a) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Then, we have

$$\det(A + B) = \det\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = 0$$

but

$$\det A + \det B = 1 + 1 = 2$$

Hence, det does not preserve addition and so is not a linear mapping.

(b) For any $p = a_0 + b_0x + c_0x^2$ and $q = a_1 + b_1x + c_1x^2$ in P_2, and $t \in \mathbb{R}$ we have

$$\begin{aligned} L(tp + q) &= L(ta_0 + a_1 + (tb_0 + b_1)x + (tc_0 + c_1)x^2) \\ &= (ta_0 + a_1 - (tb_0 + b_1)) + ((tb_0 + b_1) + (tc_0 + c_1))x^2 \\ &= t[(a_0 - b_0) + (b_0 + c_0)x^2] + (a_1 - b_1) + (b_1 + c_2)x^2 \\ &= tL(p) + L(q) \end{aligned}$$

Hence, T is linear.

(c) Observe that for any $\vec{x} \in \mathbb{R}^2$ we have

$$T(0\vec{x}) = T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

But, $0T(\vec{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence, T is not linear.

(d) For any $A_0 = \begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix}$ and $A_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ in $M(2, 2)$, and $t \in \mathbb{R}$ we have

$$M(tA_0 + A_1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = t\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = tM(A_0) + M(A_1)$$

Hence, M is linear.

A3 (a) $\mathbf{y}$ is in the range of L if there exists $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ such that

$$\begin{bmatrix} 2 \\ 0 \\ 0 \\ 3 \end{bmatrix} = L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_3 \\ 0 \\ 0 \\ x_2 + x_3 \end{bmatrix}$$

Comparing entries gives the system of linear equations

$$\begin{aligned} x_1 + x_3 &= 2 \\ x_2 + x_3 &= 3 \end{aligned}$$

Observe that the system is consistent so $\mathbf{y}$ is in the range of L. Moreover, x_3 is a free variables, so there are infinitely many vectors $\mathbf{x}$ such that $L(\mathbf{x}) = \mathbf{y}$. In particular, if we let $x_3 = t \in \mathbb{R}$, then the general solution is

$$\mathbf{x} = \begin{bmatrix} 2-t \\ 3-t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Hence, for any $t \in \mathbb{R}$ we get a different vector $\mathbf{x}$ which is mapped to $\mathbf{y}$ by L. For example, taking $t = 1$ we get $L\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right) = \mathbf{y}$.

(b) We need to determine if there exists a polynomial $a + bx + cx^2$ such that

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = L(a + bx + cx^2) = \begin{bmatrix} a+c & 0 \\ 0 & b+c \end{bmatrix}$$

This corresponds to exactly the same system of linear equations as in (a). Hence, we know that $\mathbf{y}$ is in the range of L and $L((2-t)1 + (3-t)x + tx^2) = \mathbf{y}$ for any $t \in \mathbb{R}$.

(c) We need to determine if there exists a polynomial $a + bx + cx^2$ such that

$$1 + x = L(a + bx + cx^2) = (b + c) + (-b - c)x$$

Comparing coefficients of powers of x we get the system of equations

$$\begin{aligned} b + c &= 1 \\ -b - c &= 1 \end{aligned}$$

This system is clearly inconsistent, hence $\mathbf{y}$ is not in the range of L.

(d) We need to determine if there exists $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4$ such that

$$\begin{bmatrix} -1 & -1 \\ -2 & 2 \end{bmatrix} = L(\mathbf{x}) = \begin{bmatrix} -2x_2 - 2x_3 - 2x_4 & x_1 + x_4 \\ -2x_1 - x_2 - x_4 & 2x_1 - 2x_2 - x_3 + 2x_4 \end{bmatrix}$$

Comparing entries we get the system

$$\begin{aligned} -2x_2 - 2x_3 - 2x_4 &= -1 \\ x_1 + x_4 &= -1 \\ -2x_1 - x_2 - x_4 &= -2 \\ 2x_1 - 2x_2 - x_3 + 2x_4 &= 2 \end{aligned}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{cccc|c} 0 & -2 & -2 & -2 & -1 \\ 1 & 0 & 0 & 1 & -1 \\ -2 & -1 & 0 & -1 & -2 \\ 2 & -2 & -1 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & -1 \\ 0 & -2 & -2 & -2 & -1 \\ 0 & 0 & 1 & 2 & -7/2 \\ 0 & 0 & 0 & 0 & 17/2 \end{array}\right]$$

Hence, the system is inconsistent, so $\mathbf{y}$ is not in the range of L.

A4 Alternate correct answers are possible.

(a) Every vector in the range of L has the form

$$\begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 + x_3 \end{bmatrix} = (x_1 + x_2)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Therefore, every vector in the range of L can be written as a linear combination of these two vectors. Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. Then, we have shown that $\text{Range}(L) = \text{Span}\,\mathcal{B}$. Additionally, $\mathcal{B}$ is clearly linearly independent. Hence, $\mathcal{B}$ is a basis for the range of L.

To find a basis for the nullspace of L we need to do a similar procedure. We first need to find the general form of a vector in the nullspace and then write it as a linear combination of vectors. To find the general form, we let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be any vector in the nullspace. Then, by definition of the nullspace and of L we get

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = L(\vec{x}) = \begin{bmatrix} x_1 + x_2 \\ x_1 + x_2 + x_3 \end{bmatrix}$$

Hence, we must have $x_1 + x_2 = 0$ and $x_1 + x_2 + x_3 = 0$. So, $x_3 = 0$ and $x_2 = -x_1$. Therefore, every vector in the nullspace of L has the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Therefore, the set $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$ spans the nullspace and is clearly linearly independent (it only contains one non-zero vector). Thus, it is a basis for $\text{Null}(L)$.

Then we have $\text{rank}(L) + \text{Null}(L) = 2 + 1 = \dim \mathbb{R}^3$ as predicted by the Rank-Nullity Theorem.

(b) Every vector in the range of L has the form

$$(a + b) + (a + b + c)x = (a + b)(1 + x) + cx$$

Therefore, the set $\{1 + x, x\}$ spans the range of L. It is clearly linearly linearly independent and hence is a basis for the range of L.

Let $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \text{Null}(L)$. Then,

$$0 + 0x = L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = (a + b) + (a + b + c)x$$

Thus, we get $a + b = 0$ and $a + b + c = 0$. Therefore, $c = 0$ and $b = -a$. So, every vector in the nullspace of L has the form

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ -a \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Therefore, the set $\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}\right\}$ spans the nullspace and is clearly linearly independent. Thus, it is a basis for Null(L).

Then we have rank(L) + Null(L) = 2 + 1 = dim $\mathbb{R}^3$ as predicted by the Rank-Nullity Theorem.

(c) The range of tr is $\mathbb{R}$ since we can pick $a + d$ to be any real number.

Let $\begin{bmatrix}a & b\\c & d\end{bmatrix} \in$ Null(L). Then,

$$0 = L\left(\begin{bmatrix}a & b\\c & d\end{bmatrix}\right) = a + d$$

Thus, we get $d = -a$. So, every vector in the nullspace of L has the form

$$\begin{bmatrix}a & b\\c & d\end{bmatrix} = a\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} + b\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + c\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$$

Therefore, the set $\left\{\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}\right\}$ spans the nullspace and is clearly linearly independent. Thus, it is a basis for Null(L).

We have rank(tr) + Null(tr) = 1 + 3 = dim $M(2, 2)$.

(d) Observe that the range of T is $M(2, 2)$. Therefore, we can pick any basis for $M(2, 2)$ to be the basis for the range of T. We pick the standard basis
$\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$.

Let $a + bx + cx^2 + dx^3 \in$ Null(T). Then

$$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = L(a + bx + cx^2 + dx^3) = \begin{bmatrix}a & b\\c & d\end{bmatrix}$$

Hence, $a = b = c = d = 0$. Thus, Null(T) = {0} and so a basis for Null(L) is the empty set.

We have rank(L) + Null(L) = 4 + 0 = dim P_3.

Homework Problems

B1 Show that each mapping preserves addition and scalar multiplication.

B2 (a) L is not linear.

(b) M is not linear.

(c) N is linear.

(d) L is linear.

(e) T is not linear.

B3 (a) $\mathbf{y}$ is in the range of L. We have $L\left(\begin{bmatrix}1\\-1\\-1\end{bmatrix}\right) = \mathbf{y}$.

(b) $\mathbf{y}$ is in the range of L. We have $L(1 - x - x^2) = \mathbf{y}$.

(c) $\mathbf{y}$ is in the range of L. We have $L(2 - x + 2x^2) = \mathbf{y}$.

(d) $\mathbf{y}$ is not in the range of L.

(e) $\mathbf{y}$ is in the range of L. We have $L\left(\begin{bmatrix} -3/5 & -4/5 \\ 0 & -7/5 \end{bmatrix}\right) = \mathbf{y}$.

(f) $\mathbf{y}$ is not in the range of L.

B4 Alternate correct answers are possible.

(a) A basis for Range(L) is $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$. A basis for Null($L$) is the empty set. We have rank(L) + Null(L) = $3 + 0 = \dim P_2$.

(b) A basis for Range(L) is $\{1, x\}$. A basis for Null(L) is $\{1\}$. We have rank(L) + Null(L) = $2 + 1 = \dim P_2$.

(c) A basis for Range(L) is the empty set. A basis for Null(L) is $\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$. We have rank($L$) + Null($L$) = $0 + 4 = \dim M(2, 2)$.

(d) A basis for Range(L) is $\{1 + x, x + x^2\}$. A basis for Null(L) is the empty set. We have rank(L) + Null(L) = $2 + 0 = \dim \mathbb{D}$.

(e) A basis for Range(L) is $\left\{\begin{bmatrix}1&0\\0&-1\end{bmatrix}, \begin{bmatrix}-1&1\\0&0\end{bmatrix}, \begin{bmatrix}0&-1\\1&0\end{bmatrix}\right\}$. A basis for Null($L$) is $\left\{\begin{bmatrix}1&1\\1&1\end{bmatrix}\right\}$. We have rank($L$)+ Null($L$) = $3 + 1 = \dim M(2, 2)$.

(f) A basis for Range(L) is $\{x^2\}$. A basis for Null(L) is $\left\{\begin{bmatrix}1&0\\0&-1\end{bmatrix}, \begin{bmatrix}0&1\\0&-1\end{bmatrix}, \begin{bmatrix}0&0\\1&-1\end{bmatrix}\right\}$. We have rank($L$) + Null($L$) = $1 + 3 = \dim M(2, 2)$.

(g) A basis for Range(L) is $\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$. A basis Null($L$) is the empty set. We have rank(L) + Null(L) = $4 + 0 = \dim M(2, 2)$.

(h) A basis for Range(L) is $\left\{\begin{bmatrix}-1&0\\-2&0\end{bmatrix}, \begin{bmatrix}0&2\\0&-2\end{bmatrix}, \begin{bmatrix}-2&-1\\2&-1\end{bmatrix}\right\}$. A basis Null($L$) is the empty set. We have rank(L) + Null(L) = $3 + 0 = \dim P_2$.

Conceptual Problems

D1 (1) We have $L(\mathbf{0}_{\mathbb{V}}) = L(0\mathbf{0}_{\mathbb{V}}) = 0L(\mathbf{0}_{\mathbb{V}}) = \mathbf{0}_{\mathbb{W}}$.

(2) By definition Null(L) is a subset of $\mathbb{V}$. By (1), $\mathbf{0}_{\mathbb{V}} \in$ Null(L), so Null(L) is non-empty. Let $\mathbf{x}, \mathbf{y} \in$ Null(L) so that $L(\mathbf{x}) = \mathbf{0}_{\mathbb{W}}$ and $L(\mathbf{y}) = \mathbf{0}_{\mathbb{W}}$. Then, $L(\mathbf{x}+\mathbf{y}) = L(\mathbf{x})+L(\mathbf{y}) = \mathbf{0}_{\mathbb{W}}+\mathbf{0}_{\mathbb{W}} = \mathbf{0}_{\mathbb{W}}$ so $\mathbf{x}+\mathbf{y} \in$ Null(L). Similarly, for any $t \in \mathbb{R}$ we have $L(t\mathbf{x}) = tL(\mathbf{x}) = t\mathbf{0}_{\mathbb{W}} = \mathbf{0}_{\mathbb{W}}$. Thus, $t\mathbf{x} \in$ Null(L). Hence, Null(L) is a subspace of $\mathbb{V}$.

(3) By definition Range(L) is a subset of $\mathbb{W}$. By (1), $\mathbf{0}_{\mathbb{W}} \in$ Range(L), so Range(L) is non-empty. For any $\mathbf{x}, \mathbf{y} \in$ Range(L) there exists $\mathbf{u}, \mathbf{v} \in \mathbb{V}$ such that $L(\mathbf{u}) = \mathbf{x}$ and $L(\mathbf{v}) = \mathbf{y}$. Then we get, $L(\mathbf{u}+\mathbf{v}) = L(\mathbf{u})+L(\mathbf{v}) = \mathbf{x}+\mathbf{y}$ so $\mathbf{x} + \mathbf{y} \in$ Range(L). Similarly, for any $t \in \mathbb{R}$ we have $L(t\mathbf{u}) = tL(\mathbf{u}) = t\mathbf{x}$. Thus, $t\mathbf{x} \in$ Range(L). Hence, Range(L) is a subspace of $\mathbb{W}$.

D2 (a) Observe that we have

$$\begin{aligned} L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) &= a_1 L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) + a_2 L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) + a_3 L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) \\ &= a_1 x^2 + a_2(2x) + a_3(1 + x + x^2) \\ &a_3 + (2a_2 + a_3)x + (a_1 + a_3)x^2 \end{aligned}$$

(b) Alternate correct answers are possible. One possibility is $L\left(a + bx + cx^2\right) = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$.

(c) Alternate correct answers are possible. One possibility is $L : M(2,2) \to \mathbb{R}^4$ defined $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 \\ a \\ d \\ 0 \end{bmatrix}$.

D3 (a) Assume that $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly dependent. Then there exists coefficients $t_1, \dots, t_k$ not all zero such that $t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k = \mathbf{0}_{\mathbb{V}}$. But then

$$\mathbf{0}_{\mathbb{W}} = L(\mathbf{0}_{\mathbb{V}}) = L(t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k) = t_1 L(\mathbf{v}_1) + \cdots + t_k L(\mathbf{v}_k)$$

with $t_1, \dots, t_k$ not all zero. This contradicts the fact that $L(\mathbf{v}_1), \dots, L(\mathbf{v}_k)$ is linearly independent.

(b) One simple example is to define $L : \mathbb{V} \to \mathbb{W}$ by $L(\mathbf{v}) = \mathbf{0}_{\mathbb{W}}$ for all $\mathbf{v} \in \mathbb{V}$. Then, for any set $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ we have $\{L(\mathbf{v}_1), \dots, L(\mathbf{v}_k)\}$ only contains the zero vector and hence is linearly dependent.

D4 Using the Rank-Nullity Theorem, we get that $\text{Range}(L) = \mathbb{W}$ if and only if $\text{rank}(L) = n$ if and only if $\text{nullity}(L) = 0$ if and only if $\text{Null}(L) = \{\mathbf{0}\}$.

D5 (a) Every vector $\mathbf{v}$ in the range of $M \circ L$ can be written in the form of $\mathbf{v} = M(L(\mathbf{x}))$ for some $\mathbf{x} \in \mathbb{V}$. But, $L(\mathbf{x}) = \mathbf{y}$ for some $\mathbf{y} \in \mathbb{U}$. Hence, $\mathbf{v} = M(\mathbf{y})$, so the range of $M \circ L$ is a subset of the range of M. Since ranges are always subspaces, this implies that $\text{Range}(M \circ L)$ is a subspace of $\text{Range}(M)$. Hence, $\dim \text{Range}(M \circ L) \le \dim \text{Range}(M)$ which gives $\text{rank}(M \circ L) \le \text{rank}(M)$.

(b) The nullspace of L is a subspace of the nullspace of $M \circ L$, because if $L(\mathbf{x}) = \mathbf{0}_{\mathbb{U}}$, then $M(L(\mathbf{x})) = M(\mathbf{0}_{\mathbb{U}}) = \mathbf{0}_{\mathbb{W}}$. Therefore $\text{nullity}(L) \le \text{nullity}(M \circ L)$. So, by the Rank-Nullity Theorem

$$\text{rank}(M \circ L) = n - \text{nullity}(M \circ L) \le n - \text{nullity}(L) = \text{rank}(L)$$

(c) There are many possible correct answers. One possibility is to define $L(x_1, x_2) = (x_1, 0)$ and $M(x_1, x_2) = (0, x_2)$. Then, L and M both have rank 1, but $M \circ L = M(L(x_1, x_2)) = M(x_1, 0) = (0, 0)$, so $\text{rank}(M \circ L) = 0$.

D6 Suppose that $\text{rank}(L) = r$. Let $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ be a basis for $\text{Range}(L)$. Since $\mathbf{v}_i = L(\mathbf{x}_i)$ for some $\mathbf{x}_i \in \mathbb{V}$, $M(\mathbf{v}_i) = M(L(\mathbf{x}_i)) \in \text{Range}(M \circ L)$. Thus the set $\mathcal{B} = \{M(\mathbf{v}_1), \dots, M(\mathbf{v}_r)\}$ is in the range of $M \circ L$ and we have

$$\begin{aligned} t_1 M(\mathbf{v}_1) + \cdots + t_r M(\mathbf{v}_r) &= \mathbf{0} \\ M(t_1\mathbf{v}_1 + \cdots + t_r\mathbf{v}_r) &= \mathbf{0} \\ t_1\mathbf{v}_1 + \cdots + t_r\mathbf{v}_r &= \mathbf{0} \end{aligned}$$

since the nullspace of M is $\{\mathbf{0}\}$. Hence $t_1 = \cdots = t_r = 0$ is the only solution and so $\mathcal{B}$ is linearly independent. Thus, $\text{rank}(M \circ L) \ge r = \text{rank}(L)$. Also, by D5(b) we know that $\text{rank}(M \circ L) \le \text{rank}(L)$, hence $\text{rank}(M \circ L) = \text{rank}(L)$ as required.

D7 We define $\mathbf{x}+\mathbf{y} = (x_1+y_1, x_2+y_2, \dots)$ and $t\mathbf{x} = (tx_1, tx_2, \dots)$. We have

$$(L \circ R)(\mathbf{x}) = L(0, x_1, x_2, \dots) = (x_1, x_2, \dots) = \mathbf{x}$$

but

$$(R \circ L)(\mathbf{x}) = R(x_2, x_3, \dots) = (0, x_2, x_3, \dots) \neq \mathbf{x}$$

Thus R is a right inverse for L but not a left inverse. Moreover, observe that there cannot be a left inverse for L since there is no way of recovering x_1.

4.6 Matrix of a Linear Mapping

The Matrix of L with Respect to the Basis B

Practice Problems

A1 By definition, the columns of the matrix of L with respect to the basis $\mathcal{B}$ are the $\mathcal{B}$-coordinates of the images of the basis vectors. In this question, we are given the images of the basis vectors as a linear combination of the basis vectors, so the coordinates are just the coefficients of the linear combination. To calculate $[L(\vec{x})]_{\mathcal{B}}$ we use the formula

$$[L(\vec{x})]_{\mathcal{B}} = [L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}$$

(a) We have

$$L(\vec{v}_1) = 0\vec{v}_1 + 1\vec{v}_2 \qquad \Rightarrow [L(\vec{v}_1)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$L(\vec{v}_2) = 2\vec{v}_1 + (-1)\vec{v}_2 \qquad \Rightarrow [L(\vec{v}_2)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Hence,

$$[L]_{\mathcal{B}} = \begin{bmatrix} [L(\vec{v}_1)]_{\mathcal{B}} & [L(\vec{v}_2)]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$$

So,

$$[L(\vec{x})]_{\mathcal{B}} = [L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

(b) We have

$$L(\vec{v}_1) = 2\vec{v}_1 + 0\vec{v}_2 + (-1)\vec{v}_3 \qquad \Rightarrow [L(\vec{v}_1)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

$$L(\vec{v}_2) = 2\vec{v}_1 + 0\vec{v}_2 + (-1)\vec{v}_3 \qquad \Rightarrow [L(\vec{v}_2)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

$$L(\vec{v}_3) = 0\vec{v}_1 + 4\vec{v}_2 + 5\vec{v}_3 \qquad \Rightarrow [L(\vec{v}_2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}$$

Hence,

$$[L]_{\mathcal{B}} = \begin{bmatrix}[L(\vec{v}_1)]_{\mathcal{B}} & [L(\vec{v}_2)]_{\mathcal{B}} & [L(\vec{v}_3]_{\mathcal{B}}\end{bmatrix} = \begin{bmatrix}2 & 2 & 0\\ 0 & 0 & 4\\ -1 & -1 & 5\end{bmatrix}$$

So,

$$[L(\vec{x})]_{\mathcal{B}} = [L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}2 & 2 & 0\\ 0 & 0 & 4\\ -1 & -1 & 5\end{bmatrix}\begin{bmatrix}3\\ 3\\ -1\end{bmatrix} = \begin{bmatrix}12\\ -4\\ -11\end{bmatrix}$$

A2 As in the last problem, to find the matrix of L with respect to $\mathcal{B}$ we need to find the $\mathcal{B}$-coordinates of the images of the basis vectors. In this problem, we need to write, by observation, the image of each basis vector as a linear combination of the basis vectors.

(a) We have

$$L(1,1) = \begin{bmatrix}-3\\ -3\end{bmatrix} = (-3)\begin{bmatrix}1\\ 1\end{bmatrix} + 0\begin{bmatrix}-1\\ 2\end{bmatrix} \qquad \Rightarrow [L(1,1)]_{\mathcal{B}} = \begin{bmatrix}-3\\ 0\end{bmatrix}$$

$$L(-1,2) = \begin{bmatrix}-4\\ 8\end{bmatrix} = 0\begin{bmatrix}1\\ 1\end{bmatrix} + (4)\begin{bmatrix}-1\\ 2\end{bmatrix} \qquad \Rightarrow [L(-1,2)]_{\mathcal{B}} = \begin{bmatrix}0\\ 4\end{bmatrix}$$

Hence,

$$[L]_{\mathcal{B}} = \begin{bmatrix}[L(1,1)]_{\mathcal{B}} & [L(-1,2)]_{\mathcal{B}}\end{bmatrix} = \begin{bmatrix}-3 & 0\\ 0 & 4\end{bmatrix}$$

(b) We have

$$L(1,1) = \begin{bmatrix}-1\\ 2\end{bmatrix} = 0\begin{bmatrix}1\\ 1\end{bmatrix} + 1\begin{bmatrix}-1\\ 2\end{bmatrix} \qquad \Rightarrow [L(1,1)]_{\mathcal{B}} = \begin{bmatrix}0\\ 1\end{bmatrix}$$

$$L(-1,2) = \begin{bmatrix}2\\ 2\end{bmatrix} = 2\begin{bmatrix}1\\ 1\end{bmatrix} + 0\begin{bmatrix}-1\\ 2\end{bmatrix} \qquad \Rightarrow [L(-1,2)]_{\mathcal{B}} = \begin{bmatrix}2\\ 0\end{bmatrix}$$

Hence,

$$[L]_{\mathcal{B}} = \begin{bmatrix}[L(1,1)]_{\mathcal{B}} & [L(-1,2)]_{\mathcal{B}}\end{bmatrix} = \begin{bmatrix}0 & 2\\ 1 & 0\end{bmatrix}$$

A3 (a) We have

$$L(1,1,1) = \begin{bmatrix}-1\\ 2\\ 0\end{bmatrix} = 0\begin{bmatrix}1\\ 1\\ 1\end{bmatrix} + 1\begin{bmatrix}-1\\ 2\\ 0\end{bmatrix} + 0\begin{bmatrix}0\\ -1\\ 4\end{bmatrix} \qquad \Rightarrow [L(1,1,1)]_{\mathcal{B}} = \begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$$

$$L(-1,2,0) = \begin{bmatrix}1\\ 1\\ 1\end{bmatrix} = 1\begin{bmatrix}1\\ 1\\ 1\end{bmatrix} + 0\begin{bmatrix}-1\\ 2\\ 0\end{bmatrix} + 0\begin{bmatrix}0\\ -1\\ 4\end{bmatrix} \qquad \Rightarrow [L(-1,2,0)]_{\mathcal{B}} = \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$

$$L(0,-1,4) = \begin{bmatrix}5\\ 5\\ 5\end{bmatrix} = 5\begin{bmatrix}1\\ 1\\ 1\end{bmatrix} + 0\begin{bmatrix}-1\\ 2\\ 0\end{bmatrix} + 0\begin{bmatrix}0\\ -1\\ 4\end{bmatrix} \qquad \Rightarrow [L(0,-1,4)]_{\mathcal{B}} = \begin{bmatrix}5\\ 0\\ 0\end{bmatrix}$$

Hence,

$$[L]_{\mathcal{B}} = \begin{bmatrix}[L(1,1,1)]_{\mathcal{B}} & [L(-1,2,0)]_{\mathcal{B}} & [L(0,-1,4)]_{\mathcal{B}}\end{bmatrix} = \begin{bmatrix}0 & 1 & 5\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

(b) We have

$$L(1,1,1) = \begin{bmatrix}0\\-1\\4\end{bmatrix} = 0\begin{bmatrix}1\\1\\1\end{bmatrix} + 0\begin{bmatrix}-1\\2\\0\end{bmatrix} + 1\begin{bmatrix}0\\-1\\4\end{bmatrix} \qquad \Rightarrow [L(1,1,1)]_{\mathcal{B}} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$L(-1,2,0) = \begin{bmatrix}1\\1\\1\end{bmatrix} = 1\begin{bmatrix}1\\1\\1\end{bmatrix} + 0\begin{bmatrix}-1\\2\\0\end{bmatrix} + 0\begin{bmatrix}0\\-1\\4\end{bmatrix} \qquad \Rightarrow [L(-1,2,0)]_{\mathcal{B}} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$L(0,-1,4) = \begin{bmatrix}-1\\2\\0\end{bmatrix} = 0\begin{bmatrix}1\\1\\1\end{bmatrix} + 1\begin{bmatrix}-1\\2\\0\end{bmatrix} + 0\begin{bmatrix}0\\-1\\4\end{bmatrix} \qquad \Rightarrow [L(0,-1,4)]_{\mathcal{B}} = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

Hence,

$$[L]_{\mathcal{B}} = \begin{bmatrix}[L(1,1,1)]_{\mathcal{B}} & [L(-1,2,0)]_{\mathcal{B}} & [L(0,-1,4)]_{\mathcal{B}}\end{bmatrix} = \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$$

(c) We have

$$L(1,1,1) = \begin{bmatrix}1\\1\\1\end{bmatrix} = 1\begin{bmatrix}1\\1\\1\end{bmatrix} + 0\begin{bmatrix}-1\\2\\0\end{bmatrix} + 1\begin{bmatrix}0\\-1\\4\end{bmatrix} \qquad \Rightarrow [L(1,1,1)]_{\mathcal{B}} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

Next, $L(-1,2,0) = \begin{bmatrix}0\\2\\5\end{bmatrix}$. This is not quite as easy to write as a linear combination of the basis vectors as the previous ones. Thus, if we cannot determine the coordinates by observation, then we need to row reduce the corresponding augmented matrix as in the Section 4.4. In particular, we need to determine s_1, s_2, and s_3 such that

$$\begin{bmatrix}0\\2\\5\end{bmatrix} = s_1\begin{bmatrix}1\\1\\1\end{bmatrix} + s_2\begin{bmatrix}-1\\2\\0\end{bmatrix} + s_3\begin{bmatrix}0\\-1\\4\end{bmatrix}$$

Before we solve this, we observe that we will need to repeat this procedure to write the image of the third basis vector as a linear combination of the basis vectors. To save time, we will solve them simultaneously in an doubly augmented matrix. We get

$$\left[\begin{array}{ccc|c|c} 1 & -1 & 0 & 0 & 5\\ 1 & 2 & -1 & 2 & 0\\ 1 & 0 & 4 & 5 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & 1 & 3\\ 0 & 1 & 0 & 1 & -2\\ 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

Hence,

$$[L]_{\mathcal{B}} = \begin{bmatrix}[L(1,1,1)]_{\mathcal{B}} & [L(-1,2,0)]_{\mathcal{B}} & [L(0,-1,4)]_{\mathcal{B}}\end{bmatrix} = \begin{bmatrix}1&1&3\\0&1&-2\\0&1&-1\end{bmatrix}$$

A4 (a) Consider the vector $\vec{v}_1 = \begin{bmatrix}1\\-2\end{bmatrix}$, which is the normal to the line of reflection, and the vector $\vec{v}_2 = \begin{bmatrix}2\\1\end{bmatrix}$ which is orthogonal to $\vec{v}_1$. By geometrical arguments, a basis adapted to $\text{refl}_{(1,-2)}$ is $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$. To determine the matrix of $\text{refl}_{(1,-2)}$ with respect to the basis $\mathcal{B}$, calculate the $\mathcal{B}$-coordinates of the images of the basis vectors. We observe that if we reflect the normal vector $\vec{v}_1$ over the line we will get $-\vec{v}_1$. Since the vector $\vec{v}_2$ is orthogonal to $\vec{v}_1$ it lies on the line of reflection, so reflecting it will not change the vector. Hence,

$$\begin{aligned}\operatorname{refl}_{(1,-2)} \vec{v}_1 &= -\vec{v}_1 = (-1)\vec{v}_1 + 0\vec{v}_2 \\ \operatorname{refl}_{(1,-2)} \vec{v}_1 &= \vec{v}_2 = 0\vec{v}_1 + 1\vec{v}_2\end{aligned}$$

Thus,

$$[\operatorname{refl}_{(1,-2)}]_{\mathcal{B}} = \begin{bmatrix}[\operatorname{refl}_{(1,-2)} \vec{v}_1]_{\mathcal{B}} & [\operatorname{refl}_{(1,-2)} \vec{v}_2]_{\mathcal{B}}\end{bmatrix} = \begin{bmatrix}-1 & 0 \\ 0 & 1\end{bmatrix}$$

(b) Consider the vector $\vec{v}_1 = \begin{bmatrix}2\\1\\-1\end{bmatrix}$, which is a direction vector for the projection, and the vectors $\vec{v}_2 = \begin{bmatrix}1\\0\\2\end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix}0\\1\\1\end{bmatrix}$ which are orthogonal to $\vec{v}_1$. By geometrical arguments, a basis adapted to $\operatorname{proj}_{(2,1,-1)}$ is $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. To determine the matrix of $\operatorname{proj}_{(2,1,-1)}$ with respect to the basis $\mathcal{B}$, calculate the $\mathcal{B}$-coordinates of the images of the basis vectors. We observe that the projection of $\vec{v}_1$ onto $\vec{v}_1$ is just $\vec{v}_1$, and the projection of any vector orthogonal to $\vec{v}_1$ onto $\vec{v}_1$ is $\vec{0}$. Hence,

$$\begin{aligned}\operatorname{proj}_{(2,1,-1)} \vec{v}_1 &= \vec{v}_1 = 1\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 \\ \operatorname{proj}_{(2,1,-1)} \vec{v}_2 &= \vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 \\ \operatorname{proj}_{(2,1,-1)} \vec{v}_3 &= \vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3\end{aligned}$$

Thus,

$$\begin{aligned}[\operatorname{proj}_{(2,1,-1)}]_{\mathcal{B}} &= \begin{bmatrix}[\operatorname{proj}_{(2,1,-1)} \vec{v}_1]_{\mathcal{B}} & [\operatorname{proj}_{(2,1,-1)} \vec{v}_2]_{\mathcal{B}} & [\operatorname{proj}_{(2,1,-1)} \vec{v}_3]_{\mathcal{B}}\end{bmatrix} \\ &= \begin{bmatrix}1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}\end{aligned}$$

(c) Consider the vector $\vec{v}_1 = \begin{bmatrix}-1\\-1\\1\end{bmatrix}$, which is normal to the plane of reflection, and the vectors $\vec{v}_2 = \begin{bmatrix}1\\0\\1\end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix}0\\1\\1\end{bmatrix}$, which lie in the plane and are therefore orthogonal to $\vec{v}_1$. By geometrical arguments, a basis adapted to $\operatorname{refl}_{(-1,-1,1)}$ is $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. To determine the matrix of $\operatorname{refl}_{(-1,-1,1)}$ with respect to the basis $\mathcal{B}$, calculate the $\mathcal{B}$-coordinates of the images of the basis vectors. We observe that if we reflect the normal vector $\vec{v}_1$ over the plane we will get $-\vec{v}_1$. Since the vectors $\vec{v}_2$ and $\vec{v}_3$ are orthogonal to $\vec{v}_1$ reflecting them will not change them. Hence,

$$\begin{aligned}\operatorname{refl}_{(-1,-1,1)} \vec{v}_1 &= -\vec{v}_1 = (-1)\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 \\ \operatorname{refl}_{(-1,-1,1)} \vec{v}_2 &= \vec{v}_2 = 0\vec{v}_1 + 1\vec{v}_2 + 0\vec{v}_3 \\ \operatorname{refl}_{(-1,-1,1)} \vec{v}_3 &= \vec{v}_3 = 0\vec{v}_1 + 0\vec{v}_2 + 1\vec{v}_3\end{aligned}$$

Thus,

$$[\text{refl}_{(-1,-1,1)}]_{\mathcal{B}} = \begin{bmatrix}[\text{refl}_{(-1,-1,1)}\,\vec{v}_1]_{\mathcal{B}} & [\text{refl}_{(-1,-1,1)}\,\vec{v}_2]_{\mathcal{B}} & [\text{refl}_{(-1,-1,1)}\,\vec{v}_3]_{\mathcal{B}}\end{bmatrix} = \begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

A5 (a) We need to find t_1, t_2, and t_3 such that

$$\begin{bmatrix}1\\2\\4\end{bmatrix} = t_1\begin{bmatrix}1\\0\\1\end{bmatrix} + t_2\begin{bmatrix}1\\-1\\0\end{bmatrix} + t_3\begin{bmatrix}0\\1\\2\end{bmatrix} = \begin{bmatrix}t_1+t_2\\-t_2+t_3\\1t_1+2t_3\end{bmatrix}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{ccc|c}1 & 1 & 0 & 1\\ 0 & -1 & 1 & 2\\ 1 & 0 & 2 & 4\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 0 & 0 & 2\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 1\end{array}\right]$$

Thus, $\begin{bmatrix}1\\2\\4\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}2\\-1\\1\end{bmatrix}$

(b) Using the result in part (a) and by observation we get

$$L(1,0,1) = \begin{bmatrix}1\\2\\4\end{bmatrix} = 2\begin{bmatrix}1\\0\\1\end{bmatrix} + (-1)\begin{bmatrix}1\\-1\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\\2\end{bmatrix}$$

$$L(1,-1,0) = \begin{bmatrix}0\\1\\2\end{bmatrix} = 0\begin{bmatrix}1\\0\\1\end{bmatrix} + 0\begin{bmatrix}1\\-1\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\\2\end{bmatrix}$$

$$L(0,1,2) = \begin{bmatrix}2\\-2\\0\end{bmatrix} = 0\begin{bmatrix}1\\0\\1\end{bmatrix} + 2\begin{bmatrix}1\\-1\\0\end{bmatrix} + 0\begin{bmatrix}0\\1\\2\end{bmatrix}$$

Hence,

$$[L]_{\mathcal{B}} = \begin{bmatrix}[L(1,0,1)]_{\mathcal{B}} & [L(1,-1,0)]_{\mathcal{B}} & [L(0,1,2)]_{\mathcal{B}}\end{bmatrix} = \begin{bmatrix}2 & 0 & 0\\ -1 & 0 & 2\\ 1 & 1 & 0\end{bmatrix}$$

(c) Using the result of part (a), the result of part (b), and the formula $[L(\mathbf{x})]_{\mathcal{B}} = [L]_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$ we get

$$[L(1,2,4)]_{\mathcal{B}} = [L]_{\mathcal{B}}\begin{bmatrix}1\\2\\4\end{bmatrix}_{\mathcal{B}} = \begin{bmatrix}2 & 0 & 0\\ -1 & 0 & 2\\ 1 & 1 & 0\end{bmatrix}\begin{bmatrix}2\\-1\\1\end{bmatrix} = \begin{bmatrix}4\\0\\1\end{bmatrix}$$

Thus, by definition of $\mathcal{B}$-coordinates, we have

$$L(1,2,4) = 4\begin{bmatrix}1\\0\\1\end{bmatrix} + 0\begin{bmatrix}1\\-1\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\\2\end{bmatrix} = \begin{bmatrix}4\\1\\6\end{bmatrix}$$

A6 (a) We need to find t_1, t_2, and t_3 such that

$$\begin{bmatrix} 5 \\ 3 \\ -5 \end{bmatrix} = t_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} t_1 + t_2 \\ 2t_2 + t_3 \\ -t_1 + t_3 \end{bmatrix}$$

Row reducing the corresponding augmented matrix gives

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & 2 & 1 & 3 \\ -1 & 0 & 1 & -5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

Thus, $\begin{bmatrix} 5 \\ 3 \\ -5 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix}$.

(b) We have

$$L(1,0,-1) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$L(1,2,0) = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} = (-2) \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$L(0,1,1) = \begin{bmatrix} 5 \\ 3 \\ -5 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + (-3) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Hence,

$$[L]_{\mathcal{B}} = \begin{bmatrix} [L(1,0,-1)]_{\mathcal{B}} & [L(1,2,0)]_{\mathcal{B}} & [L(0,1,1)]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 0 & -2 & 2 \\ 0 & 0 & 3 \\ 1 & 0 & -3 \end{bmatrix}$$

(c) Using the result of part (a), the result of part (b), and the formula $[L(\mathbf{x})]_{\mathcal{B}} = [L]_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$ we get

$$[L(5,3,-5)]_{\mathcal{B}} = [L]_{\mathcal{B}} \begin{bmatrix} 5 \\ 3 \\ -5 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 0 & -2 & 2 \\ 0 & 0 & 3 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} -12 \\ -9 \\ 11 \end{bmatrix}$$

Thus, by definition of $\mathcal{B}$-coordinates, we have

$$L(5,3,-5) = (-12) \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + (-9) \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 11 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -21 \\ -7 \\ 23 \end{bmatrix}$$

A7 (a) The matrix of L with respect to the standard basis is $[L]_{\mathcal{S}} = \begin{bmatrix} 1 & 3 \\ -8 & 7 \end{bmatrix}$.

The change of coordinates matrix from $\mathcal{B}$ to $\mathcal{S}$ is

$$P = \begin{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{S}} & \begin{bmatrix} 1 \\ 4 \end{bmatrix}_{\mathcal{S}} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix}$$

So, the change of coordinates matrix from $\mathcal{S}$ to $\mathcal{B}$ is $P^{-1} = \begin{bmatrix} 2 & -1/2 \\ -1 & 1/2 \end{bmatrix}$.

Hence, the $\mathcal{B}$-matrix of L is

$$[L]_{\mathcal{B}} = P^{-1}[L]_{\mathcal{S}}P = \begin{bmatrix} 2 & -1/2 \\ -1 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -8 & 7 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 11 & 16 \\ -4 & -3 \end{bmatrix}$$

(b) The matrix of L with respect to the standard basis is $[L]_{\mathcal{S}} = \begin{bmatrix} 1 & -6 \\ -4 & -1 \end{bmatrix}$.

The change of coordinates matrix from $\mathcal{B}$ to $\mathcal{S}$ is

$$P = \begin{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix}_{\mathcal{S}} & \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{S}} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$$

So, the change of coordinates matrix from $\mathcal{S}$ to $\mathcal{B}$ is $P^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$.

Hence, the $\mathcal{B}$-matrix of L is

$$[L]_{\mathcal{B}} = P^{-1}[L]_{\mathcal{S}}P = \frac{1}{5}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -6 \\ -4 & -1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}$$

(c) The matrix of L with respect to the standard basis is $[L]_{\mathcal{S}} = \begin{bmatrix} 4 & -6 \\ 2 & 8 \end{bmatrix}$.

The change of coordinates matrix from $\mathcal{B}$ to $\mathcal{S}$ is

$$P = \begin{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}_{\mathcal{S}} & \begin{bmatrix} 7 \\ 3 \end{bmatrix}_{\mathcal{S}} \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 1 & 3 \end{bmatrix}$$

So, the change of coordinates matrix from $\mathcal{S}$ to $\mathcal{B}$ is $P^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -7 \\ -1 & 3 \end{bmatrix}$.

Hence, the $\mathcal{B}$-matrix of L is

$$[L]_{\mathcal{B}} = P^{-1}[L]_{\mathcal{S}}P = \frac{1}{2}\begin{bmatrix} 3 & -7 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 4 & -6 \\ 2 & 8 \end{bmatrix}\begin{bmatrix} 3 & 7 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} -40 & -118 \\ 18 & 52 \end{bmatrix}$$

(d) The matrix of L with respect to the standard basis is $[L]_{\mathcal{S}} = \begin{bmatrix} 16 & -20 \\ 6 & -6 \end{bmatrix}$.

The change of coordinates matrix from $\mathcal{B}$ to $\mathcal{S}$ is

$$P = \begin{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix}_{\mathcal{S}} & \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{\mathcal{S}} \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$$

So, the change of coordinates matrix from $\mathcal{S}$ to $\mathcal{B}$ is $P^{-1} = \frac{1}{2}\begin{bmatrix} -2 & 4 \\ 3 & -5 \end{bmatrix}$.

Hence, the $\mathcal{B}$-matrix of L is

$$[L]_{\mathcal{B}} = P^{-1}[L]_{\mathcal{S}}P = \frac{1}{2}\begin{bmatrix} -2 & 4 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} 16 & -20 \\ 6 & -6 \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}$$

(e) The matrix of L with respect to the standard basis is $[L]_{\mathcal{S}} = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 4 & 2 \\ 1 & -1 & 5 \end{bmatrix}$.

The change of coordinates matrix from $\mathcal{B}$ to $\mathcal{S}$ is

$$P = \begin{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{S}} & \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}_{\mathcal{S}} & \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}_{\mathcal{S}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

So, the change of coordinates matrix from $\mathcal{S}$ to $\mathcal{B}$ is $P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$.

Hence, the $\mathcal{B}$-matrix of L is

$$[L]_{\mathcal{B}} = P^{-1}[L]_{\mathcal{S}}P = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 & 1 \\ 0 & 4 & 2 \\ 1 & -1 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{bmatrix}$$

(f) The matrix of L with respect to the standard basis is $[L]_{\mathcal{S}} = \begin{bmatrix} 4 & 1 & -3 \\ 16 & 4 & -18 \\ 6 & 1 & -5 \end{bmatrix}$.

The change of coordinates matrix from $\mathcal{B}$ to $\mathcal{S}$ is

$$P = \begin{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{\mathcal{S}} & \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}_{\mathcal{S}} & \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}_{\mathcal{S}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

So, the change of coordinates matrix from $\mathcal{S}$ to $\mathcal{B}$ is $P^{-1} = \begin{bmatrix} -1 & -1 & 3 \\ -1 & 0 & 1 \\ 2 & 1 & -3 \end{bmatrix}$.

Hence, the $\mathcal{B}$-matrix of L is

$$[L]_{\mathcal{B}} = P^{-1}[L]_{\mathcal{S}}P = \begin{bmatrix} -1 & -1 & 3 \\ -1 & 0 & 1 \\ 2 & 1 & -3 \end{bmatrix}\begin{bmatrix} 4 & 1 & -3 \\ 16 & 4 & -18 \\ 6 & 1 & -5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

A8 (a) We need to write find the $\mathcal{B}$-coordinates of the images of the basis vectors. So, we need to solve

$$L(1,1,1) = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = a_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + a_2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + a_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$L(0,1,1) = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = b_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b_2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + b_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$L(0,0,1) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We get three systems of linear equation with the same coefficient matrix. Hence, we row reduce the triple augmented matrix to get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 2 & 2 & 1 \\ 1 & 1 & 1 & 0 & -1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -2 & -3 & -2 \end{array}\right]$$

The columns in the augmented part are the $\mathcal{B}$-coordinates of the image of the basis vectors, hence

$$[L]_{\mathcal{B}} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ -2 & -3 & -2 \end{bmatrix}$$

(b) We need to write find the $\mathcal{B}$-coordinates of the images of the basis vectors. So, we need to solve

$$\begin{aligned} L(1+x^2) = 1 + x^2 &= a_1(1+x^2) + a_2(-1+x) + a_3(1-x+x^2) \\ L(-1+x) = -1 + x^2 &= b_1(1+x^2) + b_2(-1+x) + b_3(1-x+x^2) \\ L(1-x+x^2) = 1 &= c_1(1+x^2) + c_2(-1+x) + c_3(1-x+x^2) \end{aligned}$$

We get three systems of linear equation with the same coefficient matrix. Hence, we row reduce the triple augmented matrix to get

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right]$$

The columns in the augmented part are the $\mathcal{B}$-coordinates of the image of the basis vectors, hence

$$[L]_{\mathcal{B}} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

(c) We need to write find the $\mathcal{B}$-coordinates of the images of the basis vectors. We have

$$\begin{aligned} L(1) = 0 &= 0(1) + 0x + 0x^2 \\ L(x) = 1 &= 1(1) + 0x + 0x^2 \\ L(x^2) = 2x &= 0(1) + 2x + 0x^2 \end{aligned}$$

Hence,

$$[D]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

(d) We need to write find the $\mathcal{B}$-coordinates of the images of the basis vectors. So, we need to solve

$$\begin{aligned} L\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} &= a_1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + a_3 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ L\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} &= b_1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + b_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b_3 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ L\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} &= c_1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

We get three systems of linear equation with the same coefficient matrix. Hence, we row reduce the triple augmented matrix to get

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 & 2 & 4 \end{array}\right]$$

The columns in the augmented part are the $\mathcal{B}$-coordinates of the image of the basis vectors, hence

$$[T]_{\mathcal{B}} = \begin{bmatrix} -1 & -1 & -2 \\ 0 & 0 & -1 \\ 2 & 2 & 4 \end{bmatrix}$$

Homework Problems

B1 (a) $[L]_{\mathcal{B}} = \begin{bmatrix} 1 & 5 \\ 3 & -7 \end{bmatrix}$, $[L(\mathbf{x})]_{\mathcal{B}} = \begin{bmatrix} -6 \\ 26 \end{bmatrix}$ (b) $[L]_{\mathcal{B}} = \begin{bmatrix} 2 & 3 & -1 \\ -3 & 4 & 2 \\ 0 & -1 & 6 \end{bmatrix}$, $[L(\mathbf{x})]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -25 \\ 9 \end{bmatrix}$

B2 (a) $[L]_{\mathcal{B}} = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$ (b) $[L]_{\mathcal{B}} = \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}$ (c) $[L]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

B3 (a) $[L]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{bmatrix}$ (b) $[L]_{\mathcal{B}} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 2 & 0 \end{bmatrix}$ (c) $[L]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (d) $[L]_{\mathcal{B}} = \begin{bmatrix} 5 & 1 & 0 \\ 5 & 1 & 0 \\ -4 & 1 & 1 \end{bmatrix}$

B4 (a) $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$, $[\text{perp}_{(3,2)}]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$, $[\text{perp}_{(2,1,-2)}]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \right\}$, $[\text{refl}_{(1,2,3)}]_{\mathcal{B}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

B5 (a) $\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$

(b) $[L]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 3 \\ 5 & 0 & -1 \\ 0 & 2 & 2 \end{bmatrix}$

(c) $L(5, 2, 1) = \begin{bmatrix} 0 \\ 19 \\ 7 \end{bmatrix}$

B6 (a) $\begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$

(b) $[L]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 4 & 2 \\ 3 & 3 & 0 \end{bmatrix}$

(c) $L(1,4,4) = \begin{bmatrix} 1 \\ 19 \\ 22 \end{bmatrix}$

B7 (a) $[L]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

(b) $[L]_{\mathcal{B}} = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix}$

(c) $[L]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $[L]_{\mathcal{B}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

B8 (a) $[L]_{\mathcal{B}} = \begin{bmatrix} 333 & 198 \\ -550 & -328 \end{bmatrix}$, $[L(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 729 \\ -1206 \end{bmatrix}$

(b) $[L]_{\mathcal{B}} = \begin{bmatrix} 3 & 6 \\ -7 & -4 \end{bmatrix}$, $[L(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}$

(c) $[L]_{\mathcal{B}} = \begin{bmatrix} 0 & 2 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$, $[L(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 16 \\ 4 \\ 18 \end{bmatrix}$

(d) $[L]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$, $[L(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 0 \\ 6 \end{bmatrix}$

B9 (a) $[L]_{\mathcal{B}} = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

(b) $[L]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

(c) $[L]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $[D]_{\mathcal{B}} = \begin{bmatrix} -4/5 & 6/5 & -1/5 \\ -2/5 & -2/5 & -3/5 \\ 14/5 & -16/5 & 6/5 \end{bmatrix}$

(e) $[L]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$

Conceptual Problems

D1 We have $[L]_{\mathcal{B}} = P^{-1}[L]_{\mathcal{S}}P$ so $[L]_{\mathcal{S}} = P[L]_{\mathcal{B}}P^{-1}$. Thus,

$$[L]_{\mathcal{C}} = Q^{-1}[L]_{\mathcal{S}}Q = Q^{-1}P[L]_{\mathcal{B}}P^{-1}Q$$

D2 We have

$$\begin{aligned} A\begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix} &= \begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}\begin{bmatrix}d_1 & 0\\ 0 & d_2\end{bmatrix}\\ \begin{bmatrix}A\vec{v}_1 & A\vec{v}_2\end{bmatrix} &= \begin{bmatrix}d_1\vec{v}_1 & d_2\vec{v}_2\end{bmatrix}\end{aligned}$$

Thus, the vectors $\vec{v}_1$ and $\vec{v}_2$ must satisfy the special condition

$$A\vec{v}_1 = d_1\vec{v}_1 \quad \text{and} \quad A\vec{v}_2 = d_2\vec{v}_2$$

We shall see in Chapter 6 that such vectors are called eigenvectors of A.

D3 Suppose that $\mathbf{x} = b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n$. Then we have

$$\begin{aligned} {}_{\mathcal{C}}[L]_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} &= \begin{bmatrix}[L(\mathbf{v}_1)]_{\mathcal{C}} & \cdots & [L(\mathbf{v}_n)]_{\mathcal{C}}\end{bmatrix}\begin{bmatrix}b_1\\ \vdots\\ b_n\end{bmatrix}\\ &= b_1[L(\mathbf{v}_1)]_{\mathcal{C}} + \cdots + b_n[L(\mathbf{v}_n)]_{\mathcal{C}}\\ &= [b_1L(\mathbf{v}_1) + \cdots + b_nL(\mathbf{v}_n)]_{\mathcal{C}}\\ &= [L(b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n)]_{\mathcal{C}}\\ &= [L(\mathbf{x})]_{\mathcal{C}}\end{aligned}$$

D4 (a) We have

$$\begin{aligned} D(1) &= 0 = 0(1) + 0x\\ D(x) &= 1 = 1(1) + 0x\\ D(x^2) &= 2x = 0(1) + 2x\end{aligned}$$

Thus, ${}_{\mathcal{C}}D_{\mathcal{B}} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}$.

(b) We have

$$\begin{aligned} L(1,-1) &= x^2 = 0(1+x^2) + 1(1+x) + 1(-1-x+x^2)\\ L(1,2) &= 3 + x^2 = 3(1+x^2) - 2(1+x) - 2(-1-x+x^2)\end{aligned}$$

Thus ${}_{\mathcal{C}}L_{\mathcal{B}} = \begin{bmatrix}0 & 3\\ 1 & -2\\ 1 & -2\end{bmatrix}$.

(c) We have

$$T(2,-1) = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = -2\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$
$$T(1,2) = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} = 4\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Thus ${}_{\mathcal{C}}T_{\mathcal{B}} = \begin{bmatrix} -2 & 4 \\ 1 & 3 \\ 2 & -4 \\ 0 & 0 \end{bmatrix}$.

(d) We have

$$L(1+x^2) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$L(1+x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$L(-1+x+x^2) = \begin{bmatrix} 0 \\ 2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus ${}_{\mathcal{C}}L_{\mathcal{B}} = \begin{bmatrix} 3 & 1 & -2 \\ -1 & 0 & 2 \end{bmatrix}$.

4.7 Isomorphisms of Vector Spaces

Practice Problems

A1 (a) We define $L : P_3 \to \mathbb{R}^4$ by $L(a + bx + cx^2 + dx^3) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$.

To prove that it is an isomorphism, we must prove that it is linear, one-to-one and onto.

Linear: Let any two elements of P_3 be $p(x) = a_0 + b_0x + c_0x^2 + d_0x^3$ and $q(x) = a_1 + b_1x + c_1x^2 + d_1x^3$ and let $t \in \mathbb{R}$ then

$$\begin{aligned}
L(tp+q) &= L(t(a_0 + b_0x + c_0x^2 + d_0x^3) + (a_1 + b_1x + c_1x^2 + d_1x^3)) \\
&= L((ta_0 + a_1) + (tb_0 + b_1)x + (tc_0 + c_1)x^2 + (td_0 + d_1)x^3) \\
&= \begin{bmatrix} ta_0 + a_1 \\ tb_0 + b_1 \\ tc_0 + c_1 \\ td_0 + d_1 \end{bmatrix} \\
&= t\begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{bmatrix} + \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} \\
&= tL(p) + L(q)
\end{aligned}$$

Therefore L is linear.

One-to-one: Let $p(x) = a + bx + cx^2 + dx^3 \in \operatorname{Null}(L)$. Then

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = L(a + bx + cx^2 + dx^3) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Hence, $a = b = c = d = 0$. Thus, the only vector in $\operatorname{Null}(L)$ is $\{0\}$. Therefore, L is one-to-one by Lemma 4.7.1.

Onto: For any $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4$ we have $L(a + bx + cx^2 + dx^3) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$. Hence L is onto.

Thus, L is a linear, one-to-one, and onto mapping from P_3 to $\mathbb{R}^4$ and so it is an isomorphism.

(b) We define $L : M(2,2) \to \mathbb{R}^4$ by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$.

To prove that it is an isomorphism, we must prove that it is linear, one-to-one and onto.

Linear: Let any two elements of $M(2,2)$ be $A = \begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix}$ and $B = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and let $t \in \mathbb{R}$ then

$$\begin{aligned}
L(tA + B) &= L\left(t\begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) \\
&= L\left(\begin{bmatrix} ta_0 + a_1 & tb_0 + b_1 \\ tc_0 + c_1 & td_0 + d_1 \end{bmatrix}\right)
\end{aligned}$$

$$= \begin{bmatrix} ta_0 + a_1 \\ tb_0 + b_1 \\ tc_0 + c_1 \\ td_0 + d_1 \end{bmatrix}$$
$$= t\begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{bmatrix} + \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix}$$
$$= tL(A) + L(B)$$

Therefore L is linear.

One-to-one: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{Null}(L)$. Then

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

Hence, $a = b = c = d = 0$. Thus, the only vector in Null(L) is $\{0\}$. Therefore, L is one-to-one by Lemma 4.7.1.

Onto: For any $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4$ we have $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$. Hence L is onto.

Thus, L is a linear, one-to-one, and onto mapping from $M(2,2)$ to $\mathbb{R}^4$ and so it is an isomorphism.

(c) We define $L : P_3 \to M(2,2)$ by $L(a + bx + cx^2 + dx^3) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

To prove that it is an isomorphism, we must prove that it is linear, one-to-one and onto.

Linear: Let any two elements of P_3 be $p(x) = a_0 + b_0x + c_0x^2 + d_0x^3$ and $q(x) = a_1 + b_1x + c_1x^2 + d_1x^3$ and let $t \in \mathbb{R}$ then

$$\begin{aligned} L(tp + q) &= L(t(a_0 + b_0x + c_0x^2 + d_0x^3) + (a_1 + b_1x + c_1x^2 + d_1x^3)) \\ &= L((ta_0 + a_1) + (tb_0 + b_1)x + (tc_0 + c_1)x^2 + (td_0 + d_1)x^3) \\ &= \begin{bmatrix} ta_0 + a_1 & tb_0 + b_1 \\ tc_0 + c_1 & td_0 + d_1 \end{bmatrix} \\ &= t\begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \\ &= tL(p) + L(q) \end{aligned}$$

Therefore L is linear.

One-to-one: Let $p(x) = a + bx + cx^2 + dx^3 \in \text{Null}(L)$. Then

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = L(a + bx + cx^2 + dx^3) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Hence, $a = b = c = d = 0$. Thus, the only vector in Null(L) is $\{\mathbf{0}\}$. Therefore, L is one-to-one by Lemma 4.7.1.

Onto: For any $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^4$ we have $L(a + bx + cx^2 + dx^3) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Hence L is onto.

Thus, L is a linear, one-to-one, and onto mapping from P_3 to $M(2, 2)$ and so it is an isomorphism.

(d) We know that a general vector in $\mathbb{P}$ has the form $(x - 2)(a_1 x + a_0)$. Thus, we define $L : \mathbb{P} \to \mathbb{U}$ by $L((x - 2)(a_1 x + a_0)) = \begin{bmatrix} a_1 & 0 \\ 0 & a_0 \end{bmatrix}$.

Linear: Let any two elements of $\mathbb{P}$ be $\mathbf{a} = (x - 2)(a_1 x + a_0)$ and $\mathbf{b} = (x - 2)(b_1 x + b_0)$ and let $t \in \mathbb{R}$ then

$$\begin{aligned} L(t\mathbf{a} + \mathbf{b}) &= L(t(x - 2)(a_1 x + a_0) + (x - 2)(b_1 x + b_0)) \\ &= L((x - 2)((ta_1 + b_1)x + (ta_0 + b_0)) \\ &= \begin{bmatrix} ta_1 + b_1 & 0 \\ 0 & ta_0 + b_0 \end{bmatrix} \\ &= t \begin{bmatrix} a_1 & 0 \\ 0 & a_0 \end{bmatrix} + \begin{bmatrix} b_1 & 0 \\ 0 & b_0 \end{bmatrix} = tL(\mathbf{a}) + L(\mathbf{b}) \end{aligned}$$

Therefore L is linear.

One-to-one: Assume $L(\mathbf{a}) = L(\mathbf{b})$. Then

$$L((x - 2)(a_1 x + a_0)) = L((x - 2)(b_1 x + b_0)) \Rightarrow \begin{bmatrix} a_1 & 0 \\ 0 & a_0 \end{bmatrix} = \begin{bmatrix} b_1 & 0 \\ 0 & b_0 \end{bmatrix}$$

This gives $a_1 = b_1$ and $a_0 = b_0$ hence $\mathbf{a} = \mathbf{b}$ so L is one-to-one.

Onto: For any $\begin{bmatrix} a_1 & 0 \\ 0 & a_0 \end{bmatrix} \in \mathbb{U}$ we can pick $\mathbf{a} = (x - 2)(a_1 x + a_0) \in \mathbb{P}$ so that we have $L(\mathbf{a}) = \begin{bmatrix} a_1 & 0 \\ 0 & a_0 \end{bmatrix}$ hence L is onto.

Thus, L is an isomorphism from $\mathbb{P}$ to $\mathbb{U}$.

Homework Problems

B1 In each case verify that the given mapping is linear, one-to-one, and onto.

(a) Define $L(a + bx + cx^2 + dx^3 + ex^4) = \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}$.

(b) Define $L\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix}$.

(c) Define $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

(d) Define $L(a_1(x-1) + a_2(x^2 - x) + a_3(x^3 - x^2)) = \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix}$.

Conceptual Problems

D1 Suppose that L is one-to-one and let $\mathbf{x} \in \text{Null}(L)$. Then $L(\mathbf{x}) = \mathbf{0} = L(\mathbf{0})$. Hence, by definition of one-to-one, $\mathbf{x} = \mathbf{0}$. Thus, $\text{Null}(L) = \{\mathbf{0}\}$. On the other hand, assume $\text{Null}(L) = \{\mathbf{0}\}$. If $L(\mathbf{u}_1) = L(\mathbf{u}_2)$, then

$$\mathbf{0} = L(\mathbf{u}_1) - L(\mathbf{u}_2) = L(\mathbf{u}_1 - \mathbf{u}_2)$$

Thus, $\mathbf{u}_1 - \mathbf{u}_2 \in \text{Null}(L)$ and hence, $\mathbf{u}_1 - \mathbf{u}_2 = \vec{0}$. Therefore, $\mathbf{u}_1 = \mathbf{u}_2$ as required.

D2 (a) Let $\mathbf{x} \in \text{Null}(M \circ L)$. Then, $\mathbf{0} = (M \circ L)(\mathbf{x}) = M(L(\mathbf{x}))$. Hence, $L(\mathbf{x}) = \mathbf{0}$, since $\text{Null}(M) = \{\mathbf{0}\}$ by Lemma 1. But, $L(\mathbf{x}) = \mathbf{0}$ also implies $\mathbf{x} = \mathbf{0}$ by Lemma 1. Thus, $\text{Null}(M \circ L) = \{\mathbf{0}\}$ and so $M \circ L$ is one-to-one by Lemma 1.

(b) Let $M(x_1) = (x_1, 0)$ and $L(x_1, x_2) = (x_1)$. Then, $(M \circ L)(x_1, x_2) = (x_1, 0)$, so M is one-to-one, but $M \circ L$ is not one-to-one.

(c) It is not possible. If L is not one-to-one, then the nullspace of L is not trivial, so the nullspace of $M \circ L$ cannot be trivial.

D3 Let $\mathbf{w} \in \mathbb{W}$. Since M is onto, there is some $\mathbf{v} \in \mathbb{V}$ such that $M(\mathbf{v}) = \mathbf{w}$. But, L is also onto, so given $\mathbf{v} \in \mathbb{V}$, there is a $\mathbf{u} \in \mathbb{U}$ such that $\mathbf{v} = L(\mathbf{u})$. Thus, $\mathbf{w} = (M \circ L)(\mathbf{u})$.

D4 Since L is onto, for any $\mathbf{v} \in \mathbb{V}$ there is a $\mathbf{u} \in \mathbb{U}$ such that $L(\mathbf{u}) = \mathbf{v}$. Since L is one-to-one, there is exactly one such $\mathbf{u}$. Hence, we may define a mapping $L^{-1} : \mathbb{V} \to \mathbb{U}$ by $L^{-1}(\mathbf{v}) = \mathbf{u}$ if and only if $\mathbf{v} = L(\mathbf{u})$. It is easy to verify that this mapping is linear.

D5 If they are isomorphic, then by Exercise 3, they have bases with the same number of vectors, so they are of the same dimension.
If they are of the same dimension, let $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be a basis for $\mathbb{U}$, and $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis for $\mathbb{V}$. Define $L : \mathbb{U} \to \mathbb{V}$ by

$$L(t_1\mathbf{u}_1 + \cdots + t_n\mathbf{u}_n) = t_1\mathbf{v}_1 + \cdots + t_n\mathbf{v}_n$$

It is easy to verify that L is linear, one-to-one, and onto. Hence, $\mathbb{U}$ and $\mathbb{V}$ are isomorphic.

D6 Let $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be a basis for $\mathbb{U}$.
Suppose that L is one-to-one. By Exercise 1, $\{L(\mathbf{u}_1), \dots, L(\mathbf{u}_n)\}$ is linearly independent and hence a basis for $\mathbb{V}$. Thus, every vector in $\mathbf{v} = \mathbb{V}$ can be written as

$$\mathbf{v} = t_1L(\mathbf{u}_1) + \cdots + t_nL(\mathbf{u}_n) = L(t_1\mathbf{u}_1 + \cdots + t_n\mathbf{u}_n)$$

so L is onto.
If L is onto, then $\{L(\mathbf{u}_1), \dots, L(\mathbf{u}_n)\}$ is a spanning set for $\mathbb{V}$ and hence a basis for $\mathbb{V}$. Let $\mathbf{v} \in \text{Null}(L)$. Then,

$$\mathbf{0} = L(\mathbf{v}) = L(t_1\mathbf{u}_1 + \cdots + t_n\mathbf{u}_n) = t_1L(\mathbf{u}_1) + \cdots + t_nL(\mathbf{u}_n)$$

But, $\{L(\mathbf{u}_1), \dots, L(\mathbf{u}_n)\}$ is a basis and hence linearly independent. Thus, $t_1 = \cdots = t_n = 0$, so $\mathbf{v} = \mathbf{0}$. Thus, $\text{Null}(L) = \{\mathbf{0}\}$ and hence L is one-to-one by Lemma 1.

D7 Any plane through the origin in $\mathbb{R}^3$ has vector equation $\vec{x} = s\vec{u} + t\vec{v}$, $s, t \in \mathbb{R}$, where $\{\vec{u}, \vec{v}\}$ is linearly independent. Thus, $\{\vec{u}, \vec{v}\}$ is a basis for the plane and hence a plane through the origin is two dimensional. Thus, by Theorem 3, a plane through the origin in $\mathbb{R}^3$ is isomorphic to $\mathbb{R}^2$.

D8 Since $(\mathbf{u}_1, \mathbf{0})+(\mathbf{u}_2, \mathbf{0}) = (\mathbf{u}_1+\mathbf{u}_2, \mathbf{0})$, and $t(\mathbf{u}_1, \mathbf{0}) = (t\mathbf{u}_1, \mathbf{0})$; $\mathbb{U}\times\{\mathbf{0}_{\mathbb{V}}\}$ is a subspace of $\mathbb{U}\times\mathbb{V}$. Define an isomorphism $L : \mathbb{U} \times \{\mathbf{0}_{\mathbb{V}}\} \to \mathbb{U}$ by $L(\mathbf{u}, \mathbf{0}) = \mathbf{u}$. Check that this is linear, one-to-one, and onto.

D9 (a) Check that $L((u_1, u_2), v) = (u_1, u_2, v)$ is an isomorphism.

(b) Check that $L((u_1, \dots, u_n), (v_1, \dots, v_m)) = (u_1, \dots, u_n, v_1, \dots, v_m)$ is an isomorphism.

D10 Let $\mathbf{x} \in \mathbb{U}$ so that $L(\mathbf{x}) \in \mathbb{V}$. Then $(M \circ L)(\mathbf{x}) \in \mathbb{V}$, so $(L^{-1} \circ M \circ L)(\mathbf{x}) \in \mathbb{U}$. Thus, $L^{-1} \circ M \circ L$ is a linear mapping from $\mathbb{U}$ to $\mathbb{U}$ (a composition of linear mappings is linear).
Note that since L is invertible, both L and L^{-1} are one-to-one, and hence have trivial nullspaces.
The range of $L^{-1} \circ M \circ L$ is the "isomorphic image" in $\mathbb{U}$ under the mapping L^{-1} of the range of $M \circ L$. Since L is an isomorphism, L is onto, so the range of L is all of the domain of M; hence the range of $M \circ L$ equals the range of M. Thus the range of $L^{-1} \circ M \circ L$ is the subspace in $\mathbb{U}$ that is isomorphic (under L^{-1}) to the range of M. Similarly, the nullspace of $L^{-1} \circ M \circ L$ is isomorphic to the nullspace of M. In particular, it is the isomorphic image (under L^{-1}) of the nullspace of M.

Chapter 4 Quiz

Problems

E1 (a) The given set is a subset of $M(4, 3)$ and is non-empty since it clearly contains the zero matrix. Let A and B be any two vectors in the set. Then, $a_{11} + a_{12} + a_{13} = 0$ and $b_{11} + b_{12} + b_{13} = 0$. Then, the first row of $A + B$ satisfies

$$a_{11} + b_{11} + a_{12} + b_{12} + a_{13} + b_{13} = a_{11} + a_{12} + a_{13} + b_{11} + b_{12} + b_{13} = 0 + 0 = 0$$

so the subset is closed under addition. Similarly, for any $t \in \mathbb{R}$, the first row of tA satisfies

$$ta_{11} + ta_{12} + ta_{13} = t(a_{11} + a_{12} + a_{13}) = 0$$

so the subset is also closed under scalar multiplication. Thus, it is a subspace of $M(4, 3)$ and hence a vector space.

(b) The given set is a subset of the vector space of all polynomials and is clearly contains the zero polynomial so it is non-empty. Let $p(x)$ and $q(x)$ be in the set. Then $p(1) = 0$, $p(2) = 0$, $q(1) = 0$, and $q(2) = 0$. Hence, $p + q$ satisfies

$$(p + q)(1) = p(1) + q(1) = 0 \qquad \text{and} \qquad (p + q)(2) = p(2) + q(2) = 0$$

so the subset is closed under addition. Similarly, for any $t \in \mathbb{R}$, the first row of tp satisfies

$$(tp)(1) = tp(1) = 0 \qquad \text{and} \qquad (tp)(2) = tp(2) = 0$$

so the subset is also closed under scalar multiplication. Thus, it is a subspace and hence a vector space.

(c) The set is not a vector space since it is not closed under scalar multiplication. For example $\frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ is not in the set since it contains rational entries.

(d) The given set is a subset of $\mathbb{R}^3$ and is non-empty since it clearly contains $\vec{0}$. Let $\vec{x}$ and $\vec{y}$ be in the set. Then, $x_1 + x_2 + x_3 = 0$ and $y_1 + y_2 + y_3 = 0$. Then, $\vec{x} + \vec{y}$ satisfies

$$x_1 + y_1 + x_2 + y_2 + x_3 + y_3 = x_1 + x_2 + x_3 + y_1 + y_2 + y_3 = 0 + 0 = 0$$

so the subset is closed under addition. Similarly, for any $t \in \mathbb{R}$, $t\vec{x}$ satisfies

$$tx_1 + tx_2 + tx_3 = t(x_1 + x_2 + x_3) = 0$$

so the subset is also closed under scalar multiplication. Thus, it is a subspace of $\mathbb{R}^3$ and hence a vector space.

E2 (a) A set of five vectors in $M(2,2)$ must be linearly dependent by Theorem 4.3.4, so the set cannot be basis.

(b) Consider

$$t_1\begin{bmatrix}1 & 1\\ 2 & 1\end{bmatrix} + t_2\begin{bmatrix}0 & 1\\ 1 & -1\end{bmatrix} + t_3\begin{bmatrix}0 & 1\\ 1 & 3\end{bmatrix} + t_4\begin{bmatrix}2 & 2\\ 4 & -2\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Row reducing the coefficient matrix of the corresponding system gives

$$\begin{bmatrix}1 & 0 & 0 & 2\\ 1 & 1 & 1 & 2\\ 2 & 1 & 1 & 4\\ 1 & -1 & 3 & -2\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0 & 2\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Thus, the system has infinitely many solutions, so the set is linearly dependent and hence it is not a basis.

(c) By Theorem 4.3.4, a set of three vectors in $M(2,2)$ cannot span $M(2,2)$, so the set cannot be basis.

E3 (a) Consider

$$t_1\begin{bmatrix}1\\0\\1\\1\\3\end{bmatrix} + t_2\begin{bmatrix}1\\1\\0\\1\\1\end{bmatrix} + t_3\begin{bmatrix}3\\3\\1\\0\\2\end{bmatrix} + t_4\begin{bmatrix}1\\1\\1\\-2\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$$

Row reducing the coefficient matrix of the corresponding system gives

$$\begin{bmatrix}1 & 1 & 3 & 1\\ 0 & 1 & 3 & 1\\ 1 & 0 & 1 & 1\\ 1 & 1 & 0 & -2\\ 3 & 1 & 2 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & -2\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

Thus, $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent set. Moreover, $\vec{v}_4$ can be written as a linear combination of the $\vec{v}_1, \vec{v}_2, \vec{v}_3$, so $\mathcal{B}$ also spans $\mathbb{S}$. Hence, it is a basis for $\mathbb{S}$ and so $\dim \mathbb{S} = 3$.

(b) We need to find constants t_1, t_2, t_3 such that

$$t_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 3 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + t_3 \begin{bmatrix} 3 \\ 3 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \\ -3 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 3 & 0 \\ 1 & 1 & 3 & 2 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & -3 \\ 3 & 1 & 2 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$.

E4 (a) Let $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Then, $\{\vec{v}_1, \vec{v}_2\}$ is a basis for the plane since it is a set of two linearly independent vectors in the plane.

(b) Since $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ does not lie in the plane, the set $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent and hence a basis for $\mathbb{R}^3$.

(c) By geometric arguments, we have that

$$\begin{aligned} L(\vec{v}_1) &= \vec{v}_1 = 1\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 \\ L(\vec{v}_2) &= \vec{v}_2 = 0\vec{v}_1 + 1\vec{v}_2 + 0\vec{v}_3 \\ L(\vec{v}_3) &= -\vec{v}_3 = 0\vec{v}_1 + 0\vec{v}_2 + (-1)\vec{v}_3 \end{aligned}$$

So

$$[L]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(d) The change of coordinates matrix from $\mathcal{B}$-coordinates to $\mathcal{S}$-coordinates (standard coordinates) is

$$P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

It follows that

$$[L]_{\mathcal{S}} = P[L]_{\mathcal{B}}P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

E5 The change of coordinates matrix from $\mathcal{S}$ to $\mathcal{B}$ is

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

Hence,

$$[L]_{\mathcal{B}} = P^{-1} \begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & 1 \\ -2 & 1 & 0 \end{bmatrix} P = \begin{bmatrix} 0 & 2/3 & 11/3 \\ -1 & 2/3 & -10/3 \\ 0 & 1/3 & 1/3 \end{bmatrix}$$

E6 If $t_1 L(\mathbf{v}_1) + \cdots + t_k L(\mathbf{v}_k) = \mathbf{0}$, then

$$\mathbf{0} = L(t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k)$$

and hence $t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k \in \text{Null}(L)$. Thus,

$$t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k = \mathbf{0}$$

and hence $t_1 = \cdots = t_k = 0$ since $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is linearly independent. Thus, $\{L(\mathbf{v}_1), \dots, L(\mathbf{v}_k)\}$ is linearly independent.

E7 (a) FALSE. $\mathbb{R}^n$ is an n-dimensional subspace of $\mathbb{R}^n$.

(b) TRUE. The dimension of P_2 is 3, so a set of 4 polynomials in P_2 must be linearly dependent.

(c) FALSE. The number of components in a coordinate vector is the number of vectors in the basis. So, if $\mathcal{B}$ is a basis for a 4 dimensional subspace, then the $\mathcal{B}$-coordinate vector would have only 4 components.

(d) TRUE. Both ranks are equal to the dimension of the range of L.

(e) FALSE. If $L : P_2 \to P_2$ is a linear mapping, then the range of L is a subspace of P_2, but the columnspace of $[L]_{\mathcal{B}}$ is a subspace of $\mathbb{R}^3$. Hence, they cannot equal.

(f) FALSE. The mapping $L : \mathbb{R} \to \mathbb{R}^2$ given by $L(x_1) = (x_1, 0)$ is one-to-one, but $\dim \mathbb{R} \neq \dim \mathbb{R}^2$.

Chapter 4 Further Problems

Problems

F1 Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ be a basis for $\mathbb{S}$. Extend $\{\vec{v}_1, \dots, \vec{v}_k\}$ to a basis $C = \{\vec{v}_1, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_n\}$ for $\mathbb{V}$. Define a mapping L by

$$L(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_{k+1}\vec{v}_{k+1} + \cdots + c_n\vec{v}_n$$

For any $\vec{x}, \vec{y} \in \mathbb{V}$ and $t \in \mathbb{R}$ we have

$$\begin{aligned} L(t\vec{x} + \vec{y}) &= L((tc_1 + d_1)\vec{v}_1 + \cdots + (tc_n + d_n)\vec{v}_n) \\ &= (tc_{k+1} + d_{k+1})\vec{v}_{k+1} + \cdots + (tc_n + d_n)\vec{v}_n \\ &= t(c_{k+1}\vec{v}_{k+1} + \cdots + c_n\vec{v}_n) + d_{k+1}\vec{v}_{k+1} + \cdots + d_n\vec{v}_n \\ &= tL(\vec{x}) + L(\vec{y}) \end{aligned}$$

Hence L is linear.

If $\vec{x} \in \text{Null}(L)$, then
$$\vec{0} = L(\vec{x}) = L(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_{k+1}\vec{v}_{k+1} + \cdots + c_n\vec{v}_n$$

Hence, $c_{k+1} = \cdots = c_n = 0$, since $\{\vec{v}_{k+1}, \dots, \vec{v}_n\}$ is linearly independent. Thus, $\vec{x} \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \mathbb{S}$.
If $\vec{x} \in \mathbb{S}$, then $\vec{x} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$ and

$$L(\vec{x}) = L(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) = \vec{0}$$

Therefore, $\vec{x} \in \text{Null}(L)$.

Thus, $\text{Null}(L) = \mathbb{S}$ as required.

F2 Suppose that the $n \times n$ matrix A is row equivalent to both R and S. Denote the columns of R by $\vec{r}_j$, and the columns of S by $\vec{s}_j$. Then R is row equivalent to S. In particular, the two matrices have the same solution space. So, $R\vec{x} = \vec{0}$ if and only if $S\vec{x} = \vec{0}$.
First, note that $\vec{r}_j = \vec{0}$ if and only if $R\vec{e}_j = \vec{0}$ if and only if $S\vec{e}_j = \vec{0}$ if and only if $\vec{s}_j = \vec{0}$. Thus, R and S must have the same zero columns. Therefore, to simplify the discussion, we assume that R and S do not have any zero columns.
Then the first column of R and S must both be $\vec{e}_1$. By definition of RREF, the next column of R is either $\vec{r}_2 = \vec{e}_2$ or $\vec{r}_2 = r_{12}\vec{e}_1$. In the first case, the first two columns of R are linearly independent, so the first two columns of S must be linearly independent because $R\vec{x} = \vec{0}$ if and only if $S\vec{x} = \vec{0}$. Hence, $\vec{s}_2 = \vec{e}_2$. In the second case, we have $R(-r_{12}\vec{e}_1 + \vec{e}_2) = \vec{0}$, so
$$\vec{0} = S(-r_{12}\vec{e}_1 + \vec{e}_2) = -r_{12}\vec{s}_1 + \vec{s}_2$$

Thus, $\vec{s}_2 = -r_{12}\vec{s}_1 = -r_{12}\vec{e}_1 = \vec{r}_2$.
Suppose that the first k columns are equal and that in these k columns there are j leading ones. Thus, by definition of RREF, we know that the columns containing the leading ones are $\vec{e}_1, \dots, \vec{e}_j$. As above, either $\vec{r}_{k+1} = \vec{e}_{j+1}$ or $\vec{r}_{k+1} = r_{1(k+1)}\vec{e}_1 + \cdots + r_{j(k+1)}\vec{e}_j$. In either case, we can show that $\vec{s}_{j+1} = \vec{r}_{j+1}$ using the fact that $R\vec{x} = \vec{0}$ if and only if $S\vec{x} = \vec{0}$. Hence, by induction $R = S$.

F3 (a) The zero matrix is a magic square of weight zero, so MS_3 is non-empty. Suppose that A is a magic square of weight k and B is a magic square of weight j. Then

$$(A + B)_{11} + (A + B)_{12} + (A + B)_{13} = a_{11} + b_{11} + a_{12} + b_{12} + a_{13} + b_{13} = k + j$$

Similarly, all other row sums, column sums, and diagonal sums are $k + j$, so $A + B$ is a magic square of weight $k + j$.

By a similar calculation, all row sums, column sums, and diagonal sums of tA are tk, so tA is a magic square of weight tk. Thus MS_3 is closed under addition and scalar multiplication, so it is a subspace of $M(3, 3)$.

(b) Suppose that A and B are magic squares of weights a and b respectively. Then

$$\text{wt}(tA + B) = (ta_{11} + b_{11}) + (ta_{12} + b_{12}) + (ta_{13} + b_{13}) = t\,\text{wt}(A) + \text{wt}(B)$$

So *wt* is a linear mapping.

(c) Since $\underline{X}_1 = \begin{bmatrix} 1 & 0 & a \\ b & c & d \\ e & f & g \end{bmatrix}$ is in the nullspace of wt, all row sums are equal to zero. From the first row, $a = -1$.
The column sums are also zero, so $e = -1 - b$, $f = -c$, $g = 1 - d$. The diagonal sums are also zero, so

$1 + c + 1 - d = 0$ and $-1 + c - 1 - b = 0$. These give $d = 2 + c$ and $b = -2 + c$. Next, the second row sum is zero, so $-2 + c + c + 2 + c = 0$ which implies $c = 0$. Thus, $\underline{X}_1 = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}$. Now consider $\underline{X}_2 = \begin{bmatrix} 0 & 1 & h \\ i & j & k \\ l & m & n \end{bmatrix}$. From the first row, $h = -1$. From the columns, $l = -i$, $m = -1 - j$, $n = 1 - k$. From the diagonal sums we find that $k = 1 + j$ and $i = -1 + j$. From the second row sum it then follows that $j = 0$. Thus, $\underline{X}_2 = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$. Finally, consider $\underline{O} = \begin{bmatrix} 0 & 0 & p \\ q & r & s \\ t & u & v \end{bmatrix}$. From the first row, $p = 0$. From the columns, $t = -q$, $u = -r$, and $v = -s$. From the diagonals, $r = -v$ and $r = -t = q$. Then the second row gives $r = 0$, and $\underline{O} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Does $\mathcal{B} = \{\underline{X}_1, \underline{X}_2\}$ form a basis for this nullspace? It is easy to see that $s\underline{X}_1 + t\underline{X}_2 = \underline{O}$ if and only if $s = t = 0$, so $\mathcal{B}$ is linearly independent. Suppose that A is in Null(wt). Then

$$0 = \mathrm{wt}(A) - a_{11}\,\mathrm{wt}(\underline{X}_1) - a_{12}\,\mathrm{wt}(\underline{X}_2) = \mathrm{wt}(A - a_{11}\underline{X}_1 - a_{12}\underline{X}_2)$$

This implies that $A - a_{11}\underline{X}_1 - a_{12}\underline{X}_2 \in \mathrm{Null(wt)}$. By construction, it is of the form $\begin{bmatrix} 0 & 0 & * \\ * & * & * \\ & * & * \end{bmatrix}$. From out work above, this implies that $A - a_{11}\underline{X}_1 - a_{12}\underline{X}_2 = \underline{O}$, so

$$A = a_{11}\underline{X}_1 - a_{12}\underline{X}_2$$

Thus, $\mathcal{B}$ is a basis for Null(wt).

(d) Clearly all row, column, and diagonal sums of $\underline{J}$ are equal to 3, so $\underline{J}$ is a magic square of weight 3. Let A be any magic square of weight k. Then,

$$\mathrm{wt}\left(A - \frac{k}{3}\underline{J}\right) = \mathrm{wt}(A) - \frac{k}{3}\,\mathrm{wt}(\underline{J}) = k - k = 0$$

Therefore, $A - \frac{k}{3}\underline{J}$ is in the nullspace of wt. Thus, for some $s, t \in \mathbb{R}$ we have $A - \frac{k}{3}\underline{J} = s\underline{X}_1 + t\underline{X}_2$. That is

$$A = \frac{k}{3}\underline{J} + s\underline{X}_1 + t\underline{X}_2$$

(e) Since $\{\underline{X}_1, \underline{X}_2\}$ is linearly independent and $\underline{J} \notin \mathrm{Null(wt)}$, it follows that $\{\underline{J}, \underline{X}_1, \underline{X}_2\}$ is linearly independent. Part (d) shows that this set spans MS_3 and hence is a basis for MS_3.

(f) A is of weight 6, so $k/3 = 2$. Hence,

$$A = 2\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} + s\begin{bmatrix} 1 & 0 & -1 \\ -2 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix} + t\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

It is easy to check that $s = 1$ and $t = -1$, so the coordinates of A with respect to this basis is $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

F4 Let f and g be even functions. Then

$$(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x)$$
$$(tf)(-x) = tf(-x) = tf(x)$$

so the set of even function is a subspace of continuous functions. The proof that the odd functions form a subspace is similar.
For an arbitrary function f, write

$$f(x) = \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x))$$

Let

$$f^+ = \frac{1}{2}(f(x) + f(-x)), \quad f^- = \frac{1}{2}(f(x) - f(-x))$$

Then,

$$f^+(-x) = \frac{1}{2}(f(-x) + f(x)) = \frac{1}{2}(f(x) + f(-x)) = f^+(x)$$
$$f^-(-x) = \frac{1}{2}(f(-x) - f(x)) = -\frac{1}{2}(f(x) - f(-x)) = -f^-(x)$$

So f^+ is even and f^- is odd. Thus every function can be represented as a sum of an even function and an odd function. Thus, the subspace of even functions and the subspace of odd functions are complements of each other.

F5 (a) Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be a basis for $\mathbb{S}$ and let $\{\vec{w}_1, \dots, \vec{w}_l\}$ be a basis for some complement $\mathbb{T}$. Since $\mathbb{S} + \mathbb{T} = \mathbb{R}^n$, $\{\vec{v}_1, \dots, \vec{v}_k, \vec{w}_1, \dots, \vec{w}_l\}$ is a spanning set for $\mathbb{R}^n$. Suppose for some $s_1, \dots, s_k, t_1, \dots, t_l$

$$s_1\vec{v}_1 + \cdots + s_k\vec{v}_k + t_1\vec{w}_1 + \cdots + t_l\vec{w}_l = \vec{0}$$

Then

$$s_1\vec{v}_1 + \cdots + s_k\vec{v}_k = -t_1\vec{w}_1 - \cdots - t_l\vec{w}_l$$

and this vector lies in both $\mathbb{S}$ and $\mathbb{T}$. Since $\mathbb{S} \cup \mathbb{T} = \{\vec{0}\}$, and we know that $\{\vec{v}_1, \dots, \vec{v}_k\}$ and $\{\vec{w}_1, \dots, \vec{w}_l\}$ are both linearly independent, it follows that $s_1 = \cdots = s_k = t_1 = \cdots = t_l = 0$. Thus, $\{\vec{v}_1, \dots, \vec{v}_k, \vec{w}_1, \dots, \vec{w}_l\}$ is also linearly independent and hence a basis for $\mathbb{R}^n$. Therefore, $l = n - k$.

(b) Yes. Suppose that $\mathbb{S}$ is neither $\{\vec{0}\}$ nor $\mathbb{R}^n$. Then $\mathbb{S}$ is k-dimensional with $0 < k < n$. Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be a basis for $\mathbb{S}$, and let $\{\vec{v}_{k+1}, \dots, \vec{v}_n\}$ be a basis for some complement $\mathbb{T}$. Then $\text{Span}\{\vec{v}_k + \vec{v}_{k+1}, \vec{v}_{k+2}, \dots, \vec{v}_n\}$ is an $n-k$ dimensional subspace of $\mathbb{R}^n$, not equal to $\mathbb{T}$. Thus, it is another complement for $\mathbb{S}$. Therefore, only $\{\vec{0}\}$ and $\mathbb{R}^n$ have unique complements.

F6 Suppose that $\mathbb{T} = \text{Span}\{\vec{v}, \mathbb{S}\}$ and $\mathbb{U} = \text{Span}\{\vec{w}, \mathbb{S}\}$. Suppose that $\vec{w} \in \mathbb{T}$. Every vector in $\mathbb{T}$ can be written in the form $t\vec{v} + \vec{s}$ for some $t \in \mathbb{R}$ and $\vec{s} \in \mathbb{S}$. Hence, $\vec{w} = t\vec{v} + \vec{s}$. If $\vec{w} \notin \mathbb{S}$, t cannot be zero, so

$$\vec{v} = \frac{1}{t}(\vec{w} - \vec{s})$$

for some $\vec{s} \in \mathbb{S}$. Thus, $\vec{v} \in \mathbb{U}$.

F7 $\mathbb{S}\cup\mathbb{T}$ is a subspace of $\mathbb{S}$, so it is certainly finite dimensional. Suppose that $\dim\mathbb{S} = s$, $\dim\mathbb{T} = t$, and $\dim\mathbb{S}\cup\mathbb{T} = k$. We consider first the case where $k < t$, $s < t$. Then there is a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ for $\mathbb{S}\cup\mathbb{T}$. This can be extended to a basis for $\mathbb{S}$ by adding vectors $\{\mathbf{w}_1, \dots, \mathbf{w}_{s-k}\}$. Since $k < t$, $\mathbb{T}$ contains vectors not in $\mathbb{S}$. In fact, since $\mathbb{T}$ has dimension t, we can extend $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ to a basis for $\mathbb{T}$ by adding a linearly independent set of vectors $\{\mathbf{z}_1, \dots, \mathbf{z}_{t-k}\}$ in $\mathbb{T}$. But then

$$\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{w}_1, \dots, \mathbf{w}_{s-k}, \mathbf{z}_1, \dots, \mathbf{z}_{t-k}\}$$

is a minimal spanning set for $\mathbb{S}+\mathbb{T}$, and hence a basis, and

$$\begin{aligned}\dim(\mathbb{S}+\mathbb{T}) &= k + (s-k) + (t-k) = s + t - k\\ &= \dim\mathbb{S} + \dim\mathbb{T} - \dim\mathbb{S}\cup\mathbb{T}\end{aligned}$$

If $s = k$, then $\mathbb{S}\cup\mathbb{T} = \mathbb{S}$. Similarly, if $t = k$, $\mathbb{T} = \mathbb{T}\cup\mathbb{T}$. In either case, the result can be proven by a slight modification of the argument above.

CHAPTER 5 Determinants

5.1 Determinants in Terms of Cofactors

Practice Problems

A1 (a) $\begin{vmatrix} 2 & -4 \\ 7 & 5 \end{vmatrix} = 2(5) - (-4)7 = 38$

(b) $\begin{vmatrix} -3 & 1 \\ 2 & 1 \end{vmatrix} = (-3)(1) - 1(2) = -5$

(c) $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1(1) - 1(1) = 0$

(d) Since the third column is all zeros we get the determinant is 0.

(e) Since the matrix is upper triangular, the determinant is the product of the diagonal entries. Hence, the determinant is $1 \cdot 0 \cdot 3 = 0$.

(f) This matrix is not upper or lower triangular, so we cannot apply the theorem. However, the technique used to prove the theorem can be used to evaluate this determinant.

Expanding the determinant along the bottom row gives

$$\begin{vmatrix} 1 & 5 & -7 & 8 \\ 2 & -1 & 3 & 0 \\ -4 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = 1(-1)^{4+1} \begin{vmatrix} 5 & -7 & 8 \\ -1 & 3 & 0 \\ 2 & 0 & 0 \end{vmatrix} + 0 + 0 + 0 = (-1) \begin{vmatrix} 5 & -7 & 8 \\ -1 & 3 & 0 \\ 2 & 0 & 0 \end{vmatrix}$$

We then expand this determinant along the bottom row to get

$$\begin{aligned} \begin{vmatrix} 1 & 5 & -7 & 8 \\ 2 & -1 & 3 & 0 \\ -4 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} &= (-1)2(-1)^{3+1} \begin{vmatrix} -7 & 8 \\ 3 & 0 \end{vmatrix} \\ &= (-1)(2)[(-7)(0) - (8)(3)] = 48 \end{aligned}$$

A2 (a) We have

$$\begin{vmatrix} 3 & 4 & 0 \\ 2 & 1 & -1 \\ -4 & -1 & 2 \end{vmatrix} = 3(-1)^{1+1}\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} + 4(-1)^{1+2}\begin{vmatrix} 2 & -1 \\ -4 & 2 \end{vmatrix} + 0$$
$$= 3[1(2) - (-1)(-1)] + 4(-1)[(2(2) - (-1)(-4)]$$
$$= 3(1) - 4(0) = 3$$

(b) We have

$$\begin{vmatrix} 3 & 2 & 1 \\ -1 & 4 & 5 \\ 3 & 2 & 1 \end{vmatrix} = 3(-1)^{1+1}\begin{vmatrix} 4 & 5 \\ 2 & 1 \end{vmatrix} + 2(-1)^{1+2}\begin{vmatrix} -1 & 5 \\ 3 & 1 \end{vmatrix} + 1(-1)^{1+3}\begin{vmatrix} -1 & 4 \\ 3 & 2 \end{vmatrix}$$
$$= 3[4(1) - 5(2)] + 2(-1)[(-1)(1) - 5(3)] + 1[(-1)(2) - 4(3)]$$
$$= 3(-6) - 2(-16) + 1(-14) = 0$$

(c) We have

$$\begin{vmatrix} 2 & 1 & 0 & -1 \\ 0 & 3 & 2 & 1 \\ -4 & 0 & 2 & -2 \\ 3 & -5 & 2 & 1 \end{vmatrix} = 2(-1)^{1+1}\begin{vmatrix} 3 & 2 & 1 \\ 0 & 2 & -2 \\ -5 & 2 & 1 \end{vmatrix} + 1(-1)^{1+2}\begin{vmatrix} 0 & 2 & 1 \\ -4 & 2 & -2 \\ 3 & 2 & 1 \end{vmatrix} +$$
$$0 + (-1)(-1)^{1+4}\begin{vmatrix} 0 & 3 & 2 \\ -4 & 0 & 2 \\ 3 & -5 & 2 \end{vmatrix}$$

We now need to evaluate each of these 3×3 determinants. We have

$$\begin{vmatrix} 3 & 2 & 1 \\ 0 & 2 & -2 \\ -5 & 2 & 1 \end{vmatrix} = 3(-1)^{1+1}\begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} + 2(-1)^{1+2}\begin{vmatrix} 0 & -2 \\ -5 & 1 \end{vmatrix} + 1(-1)^{1+3}\begin{vmatrix} 0 & 2 \\ -5 & 2 \end{vmatrix}$$
$$= 3[2(1) - (-2)(2)] - 2[0(1) - (-2)(-5)] + 1[0(2) - 2(-5)]$$
$$= 3(6) - 2(-10) + 1(10) = 48$$

$$\begin{vmatrix} 0 & 2 & 1 \\ -4 & 2 & -2 \\ 3 & 2 & 1 \end{vmatrix} = 0 + 2(-1)^{1+2}\begin{vmatrix} -4 & -2 \\ 3 & 1 \end{vmatrix} + 1(-1)^{1+3}\begin{vmatrix} -4 & 2 \\ 3 & 2 \end{vmatrix}$$
$$= (-2)[(-4)(1) - (-2)(3)] + 1[(-4)(2) - 2(3)]$$
$$= (-2)(2) + 1(-14) = -18$$

$$\begin{vmatrix} 0 & 3 & 2 \\ -4 & 0 & 2 \\ 3 & -5 & 2 \end{vmatrix} = 0 + 3(-1)^{1+2}\begin{vmatrix} -4 & 2 \\ 3 & 2 \end{vmatrix} + 2(-1)^{1+3}\begin{vmatrix} -4 & 0 \\ 3 & -5 \end{vmatrix}$$
$$= (-3)[(-4)(2) - 2(3)] + 2[(-4)(-5) + 0(3)]$$
$$= (-3)(-14) + 2(20) = 82$$

Hence,

$$\begin{vmatrix} 2 & 1 & 0 & -1 \\ 0 & 3 & 2 & 1 \\ -4 & 0 & 2 & -2 \\ 3 & -5 & 2 & 1 \end{vmatrix} = 2(48) + (-1)(-18) + 1(82) = 196$$

(d) We have

$$\begin{vmatrix} 1 & 0 & 4 & 0 \\ 2 & -3 & 4 & 1 \\ -1 & 3 & 2 & 4 \\ 1 & 1 & -2 & 4 \end{vmatrix} = 1(-1)^{1+1} \begin{vmatrix} -3 & 4 & 1 \\ 3 & 2 & 4 \\ 1 & -2 & 4 \end{vmatrix} + 0 + 4(-1)^{1+3} \begin{vmatrix} 2 & -3 & 1 \\ -1 & 3 & 4 \\ 1 & 1 & 4 \end{vmatrix} + 0$$

Next, we evaluate each of the 3×3 determinants.

$$\begin{aligned} \begin{vmatrix} -3 & 4 & 1 \\ 3 & 2 & 4 \\ 1 & -2 & 4 \end{vmatrix} &= (-3)(-1)^{1+1} \begin{vmatrix} 2 & 4 \\ -2 & 4 \end{vmatrix} + 4(-1)^{1+2} \begin{vmatrix} 3 & 4 \\ 1 & 4 \end{vmatrix} + 1(-1)^{1+3} \begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix} \\ &= (-3)[(2)(4) - 4(-2)] - 4[(3(4) - 4(1)] + 1[3(-2) - 2(1)] \\ &= (-3)(16) - 4(8) + 1(-8) = -88 \end{aligned}$$

$$\begin{aligned} \begin{vmatrix} 2 & -3 & 1 \\ -1 & 3 & 4 \\ 1 & 1 & 4 \end{vmatrix} &= 2(-1)^{1+1} \begin{vmatrix} 3 & 4 \\ 1 & 4 \end{vmatrix} + (-3)(-1)^{1+2} \begin{vmatrix} -1 & 4 \\ 1 & 4 \end{vmatrix} + 1(-1)^{1+3} \begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} \\ &= 2[(3(4) - 4(1)] + 3[(-1)(4) - 4(1)] + 1[(-1)(1) - 3(1)] \\ &= 2(8) + 3(-8) + 1(-4) = -12 \end{aligned}$$

Hence,

$$\begin{vmatrix} 1 & 0 & 4 & 0 \\ 2 & -3 & 4 & 1 \\ -1 & 3 & 2 & 4 \\ 1 & 1 & -2 & 4 \end{vmatrix} = 1(-88) + 4(-12) = -136$$

A3 (a) By expanding the determinant along the third column we get that the determinant is 0.

(b) Since there are no zeros in the matrix, it does not matter which row or column we expand along. We will expand along the second column.

$$\begin{aligned} \begin{vmatrix} -5 & 2 & -4 \\ 2 & -4 & 6 \\ -6 & 2 & -3 \end{vmatrix} &= 2(-1)^{1+2} \begin{vmatrix} 2 & 6 \\ -6 & -3 \end{vmatrix} + (-4)(-1)^{2+2} \begin{vmatrix} -5 & -4 \\ -6 & -3 \end{vmatrix} + 2(-1)^{3+2} \begin{vmatrix} -5 & -4 \\ 2 & 6 \end{vmatrix} \\ &= -2[(2(-3) - 6(-6)] - 4[(-5)(-3) - (-4)(-6)] - 2[6(-5) - 2(-4)] \\ &= (-2)(30) + (-4)(-9) + (-2)(-22) = 20 \end{aligned}$$

(c) Expanding the determinant along the first column gives

$$\begin{aligned} \begin{vmatrix} 2 & 1 & 5 \\ 4 & 3 & -1 \\ 0 & 1 & -2 \end{vmatrix} &= 2(-1)^{1+1} \begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix} + 4(-1)^{2+1} \begin{vmatrix} 1 & 5 \\ 1 & -2 \end{vmatrix} + 0 \\ &= 2[(3(-2) - (-1)(1)] - 4[1(-2) - 5(1)] \\ &= 2(-5) - 4(-7) = 18 \end{aligned}$$

(d) Expanding the determinant along the third column gives

$$\begin{vmatrix} -3 & 4 & 0 & 1 \\ 4 & -1 & 0 & -6 \\ 1 & -1 & 0 & -3 \\ 4 & -2 & 3 & 6 \end{vmatrix} = 0 + 0 + 0 + 3(-1)^{4+3} \begin{vmatrix} -3 & 4 & 1 \\ 4 & -1 & -6 \\ 1 & -1 & -3 \end{vmatrix}$$

Next, we need to evaluate the 3×3 determinant. Expanding along the first column gives

$$\begin{aligned} \begin{vmatrix} -3 & 4 & 1 \\ 4 & -1 & -6 \\ 1 & -1 & -3 \end{vmatrix} &= (-3)(-1)^{1+1} \begin{vmatrix} -1 & -6 \\ -1 & -3 \end{vmatrix} + 4(-1)^{2+1} \begin{vmatrix} 4 & 1 \\ -1 & -3 \end{vmatrix} + (-1)^{3+1} \begin{vmatrix} 4 & 1 \\ -1 & -6 \end{vmatrix} \\ &= -3[(-1)(-3) - (-6)(-1)] - 4[4(-3) - 1(-1)] + [4(-6) - 1(-1)] \\ &= (-3)(-3) - 4(-11) - 23 = 30 \end{aligned}$$

Thus,

$$\begin{vmatrix} -3 & 4 & 0 & 1 \\ 4 & -1 & 0 & -6 \\ 1 & -1 & 0 & -3 \\ 4 & -2 & 3 & 6 \end{vmatrix} = (-3)(30) = -90$$

(e) Expanding the determinant along the first column gives

$$\begin{bmatrix} 0 & 6 & 1 & 2 \\ 0 & 5 & -1 & 1 \\ 3 & -5 & -3 & -5 \\ 5 & 6 & -3 & -6 \end{bmatrix} = 0 + 0 + 3(-1)^{3+1} \begin{vmatrix} 6 & 1 & 2 \\ 5 & -1 & 1 \\ 6 & -3 & -6 \end{vmatrix} + 5(-1)^{4+1} \begin{vmatrix} 6 & 1 & 2 \\ 5 & -1 & 1 \\ -5 & -3 & -5 \end{vmatrix}$$

Next, we need to evaluate the 3×3 determinants. Expanding both along the first row gives

$$\begin{aligned} \begin{vmatrix} 6 & 1 & 2 \\ 5 & -1 & 1 \\ 6 & -3 & -6 \end{vmatrix} &= 6(-1)^{1+1} \begin{vmatrix} -1 & 1 \\ -3 & -6 \end{vmatrix} + 1(-1)^{1+2} \begin{vmatrix} 5 & 1 \\ 6 & -6 \end{vmatrix} + 2(-1)^{1+3} \begin{vmatrix} 5 & -1 \\ 6 & -3 \end{vmatrix} \\ &= 6[(-1)(-6) - 1(-3)] - [5(-6) - 1(6)] + 2[5(-3) - (-1)6] \\ &= 6(9) - (-36) + 2(-9) = 72 \\ \begin{vmatrix} 6 & 1 & 2 \\ 5 & -1 & 1 \\ -5 & -3 & -5 \end{vmatrix} &= 6(-1)^{1+1} \begin{vmatrix} -1 & 1 \\ -3 & -5 \end{vmatrix} + 1(-1)^{1+2} \begin{vmatrix} 5 & 1 \\ -5 & -5 \end{vmatrix} + 2(-1)^{1+3} \begin{vmatrix} 5 & -1 \\ -5 & -3 \end{vmatrix} \\ &= 6[(-1)(-5) - 1(-3)] - [5(-5) - 1(-5)] + 2[5(-3) - (-1)(-5)] \\ &= 6(8) - (-20) + 2(-20) = 28 \end{aligned}$$

Hence,

$$\begin{bmatrix} 0 & 6 & 1 & 2 \\ 0 & 5 & -1 & 1 \\ 3 & -5 & -3 & -5 \\ 5 & 6 & -3 & -6 \end{bmatrix} = 3(72) - 5(28) = 76$$

(f) Continually expanding along the first column gives

$$\begin{vmatrix} 1 & 3 & 4 & -5 & 7 \\ 0 & -3 & 1 & 2 & 3 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 4 & 3 \end{vmatrix} = 1(-1)^{1+1} \begin{vmatrix} -3 & 1 & 2 & 3 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 4 & 3 \end{vmatrix}$$

$$= 1(-3)(-1)^{1+1} \begin{vmatrix} 4 & 1 & 0 \\ 0 & -1 & 8 \\ 0 & 4 & 3 \end{vmatrix}$$

$$= 1(-3)(4)(-1)^{1+1} \begin{vmatrix} -1 & 8 \\ 4 & 3 \end{vmatrix}$$

$$= (1)(-3)(4)[(-1)(3) - 8(4)] = 420$$

A4 (a) We get

$$\begin{vmatrix} 1 & 3 & -1 \\ 2 & 1 & 0 \\ -1 & 0 & 5 \end{vmatrix} = 3(-1)^{1+2} \begin{vmatrix} 2 & 0 \\ -1 & 5 \end{vmatrix} + 1(-1)^{2+2} \begin{vmatrix} 1 & -1 \\ -1 & 5 \end{vmatrix} + 0$$

$$= (-3)[2(5) - 0(-1)] + [1(5) - (-1)(-1)] = -26$$

$$\begin{vmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -1 & 0 & 5 \end{vmatrix} = 3(-1)^{2+1} \begin{vmatrix} 2 & -1 \\ 0 & 5 \end{vmatrix} + 1(-1)^{2+2} \begin{vmatrix} 1 & -1 \\ -1 & 5 \end{vmatrix} + 0$$

$$= (-3)[2(5) - 0(-1)] + [1(5) - (-1)(-1)] = -26$$

Thus, $\det A = \det A^T$.

(b) We get

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 2 & 5 \\ 3 & 0 & 1 & 4 \\ 4 & 5 & 1 & -2 \end{vmatrix} = 2(-1)^{1+2} \begin{vmatrix} -2 & 2 & 5 \\ 3 & 1 & 4 \\ 4 & 1 & -2 \end{vmatrix} + 0 + 0 + 5(-1)^{4+2} \begin{vmatrix} 1 & 3 & 4 \\ -2 & 2 & 5 \\ 3 & 1 & 4 \end{vmatrix}$$

We now evaluate the 3×3 determinants by expanding along the first rows to get

$$\begin{vmatrix} -2 & 2 & 5 \\ 3 & 1 & 4 \\ 4 & 1 & -2 \end{vmatrix} = (-2)(-1)^{1+1} \begin{vmatrix} 1 & 4 \\ 1 & -2 \end{vmatrix} + 2(-1)^{1+2} \begin{vmatrix} 3 & 4 \\ 4 & -2 \end{vmatrix} + 5(-1)^{1+3} \begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix}$$

$$= (-2)[(1(-2) - 4(1)] - 2[(3(-2) - 4(4)] + 5[3(1) - 1(4)] = 51$$

$$\begin{vmatrix} 1 & 3 & 4 \\ -2 & 2 & 5 \\ 3 & 1 & 4 \end{vmatrix} = 1(-1)^{1+1} \begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} + 3(-1)^{1+2} \begin{vmatrix} -2 & 5 \\ 3 & 4 \end{vmatrix} + 4(-1)^{1+3} \begin{vmatrix} -2 & 2 \\ 3 & 1 \end{vmatrix}$$

$$= [2(4) - 5(1)] - 3[(-2)(4) - 5(3)] + 4[(-2)(1) - 2(3)] = 40$$

Hence,

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 2 & 5 \\ 3 & 0 & 1 & 4 \\ 4 & 5 & 1 & -2 \end{vmatrix} = (-2)(51) + 5(40) = 98$$

For A^T we get

$$\begin{vmatrix} 1 & -2 & 3 & 4 \\ 2 & 0 & 0 & 5 \\ 3 & 2 & 1 & 1 \\ 4 & 5 & 4 & -2 \end{vmatrix} = 2(-1)^{1+2}\begin{vmatrix} -2 & 3 & 4 \\ 2 & 1 & 1 \\ 5 & 4 & -2 \end{vmatrix} + 0 + 0 + 5(-1)^{4+2}\begin{vmatrix} 1 & -2 & 3 \\ 3 & 2 & 1 \\ 4 & 5 & 4 \end{vmatrix}$$

We now evaluate the 3×3 determinants by expanding along the first columns to get

$$\begin{aligned}
\begin{vmatrix} -2 & 3 & 4 \\ 2 & 1 & 1 \\ 5 & 4 & -2 \end{vmatrix} &= (-2)(-1)^{1+1}\begin{vmatrix} 1 & 1 \\ 4 & -2 \end{vmatrix} + 2(-1)^{1+2}\begin{vmatrix} 3 & 4 \\ 4 & -2 \end{vmatrix} + 5(-1)^{1+3}\begin{vmatrix} 3 & 4 \\ 1 & 1 \end{vmatrix} \\
&= (-2)[(1(-2) - 4(1)] - 2[(3(-2) - 4(4)] + 5[3(1) - 1(4)] = 51 \\
\begin{vmatrix} 1 & -2 & 3 \\ 3 & 2 & 1 \\ 4 & 5 & 4 \end{vmatrix} &= 1(-1)^{1+1}\begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix} + 3(-1)^{1+2}\begin{vmatrix} -2 & 3 \\ 5 & 4 \end{vmatrix} + 4(-1)^{1+3}\begin{vmatrix} -2 & 3 \\ 2 & 1 \end{vmatrix} \\
&= [2(4) - 5(1)] - 3[(-2)(4) - 5(3)] + 4[(-2)(1) - 2(3)] = 40
\end{aligned}$$

Hence,

$$\begin{vmatrix} 1 & -2 & 3 & 4 \\ 2 & 0 & 0 & 5 \\ 3 & 2 & 1 & 1 \\ 4 & 5 & 4 & -2 \end{vmatrix} = (-2)(51) + 5(40) = 98 = \det A$$

A5 (a) Expanding along the first row gives

$$\det E_1 = 1(-1)^{1+1}\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + 0 + 0 = 1(0 - 1) = -1$$

(b) The matrix is upper triangular, hence $\det E_2 = 1(1)(1) = 1$.

(c) The matrix is upper triangular, hence $\det E_3 = (-3)(1)(1) = -3$.

Homework Problems

B1 (a) 0 (b) 12 (c) 2 (d) 0 (e) 6 (f) 24

B2 (a) 0 (b) -75 (c) 256 (d) 21

B3 (a) 75 (b) 78 (c) -4 (d) 184 (e)-39 (f) -294

B4 (a) -1 (b) -44

B5 (a) 5 (b) -1 (c) 1

Computer Problems

C1 (a) 51.67 (b) -1256.62 (c) 0

Conceptual Problems

D1 We will prove this for upper triangular matrices by induction on n. The proof for lower triangular matrices is similar. If A is 2×2, then

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{vmatrix} = a_{11}a_{22}$$

Assume the result holds for $(n-1)\times(n-1)$ matrices. Then for an $n \times n$ upper triangular matrix A we can expand along the first column to get

$$\det A = a_{11}(-1)^{1+1} \det A(1,1)$$

But, A_{11} is the upper triangular matrix obtained from A be deleting the first row and first column. So, by the inductive hypothesis

$$\det A = a_{11}a_{22}\cdots a_{nn}$$

as required.

D2 (a) The equation is

$$(a_2 - b_2)x_1 + (b_1 - a_1)x_2 = -(a_1b_2 - a_2b_1)$$

so it is certainly a line in $\mathbb{R}^2$ unless both $a_2 - b_2 = 0$ and $b_1 - a_1 = 0$, in which case the equation says only $0 = 0$. If $(x_1, x_2) = (a_1, a_2)$, the matrix has two equal rows, and the determinant is zero. Hence (a_1, a_2) lies on the line, and similarly (b_1, b_2) lies on the line.

(b) The equation is $\det \begin{bmatrix} x_1 & x_2 & x_3 & 1 \\ a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \end{bmatrix} = 0.$

If the points are collinear, $\vec{c} - \vec{a} = t(\vec{b} - \vec{a})$ for some t, or $\vec{c} = (1-t)\vec{a} + \vec{b}$. Since three rows of the matrix are linearly dependent, the cofactors of entries in the first row are the determinants of 3×3 matrices with three linearly dependent rows, and such determinants are zero. Hence, the equation degenerates to $0 = 0$ in this case.

5.2 Elementary Row Operations and the Determinant

Practice Problems

A1 When using row operations to simplify a determinant, it is very important to be careful how the row operation changes the determinant. Recall that adding a multiple of one row to another does not change the determinant and swapping rows multiplies the determinant by (-1). Also, multiplying a row by a scalar c multiplies the determinant by c, but always remember when using the row operation to simplify a determinant to think of 'factoring" the multiple out of the row.

(a) Since performing the row operations $R_2 - 3R_1$ and $R_3 + R_1$ do not change the determinant, we get

$$\det A = \begin{vmatrix} 1 & 2 & 4 \\ 0 & -5 & -12 \\ 0 & 5 & 6 \end{vmatrix}$$

The row operation $R_3 + R_2$ does not change the determinant, so

$$\det A = \begin{vmatrix} 1 & 2 & 4 \\ 0 & -5 & -12 \\ 0 & 0 & -6 \end{vmatrix} = 1(-5)(-6) = 30$$

Hence, A is invertible since $\det A \neq 0$.

(b) Since swapping rows multiplies the determinant by (-1) we get

$$\det A = (-1)\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & 2 & 2 \end{vmatrix}$$

Performing the row operations $R_2 - 2R_1$ and $R_3 - 3R_1$ do not change the determinant, we get

$$\det A = (-1)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & -1 \end{vmatrix}$$

Since swapping rows multiplies the determinant by (-1) we get

$$\det A = (-1)(-1)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{vmatrix} = (-1)(-1)(1)(-1)(-1) = 1$$

Hence, A is invertible since $\det A \neq 0$.

(c) Since performing the row operations $R_1 - R_2$ and $R_3 - R_2$ do not change the determinant, we get

$$\det A = \begin{vmatrix} 4 & 0 & 0 & 0 \\ 1 & 2 & -1 & 1 \\ 2 & 0 & 2 & 3 \\ -2 & 0 & 3 & 5 \end{vmatrix}$$

Next, we multiply row 1 by $\frac{1}{4}$ which has the effect of factoring out the 4. We also add row 3 to row 4 which does not change the determinant. So,

$$\det A = 4\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & -1 & 1 \\ 2 & 0 & 2 & 3 \\ 0 & 0 & 5 & 8 \end{vmatrix}$$

Next, we perform the operations $R_2 - R_1$ and $R_3 - 2R_1$ which do not change the determinant. Then

$$\det A = 4\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 5 & 8 \end{vmatrix}$$

Finally, we perform the operation $R_4 - \frac{5}{2}R_3$ to get

$$\det A = 4\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1/2 \end{vmatrix} = 4(1)(2)(2)(1/2) = 8$$

Hence, A is invertible since $\det A \neq 0$.

(d) Since performing the row operations $R_2 + 2R_1$, $R_3 - 2R_1$, and $R_4 - R_1$ do not change the determinant, we get

$$\det A = \begin{vmatrix} 1 & 1 & 3 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 4 & 2 \end{vmatrix}$$

There are two identical rows, so $\det A = 0$ by Corollary 5.2.3. Hence, A is not invertible.

(e) We first multiply the first row by $\frac{1}{5}$ which has the effect of factoring out 5 from the first row. So,

$$\det A = 5\begin{vmatrix} 1 & 2 & 1 & -1 \\ 1 & 3 & 5 & 7 \\ 1 & 2 & 6 & 3 \\ -1 & 7 & 1 & 1 \end{vmatrix}$$

The row operations $R_2 - R_1$, $R_3 - R_1$, and $R_4 + R_1$ do not change the determinant, so

$$\det A = 5\begin{vmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 4 & 8 \\ 0 & 0 & 5 & 4 \\ 0 & 9 & 2 & 0 \end{vmatrix}$$

The row operations $R_4 - 9R_2$, does not change the determinant, so

$$\det A = 5\begin{vmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 4 & 8 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & -34 & -72 \end{vmatrix}$$

Finally, the row operation $R_4 + \frac{34}{5}R_3$ also does not change the determinant, hence

$$\det A = 5\begin{vmatrix} 1 & 2 & 1 & -1 \\ 0 & 1 & 4 & 8 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & -224/5 \end{vmatrix} = 5(1)(1)(5)(-224/5) = -1120$$

Consequently, A is invertible.

A2 (a) Using the row operations $R_2 - R_1$ and $R_3 - R_1$, which do not change the determinant, and a cofactor expansion along the first column gives

$$\begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & -2 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 \\ 0 & 2 & -4 \\ 0 & 3 & 1 \end{vmatrix} = 1\begin{vmatrix} 2 & -4 \\ 3 & 1 \end{vmatrix} = 14$$

(b) The row operations $R_2 - 2R_1$ and $R_3 + R_1$ do not change the determinant, so we get

$$\begin{vmatrix} 2 & 4 & 2 \\ 4 & 2 & 1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 2 \\ 0 & -6 & -3 \\ 0 & 6 & 4 \end{vmatrix}$$

The row operation $R_3 + R_2$ also does not change the determinant, so

$$\begin{vmatrix} 2 & 4 & 2 \\ 4 & 2 & 1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 2 \\ 0 & -6 & -3 \\ 0 & 0 & 1 \end{vmatrix} = 2(-6)(1) = -12$$

(c) The row operations $R_2 - 2R_1$, $R_3 - 3R_1$, and $R_4 - R_1$ do not change the determinant, so

$$\begin{vmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 1 & 5 \\ 3 & 6 & 5 & 9 \\ 1 & 3 & 4 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 1 & 3 & 1 \end{vmatrix}$$

Using column expansions, we get

$$\begin{vmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 1 & 5 \\ 3 & 6 & 5 & 9 \\ 1 & 3 & 4 & 3 \end{vmatrix} = 1\begin{vmatrix} 0 & -1 & 1 \\ 0 & 2 & 3 \\ 1 & 3 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 2 & 3 \end{vmatrix} = -5$$

(d) Using the column operations $C_3 + C_1$ and $C_4 + 3C_1$, which do not change the determinant, and a cofactor expansion along the first row gives

$$\begin{vmatrix} 2 & 0 & -2 & -6 \\ 2 & -6 & -4 & -1 \\ -3 & -4 & 5 & 3 \\ -2 & -1 & -3 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 0 & 0 \\ 2 & -6 & -2 & 5 \\ -3 & -4 & 2 & -6 \\ -2 & -1 & -5 & -4 \end{vmatrix} = 2\begin{vmatrix} -6 & -2 & 5 \\ -4 & 2 & -6 \\ -1 & -5 & -4 \end{vmatrix}$$

Next use the row operations $R_1 - 6R_3$ and $R_2 - 4R_3$, which do not change the determinant, and a cofactor expansion along the first column to get

$$\begin{vmatrix} 2 & 0 & -2 & -6 \\ 2 & -6 & -4 & -1 \\ -3 & -4 & 5 & 3 \\ -2 & -1 & -3 & 2 \end{vmatrix} = 2\begin{vmatrix} 0 & 28 & 29 \\ 0 & 22 & 10 \\ -1 & & \\ -5 & & \\ -4 & & \end{vmatrix} = 2(-1)\begin{vmatrix} 28 & 29 \\ 22 & 10 \end{vmatrix} = 716$$

A3 (a) Using $R_1 - 2R_2$ and $R_3 - 4R_2$ and a cofactor expansion, we get

$$\det A = \begin{vmatrix} 0 & 1 & 3 \\ 1 & 1 & -1 \\ 0 & p-4 & 2 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 3 \\ p-4 & 2 \end{vmatrix} = -2 + 3(p-4) = 3p - 14$$

A is invertible when $\det A \neq 0$, so when $3p - 14 \neq 0$. Therefore, A is invertible for all $p \neq \frac{14}{3}$

(b) Using $R_1 - 2R_4$, $R_3 - R_2$, and a cofactor expansion, we get

$$\det A = \begin{vmatrix} 0 & 3 & -1 & p \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 5 & 5 \\ 1 & 0 & 1 & 0 \end{vmatrix} = (-1)\begin{vmatrix} 3 & -1 & p \\ 1 & 2 & 1 \\ 0 & 5 & 5 \end{vmatrix}$$

Using $R_1 - 3R_2$ and another cofactor expansion gives

$$\det A = (-1)\begin{vmatrix} 0 & -7 & p-3 \\ 1 & 2 & 1 \\ 0 & 5 & 5 \end{vmatrix} = (-1)(-1)\begin{vmatrix} -7 & p-3 \\ 5 & 5 \end{vmatrix} = -35 - 5(p-3) = -5p - 20$$

Therefore, A is invertible when $-5p - 20 = \det A \neq 0$. Hence, A is invertible for all $p \neq -4$

(c) Using $R_2 - R_1$, $R_3 - R_1$, $R_4 - R_1$, and a cofactor expansion gives

$$\det A = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 3 & 8 & 15 \\ 0 & 7 & 26 & p-1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 8 & 15 \\ 7 & 26 & p-1 \end{vmatrix}$$

Next, we use $R_2 - 3R_1$, $R_3 - 7R_1$, and a cofactor expansion to get

$$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 6 \\ 0 & 12 & p-22 \end{vmatrix} = \begin{vmatrix} 2 & 6 \\ 12 & p-22 \end{vmatrix} = 2(p-22) - 72 = 2p - 116$$

Consequently, A is invertible when $2p - 116 \neq 0$, hence for all $p \neq 58$.

A4 (a) We have $\det A = 13$ and $\det B = 14$. Next, we find that $AB = \begin{bmatrix} 7 & 0 \\ 4 & 26 \end{bmatrix}$, so

$$\det AB = 182 = (13)(14) = \det A \det B$$

(b) Using $R_1 + 2R_2$ and $R_3 + 3R_2$ and a cofactor expansion, we get

$$\det A = \begin{bmatrix} 0 & 8 & -1 \\ -1 & 2 & -1 \\ 0 & 6 & -1 \end{bmatrix} = (-1)(-1)\begin{vmatrix} 8 & -1 \\ 6 & -1 \end{vmatrix} = -2$$

Using $R_1 + R_2$ and $R_3 + 4R_2$ and a cofactor expansion, we get

$$\det B = \begin{vmatrix} 0 & 3 & 7 \\ -1 & 0 & 5 \\ 0 & 1 & 21 \end{vmatrix} = (-1)(-1)\begin{vmatrix} 3 & 7 \\ 1 & 21 \end{vmatrix} = 56$$

Next, we find that $AB = \begin{bmatrix} 2 & 7 & 25 \\ -7 & -4 & 7 \\ 11 & 11 & 8 \end{bmatrix}$. To evaluate $\det AB$ we use $C_2 - C_1$ and $C_3 + C_1$ to get

$$\begin{aligned} \det AB = \begin{vmatrix} 2 & 5 & 27 \\ -7 & 3 & 0 \\ 11 & 0 & 19 \end{vmatrix} &= 11\begin{vmatrix} 5 & 27 \\ 3 & 0 \end{vmatrix} + 19\begin{vmatrix} 2 & 5 \\ -7 & 3 \end{vmatrix} \\ &= -891 + 779 = -112 = (-2)(56) = \det A \det B \end{aligned}$$

A5 (a) Since rA is the matrix where each of the n rows of A must been multiplied by r, we can use Theorem 5.2.1 n times to get $\det(rA) = r^n \det A$.

(b) We have AA^{-1} is I, so

$$1 = \det I = \det AA^{-1} = (\det A)(\det A^{-1})$$

by Theorem 5.2.7. Since $\det A \neq 0$, we get $\det A^{-1} = \frac{1}{\det A}$.

(c) By Theorem 5.2.7. we have $1 = \det I = \det A^3 = (\det A)^3$. Taking cube roots of both sides gives $\det A = 1$.

Homework Problems

B1 (a) $\det A = -25$, hence A is invertible.

(b) $\det A = 0$, hence A is not invertible.

(c) $\det A = 115$, hence A is invertible.

(d) $\det A = -48$, hence A is invertible.

(e) $\det A = 448$, hence A is invertible.

(f) $\det A = 0$, hence A is not invertible.

B2 (a) -33

(b) -51

(c) -16

(d) -3

B3 (a) $\det A = p - 6$, so A is invertible for all $p \neq 6$

(b) $\det A = -p + 2$, so A is invertible for all $p \neq 2$

(c) $\det A = 10 + 10p$, so A is invertible for all $p \neq -1$

(d) $\det A = 2p - 116$, so A is invertible for all $p \neq 58$

B4 (a) $\det A = 5$, $\det B = -11$, $\det AB = \det \begin{bmatrix} -21 & 2 \\ 17 & 1 \end{bmatrix} = -55$

(b) $\det A = -23$, $\det B = 2$, $\det AB = \det \begin{bmatrix} 8 & 3 & 5 \\ -3 & 2 & 19 \\ 4 & 2 & 4 \end{bmatrix} = -46$

Conceptual Problems

D1 We have $\det A = \det A^T = \det(-A)$. Observe that $-A$ is the matrix obtained from A by multiplying each of the n rows of A by -1. Thus, by Theorem 5.2.1, we have $\det(-A) = (-1)^n \det A$. Thus, since n is odd $\det(-A) = -\det A$. Hence, $\det A = -\det A$ which implies that $\det A = 0$.

D2 By Theorem 5.2.7, we have $0 = \det AB = \det A \det B$. Therefore, we must have $\det A = 0$ or $\det B = 0$.

D3 We can say that the rows of A are linearly dependent. The range of $L(\vec{x})$ cannot be all of $\mathbb{R}^3$ and its nullspace cannot be trivial by the Invertible Matrix Theorem.

D4 By Theorem 5.2.7, we have

$$\begin{aligned}\det(P^{-1}AP) &= \det P^{-1} \det A \det P = \det A \det P^{-1} \det P \\ &= \det A \det(P^{-1}P) = \det A \det I = \det A\end{aligned}$$

D5 (a) Expanding along the first row gives

$$\det \begin{bmatrix} a+p & b+q & c+r \\ d & e & f \\ g & h & k \end{bmatrix} = (a+p)\begin{vmatrix} e & f \\ h & k \end{vmatrix} - (b+q)\begin{vmatrix} d & f \\ g & k \end{vmatrix} + (c+r)\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a\begin{vmatrix} e & f \\ h & k \end{vmatrix} - b\begin{vmatrix} d & f \\ g & k \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} + p\begin{vmatrix} e & f \\ h & k \end{vmatrix} - q\begin{vmatrix} d & f \\ g & k \end{vmatrix} + r\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} + \det \begin{bmatrix} p & q & r \\ d & e & f \\ g & h & k \end{bmatrix}$$

(b) Repeat the step of part (a) on the second row to obtain the answer

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} + \det \begin{bmatrix} p & q & r \\ d & e & f \\ g & h & k \end{bmatrix} + \det \begin{bmatrix} a & b & c \\ x & y & z \\ g & h & k \end{bmatrix} + \det \begin{bmatrix} p & q & r \\ x & y & z \\ g & h & k \end{bmatrix}$$

D6

$$\begin{aligned}\det \begin{bmatrix} a+b & p+q & u+v \\ b+c & q+r & v+w \\ c+a & r+p & w+u \end{bmatrix} &= \begin{vmatrix} a+b & p+q & u+v \\ c-a & r-p & w-u \\ c+a & r+p & w+u \end{vmatrix} = \begin{vmatrix} a+b & p+q & u+v \\ c-a & r-p & w-u \\ 2c & 2r & 2w \end{vmatrix} \\ &= 2\begin{vmatrix} a+b & p+q & u+v \\ c-a & r-p & w-u \\ c & r & w \end{vmatrix} = 2\begin{vmatrix} a+b & p+q & u+v \\ -a & -p & -u \\ c & r & w \end{vmatrix} \\ &= 2\begin{vmatrix} b & q & r \\ -a & -p & -u \\ c & r & w \end{vmatrix} = -2\begin{vmatrix} b & q & r \\ a & p & u \\ c & r & w \end{vmatrix} = 2\det \begin{bmatrix} a & p & u \\ b & q & v \\ c & r & w \end{bmatrix}\end{aligned}$$

D7

$$\det\begin{bmatrix}1 & 1 & 1\\ 1 & 1+a & 1+2a\\ 1 & (1+a)^2 & (1+2a)^2\end{bmatrix} = \begin{vmatrix}1 & 1 & 1\\ 0 & a & 2a\\ 0 & 2a+a^2 & 4a+4a^2\end{vmatrix} = \begin{vmatrix}1 & 1 & 1\\ 0 & a & 2a\\ 0 & a^2 & 4a^2\end{vmatrix} = 2a^3$$

D8 (a) Using the fact that $\det A = \det A^T$ we get

$$\det\begin{bmatrix}a_{11} & a_{12} & ra_{13}\\ a_{21} & a_{22} & ra_{23}\\ a_{31} & a_{32} & ra_{33}\end{bmatrix} = \det\begin{bmatrix}a_{11} & a_{21} & a_{31}\\ a_{12} & a_{22} & a_{32}\\ ra_{13} & ra_{23} & ra_{33}\end{bmatrix} = r\det\begin{bmatrix}a_{11} & a_{21} & a_{31}\\ a_{12} & a_{22} & a_{32}\\ a_{13} & a_{23} & a_{33}\end{bmatrix} = r\det\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$$

(b) Observe that B^T is the matrix obtained from A^T by multiplying the i-th row of A^T by a factor of r. Thus, we have

$$\det B = \det B^T = r\det A^T = r\det A$$

(c) Observe that B^T is the matrix obtained from A^T by adding a multiple of row to another. Thus, we have

$$\det B = \det B^T = \det A^T = \det A$$

5.3 Matrix Inverse by Cofactors and Cramer's Rule

Practice Problems

A1 (a) We have $\det A = \begin{vmatrix}1 & 3\\ 4 & 10\end{vmatrix} = -2$ and $\operatorname{cof} A = \begin{bmatrix}10 & -4\\ -3 & 1\end{bmatrix}$. Hence,

$$A^{-1} = \frac{1}{\det A}(\operatorname{cof} A)^T = \frac{1}{-2}\begin{bmatrix}10 & -3\\ -4 & 1\end{bmatrix}$$

(b) We have $\det A = \begin{vmatrix}3 & -5\\ 2 & -1\end{vmatrix} = 7$ and $\operatorname{cof} A = \begin{bmatrix}-1 & 5\\ -2 & 3\end{bmatrix}$. Hence,

$$A^{-1} = \frac{1}{\det A}(\operatorname{cof} A)^T = \frac{1}{7}\begin{bmatrix}-1 & -2\\ 5 & 3\end{bmatrix}$$

(c) We have

$$\det A = \begin{vmatrix}4 & 1 & 7\\ 2 & -3 & 1\\ -2 & 6 & 0\end{vmatrix} = \begin{vmatrix}0 & 13 & 7\\ 0 & 3 & 1\\ -2 & 6 & 0\end{vmatrix} = (-2)\begin{vmatrix}13 & 7\\ 3 & 1\end{vmatrix} = 16$$

and $\operatorname{cof} A = \begin{bmatrix}-6 & -2 & 6\\ 42 & 14 & -26\\ 22 & 10 & -14\end{bmatrix}$. Hence,

$$A^{-1} = \frac{1}{\det A}(\operatorname{cof} A)^T = \frac{1}{16}\begin{bmatrix}-6 & 42 & 22\\ -2 & 14 & 10\\ 6 & -26 & -14\end{bmatrix} = \frac{1}{8}\begin{bmatrix}-3 & 21 & 11\\ -1 & 7 & 5\\ 3 & -13 & -7\end{bmatrix}$$

(d) We have

$$\det A = \begin{vmatrix} 4 & 0 & -4 \\ 0 & -1 & 1 \\ -2 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 4 & 0 & 0 \\ 0 & -1 & 1 \\ -2 & 2 & -3 \end{vmatrix} = 4$$

and $\operatorname{cof} A = \begin{bmatrix} -1 & -2 & -2 \\ -8 & -12 & -8 \\ -4 & -4 & -4 \end{bmatrix}$. Hence,

$$A^{-1} = \frac{1}{\det A}(\operatorname{cof} A)^T = \frac{1}{4}\begin{bmatrix} -1 & -8 & -4 \\ -2 & -12 & -4 \\ -2 & -8 & -4 \end{bmatrix}$$

A2 (a) We get $\operatorname{cof} A = \begin{bmatrix} -1 & 5 & -2 \\ 3+t & 2-3t & -11 \\ -3 & -2-2t & -6 \end{bmatrix}$.

(b) $A(\operatorname{cof} A)^T = \begin{bmatrix} -2t-17 & 0 & 0 \\ 0 & -2t-17 & 0 \\ 0 & 0 & -2t-17 \end{bmatrix}$. So $\det A = -2t-17$, and $A^{-1} = \frac{1}{-2t-17}\begin{bmatrix} -1 & 3+t & -3 \\ 5 & 2-3t & -2-2t \\ -2 & -11 & -6 \end{bmatrix}$ provided $-2t-17 \neq 0$.

A3 (a) The coefficient matrix is $A = \begin{bmatrix} 2 & -3 \\ 3 & 5 \end{bmatrix}$, so $\det A = 19$. Hence,

$$x_1 = \frac{1}{19}\begin{vmatrix} 6 & -3 \\ 7 & 5 \end{vmatrix} = \frac{51}{19}$$

$$x_2 = \frac{1}{19}\begin{vmatrix} 2 & 6 \\ 3 & 7 \end{vmatrix} = \frac{-4}{19}$$

Thus, the solution is $\vec{x} = \begin{bmatrix} 51/19 \\ -4/19 \end{bmatrix}$.

(b) The coefficient matrix is $A = \begin{bmatrix} 3 & 3 \\ 2 & -3 \end{bmatrix}$, so $\det A = -15$. Hence,

$$x_1 = \frac{1}{-15}\begin{vmatrix} 2 & 3 \\ 5 & -3 \end{vmatrix} = \frac{-21}{-15} = \frac{7}{5}$$

$$x_2 = \frac{1}{-15}\begin{vmatrix} 3 & 2 \\ 2 & 5 \end{vmatrix} = \frac{11}{-15} = -\frac{11}{15}$$

Thus, the solution is $\vec{x} = \begin{bmatrix} 7/5 \\ -11/15 \end{bmatrix}$.

(c) The coefficient matrix is $A = \begin{bmatrix} 7 & 1 & -4 \\ -6 & -4 & 1 \\ 4 & -1 & -2 \end{bmatrix}$, so

$$\det A = \begin{vmatrix} 7 & 1 & -4 \\ -6 & -4 & 1 \\ 4 & -1 & -2 \end{vmatrix} = \begin{vmatrix} 7 & 1 & -4 \\ 22 & 0 & -15 \\ 11 & 0 & -6 \end{vmatrix} = (-1)\begin{vmatrix} 22 & -15 \\ 11 & -6 \end{vmatrix} = (-1)\begin{vmatrix} 0 & -3 \\ 11 & -6 \end{vmatrix} = -33$$

Hence,

$$x_1 = \frac{1}{-33}\begin{vmatrix} 3 & 1 & -4 \\ 0 & -4 & 1 \\ 6 & -1 & -2 \end{vmatrix} = \frac{1}{-33}\begin{vmatrix} 3 & 1 & -4 \\ 0 & -4 & 1 \\ 0 & -3 & 6 \end{vmatrix} = \frac{3}{-33}\begin{vmatrix} -4 & 1 \\ -3 & 6 \end{vmatrix} = \frac{-21}{-11} = \frac{21}{11}$$

$$x_2 = \frac{1}{-33}\begin{vmatrix} 7 & 3 & -4 \\ -6 & 0 & 1 \\ 4 & 6 & -2 \end{vmatrix} = \frac{1}{-33}\begin{vmatrix} 7 & 3 & -4 \\ -6 & 0 & 1 \\ -10 & 0 & 6 \end{vmatrix} = \frac{-3}{-33}\begin{vmatrix} -6 & 1 \\ -10 & 6 \end{vmatrix} = \frac{-26}{11}$$

$$x_3 = \frac{1}{-33}\begin{vmatrix} 7 & 1 & 3 \\ -6 & -4 & 0 \\ 4 & -1 & 6 \end{vmatrix} = \frac{1}{-33}\begin{vmatrix} 7 & 1 & 3 \\ -6 & -4 & 0 \\ -10 & -3 & 0 \end{vmatrix} = \frac{3}{-33}\begin{vmatrix} -6 & -4 \\ -10 & -3 \end{vmatrix} = \frac{-22}{-11} = 2$$

Thus, the solution is $\vec{x} = \begin{bmatrix} 21/11 \\ -26/11 \\ 2 \end{bmatrix}$.

(d) The coefficient matrix is $A = \begin{bmatrix} 2 & 3 & -5 \\ 3 & -1 & 2 \\ 5 & 4 & -6 \end{bmatrix}$, so

$$\det A = \begin{vmatrix} 2 & 3 & -5 \\ 3 & -1 & 2 \\ 5 & 4 & -6 \end{vmatrix} = \begin{vmatrix} 11 & 0 & 1 \\ 3 & -1 & 2 \\ 17 & 0 & 2 \end{vmatrix} = (-1)\begin{vmatrix} 11 & 1 \\ 17 & 2 \end{vmatrix} = -5$$

Hence,

$$x_1 = \frac{1}{-5}\begin{vmatrix} 2 & 3 & -5 \\ 1 & -1 & 2 \\ 3 & 4 & -6 \end{vmatrix} = \frac{1}{-5}\begin{vmatrix} 2 & 5 & -9 \\ 1 & 0 & 0 \\ 3 & 7 & -12 \end{vmatrix} = \frac{-1}{-5}\begin{vmatrix} 5 & -9 \\ 7 & -12 \end{vmatrix} = \frac{3}{5}$$

$$x_2 = \frac{1}{-5}\begin{vmatrix} 2 & 2 & -5 \\ 3 & 1 & 2 \\ 5 & 3 & -6 \end{vmatrix} = \frac{1}{-5}\begin{vmatrix} -4 & 3 & -9 \\ 0 & 1 & 0 \\ -4 & 0 & -12 \end{vmatrix} = \frac{1}{-5}\begin{vmatrix} -4 & -9 \\ -4 & -12 \end{vmatrix} = \frac{-12}{5}$$

$$x_3 = \frac{1}{-5}\begin{vmatrix} 2 & 3 & 2 \\ 3 & -1 & 1 \\ 5 & 4 & 3 \end{vmatrix} = \frac{1}{-5}\begin{vmatrix} -4 & 5 & 2 \\ 0 & 0 & 1 \\ -4 & 7 & 3 \end{vmatrix} = \frac{-1}{-5}\begin{vmatrix} -4 & 5 \\ -4 & 7 \end{vmatrix} = \frac{-8}{5}$$

Thus, the solution is $\vec{x} = \begin{bmatrix} 3/5 \\ -12/5 \\ -8/5 \end{bmatrix}$.

Homework Problems

B1 (a) $\frac{1}{-34}\begin{bmatrix} 2 & -4 \\ -7 & -3 \end{bmatrix}$

(b) $\frac{1}{23}\begin{bmatrix} 3 & -4 \\ 2 & 5 \end{bmatrix}$

(c) $\frac{1}{87}\begin{bmatrix} 12 & -23 & 9 \\ 12 & 6 & 9 \\ -21 & 4 & 6 \end{bmatrix}$

(d) $\begin{bmatrix} -1 & 2 & 2 \\ -2 & 5 & 4 \\ 1 & -1 & -1 \end{bmatrix}$

(e) $\frac{1}{-36}\begin{bmatrix} -6 & -18 & -12 \\ 0 & 12 & 12 \\ 0 & 42 & 24 \end{bmatrix}$

(f) $\frac{1}{-96}\begin{bmatrix} -24 & -24 & 32 & 12 \\ 24 & -24 & -32 & 12 \\ 0 & 24 & 32 & 12 \\ 24 & 0 & -32 & -24 \end{bmatrix}$

B2 (a) $\operatorname{cof} A = \begin{bmatrix} 4t+1 & -10 & 3+2t \\ -1 & 14 & -4 \\ -1-3t & 11 & 2t-3 \end{bmatrix}$

(b) $A(\operatorname{cof} A)^T = \begin{bmatrix} 1+14t & 0 & 0 \\ 0 & 1+14t & 0 \\ 0 & 0 & 1+14t \end{bmatrix}$. So $\det A = 1 + 14t$, and $A^{-1} = \frac{1}{1+14t}\begin{bmatrix} 4t+1 & -1 & -1-3t \\ -10 & 14 & 11 \\ 3+2t & -4 & 2t-3 \end{bmatrix}$ provided $1 + 14t \neq 0$.

B3 (a) $\begin{bmatrix} -107/43 \\ -49/43 \end{bmatrix}$

(b) $\begin{bmatrix} 9/4 \\ -7/4 \end{bmatrix}$

(c) $\begin{bmatrix} 30/77 \\ 2/11 \\ 57/77 \end{bmatrix}$

(d) $\begin{bmatrix} -16 \\ -32 \\ 7 \end{bmatrix}$

Conceptual Problems

D1 (a) By Cramer's Rule we have $x_i = \frac{\det N_i}{\det A}$ where N_i is the matrix obtained from A by replacing its i-th column by $\vec{a}_i$ (its i-th column). Thus if $i \neq j$, then N_i has two equal columns, so $\det N_i = 0$. Thus, $x_i = 0$ for $i \neq j$. If $i = j$, then $N_i = A$, so $x_j = 1$. Therefore, $\vec{x} = \vec{e}_j$.

(b) Since A is invertible, $A\vec{x} = \vec{a}_j$ has a unique solution. We know the linear mapping with matrix A maps $\vec{e}_j$ to $\vec{a}_j$, so $\vec{e}_j$ is the unique solution.

D2 $\det A = 2\begin{vmatrix} -1 & 3 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & 3 \end{vmatrix} = -2\begin{vmatrix} 3 & 2 \\ 0 & 3 \end{vmatrix} = -18.$

$$(A^{-1})_{23} = \frac{1}{\det A}\left((\operatorname{cof} A)^T\right)_{23} = \frac{1}{\det A}(\operatorname{cof} A)_{32} = \frac{-1}{18}(-1)^{3+2}\begin{vmatrix} 2 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{vmatrix} = \frac{18}{18} = 1$$

$$(A^{-1})_{42} = \frac{1}{\det A}\left((\operatorname{cof} A)^T\right)_{42} = \frac{1}{\det A}(\operatorname{cof} A)_{24} = \frac{-1}{18}(-1)^{2+4}\begin{vmatrix} 2 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{vmatrix} = 0$$

5.4 Area, Volume, and the Determinant

Practice Problems

A1 (a) We have $\text{Area}(\vec{u}, \vec{v}) = \left|\det\begin{bmatrix} 3 & 2 \\ 5 & 7 \end{bmatrix}\right| = |21 - 10| = 11.$

(b) We have $A\vec{u} = \begin{bmatrix} 18 \\ 19 \end{bmatrix}$ and $A\vec{v} = \begin{bmatrix} 34 \\ 20 \end{bmatrix}$.

(c) $\det A = \begin{vmatrix} -4 & 6 \\ 3 & 2 \end{vmatrix} = -26.$

(d) We can calculate the area of the parallelogram induced by the image vectors by calculating the area directly using $A\vec{u}$ and $A\vec{v}$ to get

$$\text{Area}(A\vec{u}, A\vec{v}) = \left|\det\begin{bmatrix} 18 & 34 \\ 19 & 20 \end{bmatrix}\right| = |-286| = 286$$

Or, we have

$$\text{Area}(A\vec{u}, A\vec{v}) = |\det A|\text{Area}(\vec{u}, \vec{v}) = |-26|(11) = 286$$

A2 We have $A\vec{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $A\vec{v} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$. Hence,

$$\text{Area}(A\vec{u}, A\vec{v}) = \left|\det\begin{bmatrix} 3 & 2 \\ 1 & -2 \end{bmatrix}\right| = |-8| = 8$$

A3 (a) The volume determined by $\vec{u}$, $\vec{v}$, and $\vec{w}$ is

$$\text{volume}(\vec{u}, \vec{v}, \vec{w}) = \left|\det\begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}\right| = \left|\det\begin{bmatrix} 2 & 2 & 1 \\ 3 & -1 & 5 \\ 4 & -5 & 2 \end{bmatrix}\right| = |63| = 63$$

(b) We have

$$\det A = \begin{vmatrix} 1 & -1 & 3 \\ 4 & 0 & 1 \\ 0 & 2 & 5 \end{vmatrix} = 42$$

(c) The volume of the image parallelepiped is

$$\text{volume}(A\vec{u}, A\vec{v}, A\vec{w}) = |\det A|\text{volume}(\vec{u}, \vec{v}, \vec{w}) = |42|63 = 2646$$

A4 (a) The volume determined by $\vec{u}$, $\vec{v}$, and $\vec{w}$ is

$$\text{volume}(\vec{u}, \vec{v}, \vec{w}) = \left|\det\begin{bmatrix}\vec{u} & \vec{v} & \vec{w}\end{bmatrix}\right| = \left|\det\begin{bmatrix}0 & 1 & 2\\ 2 & -5 & 1\\ 3 & 5 & 6\end{bmatrix}\right| = |-41| = 41$$

(b) We have

$$\det A = \begin{vmatrix}3 & -1 & 2\\ 3 & 2 & -1\\ 1 & 4 & 5\end{vmatrix} = 78$$

(c) The volume of the image parallelepiped is

$$\text{volume}(A\vec{u}, A\vec{v}, A\vec{w}) = |\det A|\text{volume}(\vec{u}, \vec{v}, \vec{w}) = |78|41 = 3198$$

A5 (a) The 4-volume determined by $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, and $\vec{v}_4$ is

$$\text{volume}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) = \left|\det\begin{bmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 2 & 0\\ 2 & 1 & 3 & 2\\ 0 & 3 & 0 & 5\end{bmatrix}\right| = |5| = 5$$

(b) The 4-volume determined by the images $A\vec{v}_1$, $A\vec{v}_2$, $A\vec{v}_3$, and $A\vec{v}_4$ is

$$\text{volume}(A\vec{v}_1, A\vec{v}_2, A\vec{v}_3, A\vec{v}_4) = |\det A|\text{volume}(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) = |-49||5| = 245$$

A6 The n-volume of the parallelotope induced by $\vec{v}_1, \dots, \vec{v}_n$ is $\left|\det\begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_n\end{bmatrix}\right|$. Since adding a multiple of one column to another does not change the determinant (see Problem 5.2.D8), we get that

$$\left|\det\begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_n\end{bmatrix}\right| = \left|\det\begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_n + t\vec{v}_1\end{bmatrix}\right|$$

which is the volume of the parallelotope induced by $\vec{v}_1, \dots, \vec{v}_{n-1}, \vec{v}_n + t\vec{v}_1$.

Homework Problems

B1 (a) 2

(b) $A\vec{u} = \begin{bmatrix}62\\ 32\end{bmatrix}$, $A\vec{v} = \begin{bmatrix}36\\ 18\end{bmatrix}$

(c) 18

(d) $\left|\det\begin{bmatrix}62 & 36\\ 32 & 18\end{bmatrix}\right| = |-36| = 18(2)$

B2 $A\vec{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}, A\vec{e}_2 = \begin{bmatrix}t\\1\end{bmatrix}$. Area= $\left|\det\begin{bmatrix}1 & t\\0 & 1\end{bmatrix}\right| = 1$.

B3 (a) 5

(b) 15

(c) 75

B4 (a) 47

(b) −33

(c) 1551

B5 (a) 13

(b) 273

Conceptual Problems

D1 The n-volume of the parallelotope induced by $2\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ is $\left|\det\begin{bmatrix}2\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n\end{bmatrix}\right|$. Since multiplying a column of a matrix by a factor of 2 multiplies the determinant by a factor of 2 (see Problem 5.2.D8), we get that

$$\left|\det\begin{bmatrix}2\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n\end{bmatrix}\right| = 2\left|\det\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n\end{bmatrix}\right|$$

Thus, the volume of the parallelotope induced by $\vec{v}_1, \dots, \vec{v}_n$ is half of the volumne of the parallelotope induced by $2\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$.

D2 We have $L(\vec{x}) = A\vec{x}$ and $M(\vec{x}) = B\vec{x}$. Then, $M \circ L = M(L(\vec{x})) = B(A\vec{x}) = (BA)\vec{x}$. Hence, for any $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ we have

$$\begin{aligned}\text{Volume}(BA\vec{u}, BA\vec{v}, BA\vec{w}) &= \left|\det\begin{bmatrix}BA\vec{u} & BA\vec{v} & BA\vec{w}\end{bmatrix}\right| \\ &= \left|\det\left(BA\begin{bmatrix}\vec{u} & \vec{v} & \vec{w}\end{bmatrix}\right)\right| \\ &= |\det BA|\left|\det\begin{bmatrix}\vec{u} & \vec{v} & \vec{w}\end{bmatrix}\right| \\ &= |\det BA|\text{Volume}(\vec{u}, \vec{v}, \vec{w})\end{aligned}$$

Hence, the volume is multiplied by a factor of $|\det BA|$.

Chapter 5 Quiz

Problems

E1 Expanding along the third column gives

$$\begin{vmatrix} -2 & 4 & 0 & 0 \\ 1 & -2 & 2 & 9 \\ -3 & 6 & 0 & 3 \\ 1 & -1 & 0 & 0 \end{vmatrix} = 2(-1)^{2+3} \begin{vmatrix} -2 & 4 & 0 \\ -3 & 6 & 3 \\ 1 & -1 & 0 \end{vmatrix}$$

Expanding along the third column again we get

$$\begin{vmatrix} -2 & 4 & 0 & 0 \\ 1 & -2 & 2 & 9 \\ -3 & 6 & 0 & 3 \\ 1 & -1 & 0 & 0 \end{vmatrix} = (-2)(3)(-1)^{2+3} \begin{vmatrix} -2 & 4 \\ 1 & -1 \end{vmatrix} = -12$$

E2 Since the row operations $R_2 + 2R_1$, $R_3 - R_1$, and $R_4 - R_1$ do not change the determinant we get

$$\begin{vmatrix} 3 & 2 & 7 & -8 \\ -6 & -1 & -9 & 20 \\ 3 & 8 & 21 & -17 \\ 3 & 5 & 12 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 7 & -8 \\ 0 & 3 & 5 & 4 \\ 0 & 6 & 14 & -9 \\ 0 & 3 & 5 & 9 \end{vmatrix}$$

Now using the row operations $R_3 - 2R_2$ and $R_4 - R_2$ gives

$$\begin{vmatrix} 3 & 2 & 7 & -8 \\ -6 & -1 & -9 & 20 \\ 3 & 8 & 21 & -17 \\ 3 & 5 & 12 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 7 & -8 \\ 0 & 3 & 5 & 4 \\ 0 & 0 & 4 & -17 \\ 0 & 0 & 0 & 5 \end{vmatrix} = 3(3)(4)(5) = 180$$

E3 We have

$$\begin{vmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 0 & 0 \\ 5 & 0 & 0 & 0 & 6 \end{vmatrix} = 5 \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 4 & 0 & 0 \end{vmatrix} = 5(2) \begin{vmatrix} 0 & 3 & 0 \\ 0 & 0 & 1 \\ 4 & 0 & 0 \end{vmatrix}$$

$$= 5(2)(4) \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} = 5(2)(4)(3)(1) = 120$$

E4 We have

$$\begin{vmatrix} k & 2 & 1 \\ 0 & 3 & k \\ -2 & -4 & 1 \end{vmatrix} = k(3 + 4k) + 2(2k - 3) = 4k^2 + 7k - 6$$

Hence, the matrix is not invertible whenever $k^2 + 7k - 6 = 0$. Applying the quadratic formula, we get

$$k = \frac{-7 \pm \sqrt{7^2 - 4(4)(-6)}}{2(4)} = -\frac{7}{8} \pm \frac{\sqrt{145}}{8}$$

Hence, the matrix is invertible for all $k \neq -\frac{7}{8} \pm \frac{\sqrt{145}}{8}$.

E5 (a) Since multiplying a row by a constant multiplies the determinant by the constant, we get $3(7) = 21$.

(b) We require four swaps to move the first row to the bottom and all other rows up. In particular, we keep swapping the row that starts in the top row with the row beneath it until it is at the bottom. Since swapping rows multiplies the determinant by (-1) we get $(-1)^4(7) = 7$.

(c) $2A$ multiplies each entry in A (in particular, each entry in each of the 5 rows) by 2. Hence, factoring out the 2 from each of the 5 rows we get $2^5(7) = 224$.

(d) By Theorem 5.2.7, $\frac{1}{\det A} = \frac{1}{7}$.

(e) By Theorem 5.2.7, $\det(A^TA) = \det A^T \det A = \det A \det A = 7(7) = 49$.

E6 We have

$$\det A = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 1 & 1 \\ -2 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 3 \\ 1 & 1 & 2 \\ -2 & 0 & 0 \end{vmatrix} = (-2)\begin{vmatrix} 3 & 3 \\ 1 & 2 \end{vmatrix} = -6$$

and

$$(A^{-1})_{31} = \frac{1}{\det A}((\operatorname{cof} A)^T)_{31} = \frac{1}{\det A}C_{13} = \frac{1}{-6}\begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} = \frac{2}{-6} = -\frac{1}{3}$$

E7 The coefficient matrix is $A = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 1 & -1 \\ -2 & 0 & 2 \end{vmatrix}$. We get

$$\det A = \begin{vmatrix} 2 & 3 & 1 \\ 1 & 1 & -1 \\ -2 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 3 \\ 1 & 1 & 0 \\ -2 & 0 & 0 \end{vmatrix} = (-2)\begin{vmatrix} 3 & 3 \\ 1 & 0 \end{vmatrix} = 6$$

Hence,

$$x_2 = \frac{1}{\det A}\begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & -1 \\ -2 & 1 & 2 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} 0 & 3 & 3 \\ 1 & -1 & -1 \\ 0 & -1 & 0 \end{vmatrix} = \frac{-3}{6} = -\frac{1}{2}$$

E8 (a) The volume of the parallelepiped is

$$\text{volume}(\vec{u}, \vec{v}, \vec{w}) = \left|\det \begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & 3 \\ -2 & 3 & 4 \end{bmatrix}\right| = 33$$

(b) The volume of the image parallelepiped is

$$\text{volume}(A\vec{u}, A\vec{v}, A\vec{w}) = |\det A|\text{volume}(\vec{u}, \vec{v}, \vec{w}) = |-24|(33) = 792$$

Chapter 5 Further Problems

Problems

F1 Let $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$. Since all row sums are equal to zero, we get that

$$\vec{a}_1 + \cdots + \vec{a}_n = \vec{0}$$

Therefore the columns are linearly dependent, so $\det A = 0$.

F2 From the assumption, $\det A$ and $\det A^{-1}$ are both integers. But, $\det A^{-1} = \frac{1}{\det A}$, and $\frac{1}{\det A}$ is an integer if and only if $\det A = \pm 1$.

F3 First, suppose that all three angles are less than or equal to $\frac{\pi}{2}$. Let P be the point on the line from A to B such that $\vec{CP}$ is orthogonal to $\vec{AB}$. Then, from the figure,

$$c = \|\vec{AB}\| = \|\vec{AP}\| + \|\vec{PB}\| = b\cos A + a\cos B$$

Similarly,

$$\begin{aligned} a &= c\cos B + b\cos C \\ b &= a\cos C + c\cos A \end{aligned}$$

(If one of the angles is greater than $\frac{\pi}{2}$, these equations will still hold, but a slightly different picture and calculation is required.) Rewrite the equations,

$$\begin{aligned} b\cos A + a\cos B \qquad\quad &= c \\ c\cos B + b\cos C &= a \\ c\cos A \qquad\quad + a\cos C &= b \end{aligned}$$

and solve for $\cos A$ by Cramer's Rule:

$$\cos A = \frac{\begin{vmatrix} c & a & 0 \\ a & c & b \\ b & 0 & a \end{vmatrix}}{\begin{vmatrix} b & a & 0 \\ 0 & c & b \\ c & 0 & a \end{vmatrix}} = \frac{c(ca) - a(a^2 - b^2)}{2abc} = \frac{b^2 + c^2 - a^2}{2bc}$$

F4 (a) If $a = b$, $V_3(a, b, c) = 0$ because two rows are equal. Therefore $(a - b)$ divides $V_3(a, b, c)$. Similarly, $(b - c)$ and $(c - a)$ divide $V_3(a, b, c)$. Hence, for some t,

$$V_3(a, b, c) = t(c - a)(c - b)(b - a) = t(b - a)c^2 + \cdots$$

The cofactor of c^2 in the determinant is $(b - a)$, so $t = 1$. Therefore, $V_3(a, b, c) = (c - a)(c - b)(b - a)$.

(b) The same argument shows that $(d - a)$, $(d - b)$, $(d - c)$, $(c - a)$, $(c - b)$, $(b - a)$ are factors of $V_4(a, b, c, d)$. The cofactor d^3 in the determinant is exactly $V_3(a, b, c)$, and this agrees with the coefficient of d^3 in $(d - a)(d - b)(d - c)V_3(a, b, c)$. Hence,

$$V_4(a, b, c, d) = (d - a)(d - b)(d - c)V_3(a, b, c)$$

F5 (a) We have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ 0 & a_{33} & a_{34} \\ 0 & a_{43} & a_{44} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{21} & a_{21} \\ 0 & a_{33} & a_{34} \\ 0 & a_{43} & a_{44} \end{vmatrix}$$
$$= a_{11}a_{22}\begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} - a_{12}a_{21}\begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}$$
$$= (a_{11}a_{22} - a_{12}a_{21})\begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}$$
$$= \det A_1 \det A_4$$

(b) Let

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_4 = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

Then $\det A_1 \det A_4 - \det A_2 \det A_3 = 2(10) - 3(1) = 17$. However

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 3 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -7 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 5 \end{vmatrix} = 7$$

Thus, $\det A \neq \det A_1 \det A_4 - \det A_2 \det A_3$.

F6 (a) Let $B = \begin{bmatrix} b_{44} & b_{45} \\ b_{54} & b_{55} \end{bmatrix}$. Expand the big determinant along its fifth column (which contains the second column of B), then along the fourth column (which contains the first column of B):

$$\det\left[\begin{array}{c|c} A & O_{3,2} \\ \hline O_{2,3} & B \end{array}\right] = -b_{45}\det\left[\begin{array}{c|c} A & O_{3,1} \\ \hline O_{1,3} & b_{54} \end{array}\right] + b_{55}\left[\begin{array}{c|c} A & O_{3,1} \\ \hline O_{1,3} & b_{44} \end{array}\right]$$
$$= (-b_{45}b_{54} + b_{44}b_{55})\det A = \det A \det B$$

(b) By six row interchanges (each of two rows in B with each of three rows in A), we can transform this matrix into the matrix of part (a). Hence, the answer is $(-1)^6 \det A \det B = \det A \det B$.

CHAPTER 6 Eigenvectors and Diagonalization

6.1 Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors of a Mapping

Practice Problems

A1 We have

$$A\vec{v}_1 = \begin{bmatrix} -10 & 9 & 5 \\ -10 & 9 & 5 \\ 2 & 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \\ 1 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$A\vec{v}_2 = \begin{bmatrix} -10 & 9 & 5 \\ -10 & 9 & 5 \\ 2 & 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$A\vec{v}_3 = \begin{bmatrix} -10 & 9 & 5 \\ -10 & 9 & 5 \\ 2 & 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 \\ -6 \\ 6 \end{bmatrix} = -6 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$A\vec{v}_4 = \begin{bmatrix} -10 & 9 & 5 \\ -10 & 9 & 5 \\ 2 & 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -14 \\ -14 \\ -2 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$A\vec{v}_5 = \begin{bmatrix} -10 & 9 & 5 \\ -10 & 9 & 5 \\ 2 & 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Hence, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ are not eigenvectors of A, $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda = 0$, $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda = -6$, and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue $\lambda = 4$.

A2 To find the eigenvalues of a matrix, we first find and factor the characteristic polynomial $C(\lambda) = \det(A - \lambda I)$. Then, the roots of $C(\lambda)$ are the eigenvalues of A. We find a basis for the eigenspace of each eigenvalue λ by using the method in Section 3.4 to find a basis for the nullspace of $A - \lambda I$. In particular, we are just finding a basis for the solution space of the homogeneous system $(A - \lambda I)\vec{v} = \vec{0}$. Recall that we do this by row reducing the coefficient matrix $A - \lambda I$, writing the reduced row echelon form of the coefficient matrix back into equation form, and solving by row reducing. However, with practice, one can typically find a basis for the solution space by inspection from the reduced row echelon form of the coefficient matrix. Optimally, you want to practice these so much (and think about the result) until you can do this. You will find that this saves you a lot of much needed time on tests.

We will demonstrate the full procedure of finding a basis for the nullspace of $A - \lambda I$ for the first few examples here, and we will solve the rest by inspection.

(a) We have

$$C(\lambda) = \begin{vmatrix} -\lambda & 1 \\ -6 & 5-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$$

Hence, the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$.

To find a basis for the eigenspace of $\lambda_1 = 2$ we row reduce $A - \lambda_1 I$. We get

$$A - 2I = \begin{bmatrix} -2 & 1 \\ -6 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$$

Putting this into equation form we get $x_1 - \frac{1}{2}x_2 = 0$. Thus, $x_2 = t \in \mathbb{R}$ is a free variable, and $x_1 = \frac{1}{2}x_2 = \frac{1}{2}t$, so we get the general solution is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t/2 \\ t \end{bmatrix} = t\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

Thus, a basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\right\}$. However, we could also pick any scalar multiple of this. So, an better choice might be $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$.

To find a basis for the eigenspace of $\lambda_2 = 3$ we row reduce $A - \lambda_2 I$. We get

$$A - 3I = \begin{bmatrix} -3 & 1 \\ -6 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix}$$

Writing this into equation form we get $x_1 - \frac{1}{3}x_2 = 0$. Thus, $x_2 = t \in \mathbb{R}$ is a free variable, and $x_1 = \frac{1}{3}x_2 = \frac{1}{3}t$, so we get the general solution is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t/3 \\ t \end{bmatrix} = t\begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

Thus, a basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$.

(b) We have

$$C(\lambda) = \begin{vmatrix} 1-\lambda & 3 \\ 0 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$$

Hence, the only eigenvalues is $\lambda_1 = 1$.

To find a basis for the eigenspace of $\lambda_1 = 1$ we row reduce $A - \lambda_1 I$. We get

$$A - 1I = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Writing this into equation form we get $x_2 = 0$. Thus, $x_1 = t \in \mathbb{R}$ is a free variable, and the general solution is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Thus, a basis for the eigenspace of λ_1 is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.

(c) We have

$$C(\lambda) = \begin{vmatrix} 2-\lambda & 0 \\ 0 & 3-\lambda \end{vmatrix} = (\lambda - 2)(\lambda - 3)$$

Hence, the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$.

To find a basis for the eigenspace of $\lambda_1 = 2$ we row reduce $A - \lambda_1 I$. We get

$$A - 2I = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Writing this into equation form we get $x_2 = 0$. Thus, $x_1 = t \in \mathbb{R}$ is a free variable, and the general solution is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Thus, a basis for the eigenspace of λ_1 is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$.

To find a basis for the eigenspace of $\lambda_2 = 3$ we row reduce $A - \lambda_2 I$. We get

$$A - 3I = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Writing this into equation form we get $x_1 = 0$. Thus, $x_2 = t \in \mathbb{R}$ is a free variable, and the general solution is

$$\vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Thus, a basis for the eigenspace of λ_2 is $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

(d) We have

$$C(\lambda) = \begin{vmatrix} -26-\lambda & 10 \\ -75 & 29-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda + 1)(\lambda - 4)$$

Hence, the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 4$.

For $\lambda_1 = -1$,

$$A - (-1)I = \begin{bmatrix} -25 & 10 \\ -75 & 30 \end{bmatrix} \sim \begin{bmatrix} 1 & -5/2 \\ 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace of λ_1 is $\left\{ \begin{bmatrix} 5/2 \\ 1 \end{bmatrix} \right\}$, or we could pick $\left\{ \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$.

For $\lambda_2 = 4$,

$$A - 4I = \begin{bmatrix} -30 & 10 \\ -75 & 25 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace of λ_2 is $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$.

(e) We have

$$C(\lambda) = \begin{vmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix} = \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2)$$

Hence, the eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = -2$.

For $\lambda_1 = 5$,

$$A - 5I = \begin{bmatrix} -4 & 3 \\ 4 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/4 \\ 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace of λ_1 is $\left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$.

For $\lambda_2 = -2$,

$$A - (-2)I = \begin{bmatrix} 3 & 3 \\ 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace of λ_2 is $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

(f) We have

$$C(\lambda) = \begin{vmatrix} 3-\lambda & -3 \\ 6 & -6-\lambda \end{vmatrix} = \lambda^2 + 3\lambda = \lambda(\lambda + 3)$$

Hence, the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -3$.

For $\lambda_1 = 0$,

$$A - 0I = \begin{bmatrix} 3 & -3 \\ 6 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace of λ_1 is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

For $\lambda_2 = -3$,

$$A - (-3)I = \begin{bmatrix} 6 & -3 \\ 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace of λ_2 is $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

A3 (a) From what we learned in question A2 (b) and (c), we see by inspection that the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$. Moreover, each appear as a single root of the characteristic polynomial, so the algebraic multiplicity of each eigenvalue is 1. If you are not confident with finding the eigenvalues by inspection at this point, find and factor the characteristic polynomial to verify the result.

For $\lambda_1 = 2$,

$$A - 2I = \begin{bmatrix} 0 & -2 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace of λ_1 is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. Since a basis for the eigenspace contains a single vector, the dimension of the eigenspace is 1. Hence, the geometric multiplicity of $\lambda_1 = 2$ is 1.

For $\lambda_2 = 3$,

$$A - 3I = \begin{bmatrix} -1 & -2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix}-2\\1\end{bmatrix}\right\}$. Since a basis for the eigenspace contains a single vector, the dimension of the eigenspace is 1. Hence, the geometric multiplicity of $\lambda_2 = 3$ is 1.

(b) From what we learned in question A2 (b) and (c), we see by inspection that $\lambda_1 = 2$ is a double root of the characteristic polynomial. Hence, it has algebraic multiplicity 2. Verify this result by finding and factoring the characteristic polynomial.

For $\lambda_1 = 2$,

$$A - 2I = \begin{bmatrix}0 & -2\\0 & 0\end{bmatrix} \sim \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$$

Thus, a basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$. Since a basis for the eigenspace contains a single vector, the dimension of the eigenspace is 1. Hence, the geometric multiplicity of $\lambda_1 = 2$ is 1.

(c) We have

$$C(\lambda) = \begin{vmatrix}1-\lambda & 1\\-1 & 3-\lambda\end{vmatrix} = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2$$

Hence, the eigenvalue is $\lambda_1 = 2$. Since it is a double root of the characteristic polynomial we have the algebraic multiplicity is 2.

For $\lambda_1 = 2$,

$$A - 2I = \begin{bmatrix}-1 & 1\\-1 & 1\end{bmatrix} \sim \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}$$

Thus, a basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$. Since a basis for the eigenspace contains a single vector, the dimension of the eigenspace is 1. Hence, the geometric multiplicity of $\lambda_1 = 2$ is 1.

(d) We have

$$C(\lambda) = \begin{vmatrix}-\lambda & -5 & 3\\-2 & -6-\lambda & 6\\-2 & -7 & 7-\lambda\end{vmatrix}$$

Instead of just expanding the determinant along some row or column, we use row and column operations to simplify the determinant. This not only makes calculating the determinant easier, but also can make the charactersitic polynomial easier to factor. We get

$$\begin{aligned}C(\lambda) &= \begin{vmatrix}-\lambda & -5 & 3\\-2 & -6-\lambda & 6\\0 & -1+\lambda & 1-\lambda\end{vmatrix} = \begin{vmatrix}-\lambda & -5 & -2\\-2 & -6-\lambda & -\lambda\\0 & -1+\lambda & 0\end{vmatrix}\\ &= -(-1+\lambda)(\lambda^2-4) = -(\lambda-1)(\lambda-2)(\lambda+2)\end{aligned}$$

Thus, the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = -2$, each with algebraic multiplicity 1.

For $\lambda_1 = 1$,

$$A - I = \begin{bmatrix}-1 & -5 & 3\\-2 & -7 & 6\\-2 & -7 & 6\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -3\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$$

Thus, a basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}3\\0\\1\end{bmatrix}\right\}$. Since a basis for the eigenspace contains a single vector, the dimension of the eigenspace is 1. Hence, the geometric multiplicity of $\lambda_1 = 1$ is 1.

For $\lambda_2 = 2$,

$$A - 2I = \begin{bmatrix} -2 & -5 & 3 \\ -2 & -8 & 6 \\ -2 & -7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace of λ_2 is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$. Since a basis for the eigenspace contains a single vector, the dimension of the eigenspace is 1. Hence, the geometric multiplicity of $\lambda_2 = 2$ is 1.

For $\lambda_3 = -2$,

$$A - (-2)I = \begin{bmatrix} 2 & -5 & 3 \\ -2 & -4 & 6 \\ -2 & -7 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace of λ_3 is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. Since a basis for the eigenspace contains a single vector, the dimension of the eigenspace is 1. Hence, the geometric multiplicity of $\lambda_3 = -2$ is 1.

(e) We have

$$C(\lambda) = \begin{vmatrix} 2-\lambda & 2 & 2 \\ 2 & 2-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 2 & 2 \\ 2 & 2-\lambda & 2 \\ 0 & \lambda & -\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 4 & 2 \\ 2 & 4-\lambda & 2 \\ 0 & 0 & -\lambda \end{vmatrix}$$
$$= -\lambda(\lambda^2 - 6\lambda) = -\lambda^2(\lambda - 6)$$

Thus, the eigenvalues are $\lambda_1 = 0$ with algebraic multiplicity 2 and $\lambda_2 = 6$ with algebraic multiplicity 1.

For $\lambda_1 = 0$,

$$A - 0I = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace of λ_1 is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$. The dimension of the eigenspace is 2, so the geometric multiplicity of $\lambda_1 = 0$ is 2.

For $\lambda_2 = 6$,

$$A - 6I = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace of λ_2 is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. The dimension of the eigenspace is 1, so the geometric multiplicity of $\lambda_2 = 6$ is 1.

(f) We have

$$C(\lambda) = \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & 1 \\ 0 & -2+\lambda & 2-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & 2 & 1 \\ 1 & 4-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$
$$= (2-\lambda)(\lambda^2 - 7\lambda + 10) = -(\lambda-2)(\lambda-2)(\lambda-5)$$

Thus, the eigenvalues are $\lambda_1 = 2$ with algebraic multiplicity 2 and $\lambda_2 = 5$ with algebraic multiplicity 1.
For $\lambda_1 = 2$,

$$A - 2I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace of λ_1 is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$. The dimension of the eigenspace is 2, so the geometric multiplicity of $\lambda_1 = 2$ is 2.
For $\lambda_2 = 5$,

$$A - 5I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace of λ_2 is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. The dimension of the eigenspace is 1, so the geometric multiplicity of $\lambda_2 = 5$ is 1.

Homework Problems

B1 $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ are not eigenvectors of A, while $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ are eigenvectors of A with corresponding eigenvalues -2 and -3 respectively.

B2 (a) The eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 6$. The eigenspace of λ_1 is Span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. The eigenspace of λ_2 is Span $\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$.

(b) The only eigenvalue is $\lambda = 3$. The eigenspace of λ is Span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

(c) The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 4$. The eigenspace of λ_1 is Span $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$. The eigenspace of λ_2 is Span $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$.

(d) The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -4$. The eigenspace of λ_1 is Span$\left\{\begin{bmatrix}-2\\1\end{bmatrix}\right\}$. The eigenspace of λ_2 is Span$\left\{\begin{bmatrix}1\\-3\end{bmatrix}\right\}$.

(e) The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$. The eigenspace of λ_1 is Span$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$. The eigenspace of λ_2 is Span$\left\{\begin{bmatrix}3\\1\\0\end{bmatrix}\right\}$. The eigenspace of λ_3 is Span$\left\{\begin{bmatrix}13\\7\\1\end{bmatrix}\right\}$.

(f) The eigenvalues are $\lambda_1 = 2$, $\lambda_2 = -2$, and $\lambda_3 = -1$. The eigenspace of λ_1 is Span$\left\{\begin{bmatrix}2\\1\\1\end{bmatrix}\right\}$. The eigenspace of λ_2 is Span$\left\{\begin{bmatrix}0\\-1\\1\end{bmatrix}\right\}$. The eigenspace of λ_3 is Span$\left\{\begin{bmatrix}2\\0\\1\end{bmatrix}\right\}$.

B3 (a) $\lambda_1 = 3$ has algebraic multiplicity 1; a basis for its eigenspace is $\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$ so it has geometric multiplicity 1. $\lambda_2 = 7$ has algebraic multiplicity 1; a basis for its eigenspace is $\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$ so it has geometric multiplicity 1.

(b) $\lambda_1 = -3$ has algebraic multiplicity 2; a basis for its eigenspace is $\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$ so it has geometric multiplicity 1.

(c) $\lambda_1 = 8$ has algebraic multiplicity 1; a basis for its eigenspace is $\left\{\begin{bmatrix}3\\1\end{bmatrix}\right\}$ so it has geometric multiplicity 1. $\lambda_2 = 1$ has algebraic multiplicity 1; a basis for its eigenspace is $\left\{\begin{bmatrix}-1\\2\end{bmatrix}\right\}$ so it has geometric multiplicity 1.

(d) $\lambda_1 = -4$ has algebraic multiplicity 1; a basis for its eigenspace is $\left\{\begin{bmatrix}6\\4\\1\end{bmatrix}\right\}$ so it has geometric multiplicity 1. $\lambda_2 = -10$ has algebraic multiplicity 1; a basis for its eigenspace is $\left\{\begin{bmatrix}0\\-2\\1\end{bmatrix}\right\}$ so it has geometric multiplicity 1. $\lambda_3 = 2$ has algebraic multiplicity 1; a basis for its eigenspace is $\left\{\begin{bmatrix}0\\2\\5\end{bmatrix}\right\}$ so it has geometric multiplicity 1.

(e) $\lambda_1 = 2$ has algebraic multiplicity 2; a basis for its eigenspace is $\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$ so it has geometric multiplicity 2. $\lambda_2 = 8$ has algebraic multiplicity 1; a basis for its eigenspace is $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ so it has geometric multiplicity 1.

(f) $\lambda_1 = 5$ has algebraic multiplicity 1; a basis for its eigenspace is $\left\{\begin{bmatrix}2\\-1\\3\end{bmatrix}\right\}$ so it has geometric multiplicity 1.

$\lambda_2 = -16$ has algebraic multiplicity 1; a basis for its eigenspace is $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$ so it has geometric multiplicity 1.

$\lambda_3 = -8$ has algebraic multiplicity 1; a basis for its eigenspace is $\left\{\begin{bmatrix}21\\61\\64\end{bmatrix}\right\}$ so it has geometric multiplicity 1.

Computer Problems

C1 (a) The eigenvalues are $\lambda_1 = 9$, $\lambda_2 = \sqrt{19}$, and $\lambda_3 = -\sqrt{19}$. The eigenspace of λ_1 is Span $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$. The eigenspace of λ_2 is Span $\left\{\begin{bmatrix}-\frac{67+8\sqrt{19}}{(\sqrt{19}+5)(\sqrt{19}-9)}\\ \frac{3}{\sqrt{19}+5}\\ 1\end{bmatrix}\right\}$. The eigenspace of λ_3 is Span $\left\{\begin{bmatrix}-\frac{67-8\sqrt{19}}{(5-\sqrt{19})(-\sqrt{19}-9)}\\ \frac{3}{5-\sqrt{19}}\\ 1\end{bmatrix}\right\}$.

(b) The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 4 + \sqrt{7}$, and $\lambda_3 = 4 - \sqrt{7}$. The eigenspace of λ_1 is Span $\left\{\begin{bmatrix}17\\16\\3\end{bmatrix}\right\}$. The eigenspace of λ_2 is Span $\left\{\begin{bmatrix}\frac{2(29+4\sqrt{7})}{(17+7\sqrt{7})(-1+\sqrt{7})}\\ \frac{2(1+2\sqrt{7})}{17+7\sqrt{7}}\\ 1\end{bmatrix}\right\}$. The eigenspace of λ_3 is Span $\left\{\begin{bmatrix}\frac{2(29-4\sqrt{7})}{(17-7\sqrt{7})(-1-\sqrt{7})}\\ \frac{2(1-2\sqrt{7})}{17-7\sqrt{7}}\\ 1\end{bmatrix}\right\}$.

(c) The eigenvalues are $\lambda_1 = 0$ (with algebraic multiplicity 2), $\lambda_2 = 7$, and $\lambda_3 = 4$. The eigenspace of λ_1 is Span $\left\{\begin{bmatrix}-3\\3\\1\\1\end{bmatrix}\right\}$. The eigenspace of λ_2 is Span $\left\{\begin{bmatrix}1\\-1\\2\\2\end{bmatrix}\right\}$. The eigenspace of λ_3 is Span $\left\{\begin{bmatrix}1\\-4\\2\\2\end{bmatrix}\right\}$.

C2 $\begin{bmatrix}1.21\\-0.34\\0.87\end{bmatrix}$ is an eigenvector with eigenvalue 5. $\begin{bmatrix}1.31\\2.15\\-0.21\end{bmatrix}$ is an eigenvector with eigenvalue 0.4. $\begin{bmatrix}-1.85\\0.67\\2.10\end{bmatrix}$ is an eigenvector with eigenvalue 0.6.

Conceptual Problems

D1 Since $A\vec{v} = \lambda\vec{v}$ and $B\vec{v} = \mu\vec{v}$, it follows that

$$(A + B)\vec{v} = A\vec{v} + B\vec{v} = \lambda\vec{v} + \mu\vec{v} = (\lambda + \mu)\vec{v}$$
$$(AB)\vec{v} = A(B\vec{v}) = A(\mu\vec{v}) = \mu A\vec{v} = \mu\lambda\vec{v}$$

Hence, $\vec{v}$ is an eigenvector of $A + B$ with eigenvalue $\lambda + \mu$, and an eigenvector of AB with eigenvalue $\mu\lambda$.

D2 (a) Since $A\vec{v} = \lambda\vec{v}$, it follows that

$$A^2\vec{v} = A(A\vec{v}) = A(\lambda\vec{v}) = \lambda A\vec{v} = \lambda^2\vec{v}$$

Then, by induction,

$$A^n\vec{v} = A^{n-1}(A\vec{v}) = A^{n-1}(\lambda\vec{v}) = \lambda A^{n-1}\vec{v} = \lambda^n\vec{v}$$

Hence, λ^n is an eigenvalue of A^n with eigenvector $\vec{v}$.

(b) Consider the matrix of rotation in $\mathbb{R}^2$ through angle $2\pi/3$:

$$A = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

Then A leaves no direction unchanged so it has no real eigenvectors or eigenvalues. But, $A^3 = I$ has eigenvalue 1, and every non-zero vector in $\mathbb{R}^2$ is an eigenvector of A^3.

D3 If A is invertible, and $A\vec{v} = \lambda\vec{v}$, then

$$\vec{v} = A^{-1}A\vec{v} = A^{-1}(\lambda\vec{v}) = \lambda A^{-1}\vec{v}$$

So $A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$. Hence, $\vec{v}$ is an eigenvector of A^{-1} with eigenvalue $1/\lambda$.

D4 (a) If $\operatorname{rank}(A) < n$, then $0 = \det A = \det(A - 0I)$, so 0 is an eigenvalue. The geometric multiplicity of $\lambda = 0$ is the dimension of its eigenspace, which is the dimension of the nullspace of $A - \lambda I = 0$. Thus, by the Rank-Nullity Theorem, the geometric multiplicity of $\lambda = 0$ is $n - \operatorname{rank} A = n - r$.

(b) The matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, has rank 2, so the geometric multiplicity of $\lambda = 0$ is $n - \operatorname{rank} A = 3 - 2 = 1$. But, the characteristic polynomial is $C(\lambda) = -\lambda^3$, so the algebraic multiplicity of $\lambda = 0$ is 3.

D5 By definition of matrix multiplication, we have

$$(A\vec{v})_i = \sum_{k=1}^{n} a_{ik}\vec{v}_k = \sum_{k=1}^{n} a_{ik}(1) = c$$

for $1 \leq i \leq n$. Thus,

$$A\vec{v} = \begin{bmatrix} c \\ \vdots \\ c \end{bmatrix} = c\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

So $\vec{v}$ is an eigenvector of A.

6.2 Diagonalization

Practice Problems

A1 (a) We have

$$\begin{bmatrix} 11 & 6 \\ 9 & -4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 28 \\ 14 \end{bmatrix} = 14\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 11 & 6 \\ 9 & -4 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ -21 \end{bmatrix} = -7\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Thus, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue 14, and $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ is an eigenvector of A with eigenvalue -7. Using the methods of Section 3.5 we get that $P^{-1} = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$. Hence,

$$P^{-1}AP = \frac{1}{7}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 11 & 6 \\ 9 & -4 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & -7 \end{bmatrix}$$

(b) We have

$$\begin{bmatrix} 6 & 5 \\ 3 & -7 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \end{bmatrix} \neq \lambda\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 6 & 5 \\ 3 & -7 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 17 \\ -1 \end{bmatrix} \neq \lambda\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Hence, the columns of P are not eigenvectors of A, so P does not diagonalize A.

(c) We have

$$\begin{bmatrix} 5 & -8 \\ 4 & -7 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 5 & -8 \\ 4 & -7 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix} = -3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Thus, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue 1, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A with eigenvalue -3. Using the methods of Section 3.5 we get that $P^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. Hence,

$$P^{-1}AP = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 5 & -8 \\ 4 & -7 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$$

(d) We have

$$\begin{bmatrix} 2 & 4 & 4 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} = -2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 4 & 4 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = -2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & 4 & 4 \\ 4 & 2 & 4 \\ 4 & 4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix} = 10\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Thus, $\begin{bmatrix}-1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}-1\\0\\1\end{bmatrix}$ are both eigenvectors of A with eigenvalue -2, and $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is an eigenvector of A with eigenvalue 10. Using the methods of Section 3.5 we get that $P^{-1} = \frac{1}{3}\begin{bmatrix}-1 & 2 & -1\\-1 & -1 & 2\\1 & 1 & 1\end{bmatrix}$. Hence,

$$P^{-1}AP = \frac{1}{3}\begin{bmatrix}-1 & 2 & -1\\-1 & -1 & 2\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}2 & 4 & 4\\4 & 2 & 4\\4 & 4 & 2\end{bmatrix}\begin{bmatrix}-1 & -1 & 1\\1 & 0 & 1\\0 & 1 & 1\end{bmatrix} = \begin{bmatrix}-2 & 0 & 0\\0 & -2 & 0\\0 & 0 & 10\end{bmatrix}$$

A2 Alternate correct answers are possible.

(a) We have

$$C(\lambda) = \det(A - \lambda I) = \begin{vmatrix}3-\lambda & 2\\5 & 6-\lambda\end{vmatrix} = \lambda^2 - 9\lambda + 8 = (\lambda - 8)(\lambda - 1)$$

Hence, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 8$ each with algebraic multiplicity 1.
For $\lambda_1 = 1$ we get

$$A - I = \begin{bmatrix}2 & 2\\5 & 5\end{bmatrix} \sim \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$$

Thus, a basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}-1\\1\end{bmatrix}\right\}$ so the geometric multiplicity is 1.
For $\lambda_2 = 8$ we get

$$A - 8I = \begin{bmatrix}-5 & 2\\5 & -2\end{bmatrix} \sim \begin{bmatrix}1 & -2/5\\0 & 0\end{bmatrix}$$

Thus, a basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix}2\\5\end{bmatrix}\right\}$ so the geometric multiplicity is 1.

So, by Corollary 6.2.3, A is diagonalizable. In particular, $\left\{\begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}2\\5\end{bmatrix}\right\}$ forms a basis for $\mathbb{R}^2$ of eigenvectors of A. We take these basis vectors to be the columns of P. Then $P = \begin{bmatrix}-1 & 2\\1 & 5\end{bmatrix}$ diagonalizes A to $P^{-1}AP = \begin{bmatrix}1 & 0\\0 & 8\end{bmatrix} = D$.

(b) We have

$$C(\lambda) = \begin{vmatrix}-2-\lambda & 3\\4 & -3-\lambda\end{vmatrix} = \lambda^2 + 5\lambda - 6 = (\lambda + 6)(\lambda - 1)$$

Hence, eigenvalues are $\lambda_1 = -6$ and $\lambda_2 = 1$ each with algebraic multiplicity 1.
For $\lambda_1 = -6$ we get

$$A - (-6)I = \begin{bmatrix}4 & 3\\4 & 3\end{bmatrix} \sim \begin{bmatrix}1 & 3/4\\0 & 0\end{bmatrix}$$

Thus, a basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}-3\\4\end{bmatrix}\right\}$ so the geometric multiplicity is 1.

For $\lambda_2 = 1$ we get

$$A - I = \begin{bmatrix}-3 & 3\\4 & -4\end{bmatrix} \sim \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}$$

Thus, a basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ so the geometric multiplicity is 1.

So, by Corollary 6.2.3, A is diagonalizable. In particular, $\left\{\begin{bmatrix}-3\\4\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}\right\}$ forms a basis for $\mathbb{R}^2$ of eigenvectors of A. We take these basis vectors to be the columns of P. Then $P=\begin{bmatrix}-3&1\\4&1\end{bmatrix}$ diagonalizes A to $P^{-1}AP=\begin{bmatrix}-6&0\\0&1\end{bmatrix}=D$.

(c) We have

$$C(\lambda)=\begin{vmatrix}3-\lambda&6\\-5&-3-\lambda\end{vmatrix}=\lambda^2+21$$

Thus, by the quadratic formuls, we get that

$$\lambda=\frac{0\pm\sqrt{-48}}{2}=\pm\sqrt{12}i$$

Since A has complex eigenvalues it is not diagonalizable over $\mathbb{R}$.

(d) We have

$$C(\lambda)=\begin{vmatrix}-\lambda&1&0\\1&-\lambda&1\\1&1&1-\lambda\end{vmatrix}=-\lambda(\lambda+1)(\lambda-2)$$

Hence, the eigenvalues are $\lambda_1=-1$, $\lambda_2=2$, and $\lambda_3=0$ each with algebraic multiplicity 1.

For $\lambda_1=-1$ we get

$$A-(-1)I=\begin{bmatrix}1&1&0\\1&1&1\\1&1&2\end{bmatrix}\sim\begin{bmatrix}1&1&0\\0&0&1\\0&0&0\end{bmatrix}$$

Thus, a basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix}\right\}$ so the geometric multiplicity is 1.

For $\lambda_2=2$ we get

$$A-2I=\begin{bmatrix}-2&1&0\\1&-2&1\\1&1&-1\end{bmatrix}\sim\begin{bmatrix}1&0&-1/3\\0&1&-2/3\\0&0&0\end{bmatrix}$$

Thus, a basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$ so the geometric multiplicity is 1.

For $\lambda_3=0$ we get

$$A-0I=\begin{bmatrix}0&1&0\\1&0&1\\1&1&1\end{bmatrix}\sim\begin{bmatrix}1&0&1\\0&1&0\\0&0&0\end{bmatrix}$$

Thus, a basis for the eigenspace of λ_3 is $\left\{\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$ so the geometric multiplicity is 1.

So, by Corollary 6.2.3, A is diagonalizable. In particular, $\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$ forms a basis for $\mathbb{R}^3$ of eigenvectors of A. We take these basis vectors to be the columns of P. Then $P = \begin{bmatrix}-1 & 1 & -1\\1 & 2 & 0\\0 & 3 & 1\end{bmatrix}$ diagonalizes A to $P^{-1}AP = \begin{bmatrix}-1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 0\end{bmatrix} = D$.

(e) We have

$$C(\lambda) = \begin{vmatrix}6-\lambda & -9 & -5\\-4 & 9-\lambda & 4\\9 & -17 & -8-\lambda\end{vmatrix} = -(\lambda-1)^2(\lambda-5)$$

So, the eigenvalues are $\lambda_1 = 1$ with algebraic multiplicity 2 and $\lambda_2 = 5$ with algebraic multiplicity 1. For $\lambda_1 = 1$ we get

$$A - I = \begin{bmatrix}5 & -9 & -5\\-4 & 8 & 4\\9 & -17 & -9\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -1\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$$

Thus, a basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$ so the geometric multiplicity is 1.

For $\lambda_2 = 5$ we get

$$A - 5I = \begin{bmatrix}1 & -9 & -5\\-4 & 4 & 4\\9 & -17 & -13\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -1/2\\0 & 1 & 1/2\\0 & 0 & 0\end{bmatrix}$$

Thus, a basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}\right\}$ so the geometric multiplicity is 1.

Therefore, by Corollary 6.2.3, A is not diagonalizable since the geometric multiplicity of λ_1 does not equal its algebraic multiplicity.

(f) We have

$$C(\lambda) = \begin{vmatrix}-2-\lambda & 7 & 3\\-1 & 2-\lambda & 1\\0 & 2 & 1-\lambda\end{vmatrix} = -(\lambda-1)(\lambda^2+1)$$

So, the eigenvalues of A are 1, i, and $-i$. Hence, A is not diagonalizable over $\mathbb{R}$.

(g) We have

$$C(\lambda) = \begin{vmatrix}-1-\lambda & 6 & 3\\3 & -4-\lambda & -3\\-6 & 12 & 8-\lambda\end{vmatrix} = -(\lambda-2)^2(\lambda+1)$$

Thus, the eigenvalues are $\lambda_1 = 2$ with algebraic multiplicity 2 and $\lambda_2 = -1$ with algebraic multiplicity 1. For $\lambda_1 = 2$ we get

$$A - 2I = \begin{bmatrix}-3 & 6 & 3\\3 & -6 & -3\\-6 & 12 & 6\end{bmatrix} \sim \begin{bmatrix}1 & -2 & -1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

So, a basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}2\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$ so the geometric multiplicity is 2.

For $\lambda_2 = -1$ we get

$$A-(-1)I = \begin{bmatrix}0 & 6 & 3\\ 3 & -3 & -3\\ -6 & 12 & 9\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -1/2\\ 0 & 1 & 1/2\\ 0 & 0 & 0\end{bmatrix}$$

A basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}\right\}$ so the geometric multiplicity is 1.

Therefore by Corollary 6.2.3, A is diagonalizable. In particular, $\left\{\begin{bmatrix}2\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}1\\-1\\2\end{bmatrix}\right\}$ forms a basis for $\mathbb{R}^3$ of eigenvectors of A. We take these basis vectors to be the columns of P. Then $P = \begin{bmatrix}2 & 1 & 1\\ 1 & 0 & -1\\ 0 & 1 & 2\end{bmatrix}$ diagonalizes A to $P^{-1}AP = \begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & -1\end{bmatrix} = D$.

A3 Alternate correct answers are possible.

(a) We have

$$C(\lambda) = \begin{vmatrix}3-\lambda & 0\\ -3 & 3-\lambda\end{vmatrix} = (\lambda-3)^2$$

The only eigenvalue is $\lambda_1 = 3$ with algebraic multiplicity 2.

For $\lambda_1 = 3$ we get

$$A-3I = \begin{bmatrix}0 & 0\\ -3 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$$

Hence, a basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$ so the geometric multiplicity is 1.

Therefore, by Corollary 6.2.3, A is not diagonalizable since the geometric multiplicity of λ_1 does not equal its algebraic multiplicity.

(b) We have

$$C(\lambda) = \begin{vmatrix}4-\lambda & 4\\ 4 & 4-\lambda\end{vmatrix} = \lambda(\lambda-8)$$

The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 8$ each with algebraic multiplicity 1.

For $\lambda_1 = 0$ we get

$$A-0I = \begin{bmatrix}4 & 4\\ 4 & 4\end{bmatrix} \sim \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$$

Hence, a basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}-1\\1\end{bmatrix}\right\}$ so the geometric multiplicity is 1.

For $\lambda_2 = 8$ we get

$$A-8I = \begin{bmatrix}-4 & 4\\ 4 & -4\end{bmatrix} \sim \begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}$$

So, a basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ so the geometric multiplicity is 1.

Therefore, by Corollary 6.2.3, A is diagonalizable with $P = \begin{bmatrix}-1 & 1\\1 & 1\end{bmatrix}$ and $D = \begin{bmatrix}0 & 0\\0 & 8\end{bmatrix}$.

(c) We have

$$C(\lambda) = \begin{vmatrix}-2-\lambda & 5\\5 & -2-\lambda\end{vmatrix} = (\lambda-3)(\lambda+7)$$

The eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -7$ each with algebraic multiplicity 1.

For $\lambda_1 = 3$ we get

$$A - 3I = \begin{bmatrix}-5 & 5\\5 & -5\end{bmatrix} \sim \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}$$

A basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ so the geometric multiplicity is 1.

For $\lambda_2 = 7$ we get

$$A - 7I = \begin{bmatrix}5 & 5\\5 & 5\end{bmatrix} \sim \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$$

So, a basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix}-1\\1\end{bmatrix}\right\}$ so the geometric multiplicity is 1.

Therefore, by Corollary 6.2.3, A is diagonalizable with $P = \begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$ and $D = \begin{bmatrix}3 & 0\\0 & -7\end{bmatrix}$.

(d) We have

$$C(\lambda) = \begin{vmatrix}-\lambda & 6 & -8\\-2 & 4-\lambda & -4\\-2 & 2 & -2-\lambda\end{vmatrix} = -(\lambda-2)^2(\lambda+2)$$

The eigenvalues are $\lambda_1 = 2$ with algebraic multiplicity 2 and $\lambda_2 = -2$ with algebraic multiplicity 1.

For $\lambda_1 = 2$ we get

$$A - 2I = \begin{bmatrix}-2 & 6 & -8\\-2 & 2 & -4\\-2 & 2 & -4\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 1\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix}$$

Thus, a basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}-1\\1\\1\end{bmatrix}\right\}$ so the geometric multiplicity is 1.

For $\lambda_2 = -2$ we get

$$A - (-2)I = \begin{bmatrix}2 & 6 & -8\\-2 & 6 & -4\\-2 & 2 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -1\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix}$$

A basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$ so the geometric multiplicity is 1.

Therefore, by Corollary 6.2.3, A is not diagonalizable since the geometric multiplicity of λ_1 does not equal its algebraic multiplicity.

(e) We have

$$C(\lambda) = \begin{vmatrix} -\lambda & 2 & 2 \\ 2 & -\lambda & 2 \\ 2 & 2 & -\lambda \end{vmatrix} = -(\lambda+2)^2(\lambda-4)$$

The eigenvalues are $\lambda_1 = -2$ with algebraic multiplicity 2 and $\lambda_2 = 4$ with algebraic multiplicity 1. For $\lambda_1 = -2$ we get

$$A - (-2)I = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for the eigenspace of λ_1 is $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ so the geometric multiplicity is 2.

For $\lambda_2 = 4$ we get

$$A - 4I = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for the eigenspace of λ_2 is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ so the geometric multiplicity is 1.

Therefore, by Corollary 6.2.3, A is diagonalizable with $P = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

(f) We have

$$C(\lambda) = \begin{vmatrix} 2-\lambda & 0 & 0 \\ -1 & -\lambda & 1 \\ -1 & -2 & 3-\lambda \end{vmatrix} = -(\lambda-2)^2(\lambda-1)$$

The eigenvalues are $\lambda_1 = 2$ with algebraic multiplicity 2 and $\lambda_2 = 1$ with algebraic multiplicity 1. For $\lambda_1 = 2$ we get

$$A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ -1 & -2 & 1 \\ -1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for the eigenspace of λ_1 is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ so the geometric multiplicity is 2.

For $\lambda_2 = 1$ we get

$$A - I = \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 1 \\ -1 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for the eigenspace of λ_2 is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ so the geometric multiplicity is 1.

Therefore, by Corollary 6.2.3, A is diagonalizable with $P = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(g) We have

$$C(\lambda) = \begin{vmatrix} -3-\lambda & -3 & 5 \\ 13 & 10-\lambda & -13 \\ 3 & 2 & -1-\lambda \end{vmatrix} = -(\lambda-2)(\lambda^2-4x+5)$$

So, the eigenvalues of are 2, $2+i$ and $2-i$. Hence, A is not diagonalizable over $\mathbb{R}$.

Homework Problems

B1 (a) P does not diagonalize A.

(b) $\begin{bmatrix}1\\1\end{bmatrix}$ is an eigenvector with eigenvalue 4 and $\begin{bmatrix}1\\-1\end{bmatrix}$ is an eigenvector with eigenvalue -2. $P^{-1} = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$, $P^{-1}AP = \begin{bmatrix}4 & 0\\0 & -2\end{bmatrix}$.

(c) $\begin{bmatrix}2\\3\end{bmatrix}$ is an eigenvector with eigenvalue 4 and $\begin{bmatrix}1\\-1\end{bmatrix}$ is an eigenvector with eigenvalue -1. $P^{-1} = \frac{1}{5}\begin{bmatrix}1 & 1\\3 & -2\end{bmatrix}$, $P^{-1}AP = \begin{bmatrix}4 & 0\\0 & -1\end{bmatrix}$.

(d) $\begin{bmatrix}-1\\1\\3\end{bmatrix}$ is an eigenvector with eigenvalue 3, $\begin{bmatrix}-1\\2\\3\end{bmatrix}$ is an eigenvector with eigenvalue 1, and $\begin{bmatrix}1\\-1\\-2\end{bmatrix}$ is an eigenvector with eigenvalue -1. $P^{-1} = \begin{bmatrix}1 & -1 & 1\\1 & 1 & 0\\3 & 0 & 1\end{bmatrix}$, $P^{-1}AP = \begin{bmatrix}3 & 0 & 0\\0 & 1 & 0\\0 & 0 & -1\end{bmatrix}$

B2 Alternate correct answers are possible.

(a) The eigenvalues are $\lambda_1 = -5$ and $\lambda_2 = 2$ each with algebraic multiplicity 1. A basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}-3\\1\end{bmatrix}\right\}$ so the geometric multiplicity is 1. A basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}$ so the geometric multiplicity is 1. Therefore, by Corollary 3, A is diagonalizable with $P = \begin{bmatrix}-3 & 1\\1 & 2\end{bmatrix}$ and $D = \begin{bmatrix}-5 & 0\\0 & 2\end{bmatrix}$.

(b) The eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = 3$ each with algebraic multiplicity 1. A basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$ so the geometric multiplicity is 1. A basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix}-1\\1\end{bmatrix}\right\}$ so the geometric multiplicity is 1. Therefore, by Corollary 3, A is diagonalizable with $P = \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}$ and $D = \begin{bmatrix}5 & 0\\0 & 3\end{bmatrix}$.

(c) The only eigenvalue is $\lambda_1 = 5$ with algebraic multiplicity 2. A basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$ so the geometric multiplicity is 1. Therefore, by Corollary 3, A is not diagonalizable since the geometric multiplicity of λ_1 does not equal its algebraic multiplicity.

(d) The eigenvalues are $\lambda_1 = -2$ with algebraic multiplicity 2 and $\lambda_2 = 4$ with algebraic multiplicity 1. A basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$ so the geometric multiplicity is 2. A basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$ so the geometric multiplicity is 1. Therefore, by Corollary 3, A is diagonalizable with $P = \begin{bmatrix}-2 & -1 & 1\\1 & 0 & 0\\0 & 1 & 1\end{bmatrix}$ and $D = \begin{bmatrix}-2 & 0 & 0\\0 & -2 & 0\\0 & 0 & 4\end{bmatrix}$.

(e) The eigenvalues are $\lambda_1 = -2$ with algebraic multiplicity 2 and $\lambda_2 = 6$ with algebraic multiplicity 1. A basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$ so the geometric multiplicity is 1. A basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix}0\\1\\-2\end{bmatrix}\right\}$ so the geometric multiplicity is 1. Therefore, by Corollary 3, A is not diagonalizable since the geometric multiplicity of λ_1 does not equal its algebraic multiplicity.

(f) The eigenvalues of are 0 and $\pm\sqrt{2}i$. Hence, A is not diagonalizable over $\mathbb{R}$.

(g) The eigenvalues are $\lambda_1 = 6$, $\lambda_2 = -2$, and $\lambda_3 = 1$ each with algebraic multiplicity 1. A basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}-1\\2\\1\end{bmatrix}\right\}$ so the geometric multiplicity is 1. A basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix}-1\\-6\\9\end{bmatrix}\right\}$ so the geometric multiplicity is 1. A basis for the eigenspace of λ_3 is $\left\{\begin{bmatrix}4\\-3\\6\end{bmatrix}\right\}$ so the geometric multiplicity is 1. Therefore, by Corollary 3, A is diagonalizable with $P = \begin{bmatrix}-1 & -1 & 4\\2 & -6 & -3\\1 & 9 & 6\end{bmatrix}$ and $D = \begin{bmatrix}6 & 0 & 0\\0 & -2 & 0\\0 & 0 & 1\end{bmatrix}$.

B3 Alternate correct answers are possible.

(a) The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 9$ each with algebraic multiplicity 1. A basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}-1\\1\end{bmatrix}\right\}$ so the geometric multiplicity is 1. A basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix}3\\5\end{bmatrix}\right\}$ so the geometric multiplicity is 1. Therefore, by Corollary 3, A is diagonalizable with $P = \begin{bmatrix}-1 & 3\\1 & 5\end{bmatrix}$ and $D = \begin{bmatrix}1 & 0\\0 & 9\end{bmatrix}$.

(b) The eigenvalues are $\lambda_1 = 11$ and $\lambda_2 = 0$ each with algebraic multiplicity 1. A basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}-3\\4\end{bmatrix}\right\}$ so the geometric multiplicity is 1. A basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}$ so the geometric multiplicity is 1. Therefore, by Corollary 3, A is diagonalizable with $P = \begin{bmatrix}-3 & 2\\4 & 1\end{bmatrix}$ and $D = \begin{bmatrix}11 & 0\\0 & 0\end{bmatrix}$.

(c) The eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = -3$ each with algebraic multiplicity 1. A basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}-1\\1\end{bmatrix}\right\}$ so the geometric multiplicity is 1. A basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ so the geometric multiplicity is 1. Therefore, by Corollary 3, A is diagonalizable with $P = \begin{bmatrix}-1 & 1\\1 & 1\end{bmatrix}$ and $D = \begin{bmatrix}5 & 0\\0 & -3\end{bmatrix}$.

(d) The eigenvalues are $\lambda_1 = -1$ with algebraic multiplicity 2 and $\lambda_2 = 0$ with algebraic multiplicity 1. A basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}1\\3\\0\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}\right\}$ so the geometric multiplicity is 2. A basis for the eigenspace of λ_2 is

$\left\{\begin{bmatrix}2\\1\\1\end{bmatrix}\right\}$ so the geometric multiplicity is 1. Therefore, by Corollary 3, A is diagonalizable with $P = \begin{bmatrix}1&2&2\\3&0&1\\0&1&1\end{bmatrix}$ and $D = \begin{bmatrix}-1&0&0\\0&-1&0\\0&0&0\end{bmatrix}$.

(e) The only eigenvalue is $\lambda_1 = 1$ with algebraic multiplicity 3. A basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\1\end{bmatrix}\right\}$ so the geometric multiplicity is 2. Therefore, by Corollary 3, A is not diagonalizable since the geometric multiplicity of λ_1 does not equal its algebraic multiplicity.

(f) The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 5$, and $\lambda_3 = -2$ each with algebraic multiplicity 1. A basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}-2\\1\\0\end{bmatrix}\right\}$ so the geometric multiplicity is 1. A basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix}1\\2\\0\end{bmatrix}\right\}$ so the geometric multiplicity is 1. A basis for the eigenspace of λ_3 is $\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ so the geometric multiplicity is 1. Therefore, by Corollary 3, A is diagonalizable with $P = \begin{bmatrix}-2&1&0\\1&2&0\\0&0&1\end{bmatrix}$ and $D = \begin{bmatrix}0&0&0\\0&5&0\\0&0&-2\end{bmatrix}$.

(g) The eigenvalues of are 1 and $1 \pm 2i$. Hence, A is not diagonalizable over $\mathbb{R}$.

Conceptual Problems

D1 Observe that if $P^{-1}AP = B$ for some invertible matrix P, then

$$\begin{aligned}\det(B - \lambda I) &= \det(P^{-1}AP - \lambda P^{-1}P) = \det\left(P^{-1}(A - \lambda I)P\right)\\ &= \det P^{-1}\det(A - \lambda I)\det P = \det(A - \lambda I)\end{aligned}$$

Thus, since A and B have the same characteristic polynomial, they have the same eigenvalues.

D2 If A and B are similar, then there exists an invertible matrix P such that $P^{-1}AP = B$. First, observe that if $\vec{y} \in \mathrm{Col}(B)$, then there exist $\vec{x}$ such that $\vec{y} = B\vec{x} = P^{-1}AP\vec{x}$. Since P is invertible, there exists a vector $\vec{z}$ such that $\vec{z} = P\vec{x}$, hence $\vec{y} = P^{-1}A\vec{z}$. So $\vec{y} \in \mathrm{Col}(P^{-1}A)$. Therefore $\mathrm{rank}(P^{-1}A) \geq \mathrm{rank}(B)$.
Let $\vec{x} \in \mathrm{Null}(A)$ so that $A\vec{x} = \vec{0}$. Then $P^{-1}A\vec{x} = P^{-1}\vec{0} = \vec{0}$. Thus $\vec{x} \in \mathrm{Null}(P^{-1}A)$. Hence, $\mathrm{nullity}(P^{-1}A) \geq \mathrm{nullity}(A)$ which implies that $\mathrm{rank}(A) \geq \mathrm{rank}(P^{-1}A)$ by the Rank Theorem.
Therefore, we have shown that $\mathrm{rank}(B) \leq \mathrm{rank}(P^{-1}A) \leq \mathrm{rank}(A)$. However, we can also write $A = Q^{-1}BQ$ where $Q = P^{-1}$. Thus, by the same argument, $\mathrm{rank}(B) \geq \mathrm{rank}(A)$ and hence $\mathrm{rank}(A) = \mathrm{rank}(B)$.

D3 (a)

$$\mathrm{tr}(AB) = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki} = \sum_{k=1}^{n}\sum_{i=1}^{n} b_{ki}a_{ik} = \mathrm{tr}(BA)$$

(b)

$$\mathrm{tr}(B) = \mathrm{tr}(P^{-1}AP) = \mathrm{tr}(P^{-1}(AP)) = \mathrm{tr}((AP)P^{-1}) = \mathrm{tr}(APP^{-1}) = \mathrm{tr}(A)$$

D4 (a) If $P^{-1}AP = D$, then multiplying on the left by P gives $AP = PD$. Now, multiplying on the right by P^{-1} gives $A = PDP^{-1}$ as required.

(b) Let $P = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Then we get $A = PDP^{-1} = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$.

(c) Let $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Then we get $A = PDP^{-1} = \begin{bmatrix} -1 & 2 & 3 \\ -5 & 8 & 5 \\ 6 & -8 & -4 \end{bmatrix}$.

D5 (a) We prove the result by induction on k. If $k = 1$, then $P^{-1}AP = D$ implies $A = PDP^{-1}$ and so the result holds. Assume the result is true for some k. We then have

$$A^{k+1} = A^kA = (PD^kP^{-1})(PDP^{-1}) = PD^kP^{-1}PDP^{-1} = PD^kIDP^{-1} = PD^{k+1}P^{-1}$$

as required.

(b) The characteristic polynomial of A is

$$C(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 6 & 3 \\ 3 & -4-\lambda & -3 \\ -6 & 12 & 8-\lambda \end{vmatrix} = -\lambda^3 + 3\lambda^2 - 4$$
$$= -(\lambda - 2)^2(\lambda + 1)$$

Hence, $\lambda_1 = 2$ is an eigenvalue with algebraic multiplicity 2 and $\lambda_2 = -1$ is an eigenvalue with algebraic multiplicity 1.

For $\lambda_1 = 2$ we get $A - \lambda_1 I = \begin{bmatrix} -3 & 6 & 3 \\ 3 & -6 & -3 \\ -6 & 12 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus, a basis for the eigenspace is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$. Hence, the geometric multiplicity of λ_1 is 2.

For $\lambda_2 = -1$ we get $A - \lambda_2 I = \begin{bmatrix} 0 & 6 & 3 \\ 3 & -3 & -3 \\ -6 & 12 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$. Therefore $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$ is a basis for the eigenspace. Hence, the geometric multiplicity of λ_2 is 1.

For every eigenvalue of A the geometric multiplicity equals the algebraic multiplicity, so A is diagonalizable. We can take $P = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$ and get $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. We now find that $P^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ -2 & 4 & 3 \\ 1 & -2 & -1 \end{bmatrix}$ and

$$A^5 = PD^5P^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 32 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -2 & 4 & 3 \\ 1 & -2 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 66 & 33 \\ 33 & -34 & -33 \\ -66 & 132 & 98 \end{bmatrix}$$

D6 (a) Suppose that A is an $n \times n$ matrix and $P^{-1}AP = D$. Then, the diagonal entries of D are the eigenvalues $\lambda_1, \dots, \lambda_n$ of A. Since A is similar to D, they both have the same trace by Theorem 1. Thus, $\operatorname{tr} A = \operatorname{tr} D = \lambda_1 + \cdots + \lambda_n$.

(b) Observe that

$$A - bI = \begin{bmatrix} a+b-b & a & a \\ a & a+b-b & a \\ a & a & a+b-b \end{bmatrix} = \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $\lambda_1 = b$ is an eigenvalue with geometric multiplicity 2. Therefore, the algebraic multiplicity of λ_1 is at least 2. From part (a), we have

$$\lambda_1 + \lambda_1 + \lambda_2 = \operatorname{tr} A = (a+b) + (a+b) + (a+b) = 3a + 3b$$

Thus, $\lambda_2 = 3a + b$. Hence, if $a = 0$, we have $\lambda_1 = b$ is an eigenvalue with algebraic multiplicity 3 and geometric multiplicity 3, or if $a \neq 0$, we have $\lambda_1 = b$ is an eigenvalue with algebraic and geometric multiplicity 2 and $\lambda_2 = 3a + b$ is an eigenvalue with algebraic and geometric multiplicity 1.

D7 (a) If A is an $n \times n$ diagonalizable matrix, then A is similar to a diagonal matrix $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Hence, by Theorem 6.2.1 $\det A = \det D = \lambda_1 \cdots \lambda_n$.

(b) For any polynomial $p(x)$, the constant term is $p(0)$. Thus, for the characteristic polynomial of A, the constant term is $C(0) = \det(A - 0I) = \det A$.

(c) By the Fundamental Theorem of Algebra, a polynomial with real coefficients can be written as a product of first degree factors (possible using complex and repeated zeros of the polynomial). In particular,

$$\det(A - \lambda I) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

where the λ_i may be complex or repeated. Set $\lambda = 0$:

$$\det A = (-1)^n (-\lambda_1)(-\lambda_2) \cdots (-\lambda_n) = \lambda_1 \cdots \lambda_n$$

(Note that the product of the characteristic roots is real even if some of them are complex.)

D8 By Problem D7, 0 is an eigenvalue of A if and only if $\det A = 0$. But $\det A = 0$ if and only if A is not invertible.

D9 Let $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ so that $P^{-1}AP = D$. Hence,

$$P^{-1}(A - \lambda_1 I)P = P^{-1}AP - \lambda_1 I = D - \lambda_1 I = \operatorname{diag}(0, \lambda_2 - \lambda_1, \cdots, \lambda_n - \lambda_1)$$

Since $A - \lambda_1 I$ is diagonalized to this form, its eigenvalues are $0, \lambda_2 - \lambda_1, \dots, \lambda_n - \lambda_1$.

6.3 Powers of Matrices and the Markov Process

Practice Problems

A1 (a) Since the columns in A do not sum to 1, A is not a Markov matrix.

(b) Since the columns in B sum to 1 it is a Markov matrix. If $\lambda = 1$, then

$$B - \lambda I = \begin{bmatrix} -0.7 & 0.6 \\ 0.7 & -0.6 \end{bmatrix} \sim \begin{bmatrix} 1 & -6/7 \\ 0 & 0 \end{bmatrix}$$

Therefore, an eigenvector corresponding to λ_1 is $\begin{bmatrix} 6/7 \\ 1 \end{bmatrix}$. The components in the state vector must sum to 1, so the invariant state corresponding to λ_1 is

$$\frac{1}{\frac{6}{7}+1}\begin{bmatrix} 6/7 \\ 1 \end{bmatrix} = \begin{bmatrix} 6/13 \\ 7/13 \end{bmatrix}$$

(c) Since the columns in C do not sum to 1, A is not a Markov matrix.

(d) Since the columns in D sum to 1 it is a Markov matrix. If $\lambda = 1$, then

$$D - \lambda I = \begin{bmatrix} -0.1 & 0.1 & 0 \\ 0 & -0.1 & 0.1 \\ 0.1 & 0 & -0.1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, an eigenvector corresponding to λ_1 is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The components in the state vector must sum to 1, so the invariant state corresponding to λ_1 is

$$\frac{1}{1+1+1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

A2 (a) Let R_m be the fraction of people dwelling in rural areas (as a decimal), and U_m be the number of people dwelling in urban areas. Then $R_m + U_m = 1$, and

$$\begin{aligned} R_{m+1} &= 0.85R_m + 0.05U_m \\ U_{m+1} &= 0.15R_m + 0.95U_m \end{aligned}$$

Or, in vector form,

$$\begin{bmatrix} R_{m+1} \\ U_{m+1} \end{bmatrix} = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}\begin{bmatrix} R_m \\ U_m \end{bmatrix}$$

The transition matrix is $T = \begin{bmatrix} 0.85 & 0.05 \\ 0.15 & 0.95 \end{bmatrix}$. Since T is a Markov matrix, it necessarily has an eigenvalue $\lambda_1 = 1$. An eigenvector corresponding to λ_1 is $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, the state vector is then $\begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}$, which is fixed under the transformation with matrix T. If follow that in the long run, 25% of the population will be rural dwellers and 75% will be urban dwellers.

(b) Another eigenvalue of T is $\lambda_2 = 0.8$, with corresponding eigenvector $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. It follows that T is diagonalized by $P = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}$. We get $P^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}$. Since these eigenvectors form a basis for $\mathbb{R}^2$,

$$\begin{bmatrix} R_0 \\ U_0 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = P\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

So

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = P^{-1}\begin{bmatrix} R_0 \\ U_0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} R_0 + U_0 \\ -3R_0 + U_0 \end{bmatrix}$$

By linearity,

$$\begin{aligned} T^m\begin{bmatrix} R_0 \\ U_0 \end{bmatrix} &= c_1T^m\vec{v}_1 + c_2T^m\vec{v}_2 \\ &= c_1\lambda_1^m\vec{v}_1 + c_2\lambda_2^m\vec{v}_2 \\ &= \frac{1}{4}(R_0+U_0)\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \frac{1}{4}(-3R_0+U_0)(0.8)^m\begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned}$$

After 50 years, or 5 decades,

$$\begin{aligned} T^5\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} &= \frac{1}{4}(0.5+0.5)\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \frac{1}{4}(-1)(0.8)^5\begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &\approx \frac{1}{4}\begin{bmatrix} 1+0.3277 \\ 3-0.3277 \end{bmatrix} \\ &\approx \begin{bmatrix} 0.33 \\ 0.67 \end{bmatrix} \end{aligned}$$

Therefore, after 5 decades, approximately 33% of the population will be rural dwellers and 67% will be urban dwellers.

A3 Let A_m be the number of cars returned to the airport (as a fraction), and let T_m and C_m be the number of cars returned to the train station and city centre, respectively. Then $A_m + T_m + C_m = 1$, and

$$\begin{aligned} A_{m+1} &= \frac{8}{10}A_m + \frac{3}{10}T_m + \frac{3}{10}C_m \\ T_{m+1} &= \frac{1}{10}A_m + \frac{6}{10}T_m + \frac{1}{10}C_m \\ C_{m+1} &= \frac{1}{10}A_m + \frac{1}{10}T_m + \frac{6}{10}C_m \end{aligned}$$

Or in vector form

$$\begin{bmatrix} A_{m+1} \\ T_{m+1} \\ C_{m+1} \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 8 & 3 & 3 \\ 1 & 6 & 1 \\ 1 & 1 & 6 \end{bmatrix}\begin{bmatrix} A_m \\ T_m \\ C_m \end{bmatrix}$$

The transition matrix is $T = \frac{1}{10}\begin{bmatrix} 8 & 3 & 3 \\ 1 & 6 & 1 \\ 1 & 1 & 6 \end{bmatrix}$. Since T is a Markov matrix, it necessarily has an eigenvalue $\lambda_1 = 1$. An eigenvector corresponding to λ_1 is $\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, the state vector is then $\begin{bmatrix} 3/5 \\ 1/5 \\ 1/5 \end{bmatrix}$, which is fixed under the transformation with matrix T. If follow that in the long run, 60% of the cars will be at the airport, 20% of the cars will be at the train station, and 20% of the cars will be at the city centre.

A4 (a) Let $A = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$. If $\vec{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then $\vec{y}_0 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \approx \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}$;

$$\vec{x}_1 = A\vec{y}_0 \approx \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix} \approx \begin{bmatrix} 3.535 \\ -1.414 \end{bmatrix}$$

$$\vec{y}_1 = \frac{1}{\|\vec{x}_1\|}\vec{x}_1 \approx \begin{bmatrix} 0.929 \\ -0.371 \end{bmatrix}$$

$$\vec{x}_2 = A\vec{y}_1 \approx \begin{bmatrix} 4.645 \\ 0.742 \end{bmatrix}, \qquad \vec{y}_2 = \begin{bmatrix} 0.987 \\ 0.158 \end{bmatrix}$$

$$\vec{x}_3 = A\vec{y}_2 \approx \begin{bmatrix} 4.935 \\ -0.316 \end{bmatrix}, \qquad \vec{y}_3 = \begin{bmatrix} 0.998 \\ -0.064 \end{bmatrix}$$

$$\vec{x}_4 = A\vec{y}_3 \approx \begin{bmatrix} 4.990 \\ 0.128 \end{bmatrix}, \qquad \vec{y}_4 = \begin{bmatrix} 1.000 \\ 0.0256 \end{bmatrix}$$

$$\vec{x}_5 = A\vec{y}_4 \approx \begin{bmatrix} 5.000 \\ -0.0512 \end{bmatrix}, \qquad \vec{y}_5 = \begin{bmatrix} 1.000 \\ -0.0102 \end{bmatrix}$$

$$\vec{x}_6 = A\vec{y}_5 \approx \begin{bmatrix} 5.000 \\ 0.0204 \end{bmatrix}, \qquad \vec{y}_6 = \begin{bmatrix} 1.000 \\ 0.0041 \end{bmatrix}$$

It appears that $y_m \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of A. The corresponding dominant eigenvalue is $\lambda = 5$.

(b) Let $A = \begin{bmatrix} 27 & 84 \\ -7 & -22 \end{bmatrix}$. If $\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $\vec{y}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$;

$$\vec{x}_1 = A\vec{y}_0 \approx \begin{bmatrix} 27 & 84 \\ -7 & -22 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 27 \\ -7 \end{bmatrix}$$

$$\vec{y}_1 = \frac{1}{\|\vec{x}_1\|}\vec{x}_1 \approx \begin{bmatrix} 0.968 \\ -0.251 \end{bmatrix}$$

$$\vec{x}_2 = A\vec{y}_1 \approx \begin{bmatrix} 5.052 \\ -1.254 \end{bmatrix}, \qquad \vec{y}_2 = \begin{bmatrix} 0.971 \\ -0.241 \end{bmatrix}$$

$$\vec{x}_3 = A\vec{y}_2 \approx \begin{bmatrix} 5.973 \\ -1.495 \end{bmatrix}, \qquad \vec{y}_3 = \begin{bmatrix} 0.9701 \\ -0.2428 \end{bmatrix}$$

$$\vec{x}_4 = A\vec{y}_3 \approx \begin{bmatrix} 5.7975 \\ -1.4991 \end{bmatrix}, \qquad \vec{y}_4 = \begin{bmatrix} 0.9702 \\ -0.2425 \end{bmatrix}$$

It appears that $y_m \to \begin{bmatrix} 0.970 \\ -0.242 \end{bmatrix}$, so $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$ is an eigenvector of A. The corresponding dominant eigenvalue is $\lambda = 6$.

Homework Problems

B1 (a) A is not a Markov matrix.

(b) B is a Markov matrix. The invariant state is $\begin{bmatrix} 6/11 \\ 5/11 \end{bmatrix}$.

(c) C is a Markov matrix. The invariant state is $\begin{bmatrix} 7/13 \\ 2/13 \\ 4/13 \end{bmatrix}$.

(d) D is not a Markov matrix.

B2 (a) In the long run, 53% of the customers will deal with Johnson and 47% will deal with Thomson.

(b) $T^m \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 8 \\ 7 \end{bmatrix} + \frac{1}{15}(4.25)(-0.5)^m \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

B3 In the long run, 54% of the bicycles will be at the residence, 29% will be at the library, and 17% will be at the athletic centre.

B4 The dominant eigenvalue is $\lambda = 8$.

Computer Problems

C1 The dominant eigenvalue is $\lambda = 5$.

Conceptual Problems

D1 (a) The characteristic equation is

$$\det \begin{bmatrix} t_{11} - \lambda & t_{12} \\ t_{21} & t_{22} - \lambda \end{bmatrix} = \lambda^2 - (t_{11} + t_{22})\lambda + (t_{11}t_{22} - t_{12}t_{21})$$

However, $(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$. So, we must have $\lambda_1 + \lambda_2 = t_{11} + t_{22}$. Since $\lambda_1 = 1$, $\lambda_2 = t_{11} + t_{22} - 1$.

(b) The transition matrix can be written as $\begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix}$. We could easily verify that the given vector is an eigenvector corresponding to $\lambda = 1$, but it is easy to determine the eigenvector too. For $\lambda = 1$,

$$\begin{bmatrix} 1-a & b \\ a & 1-b \end{bmatrix} - I = \begin{bmatrix} -a & b \\ a & -b \end{bmatrix} \sim \begin{bmatrix} -a & b \\ 0 & 0 \end{bmatrix}$$

So, a corresponding eigenvector is any multiple of $\begin{bmatrix} b \\ a \end{bmatrix}$. Since we require the sum of the components to be 1, we take the eigenvector to be $\begin{bmatrix} b/(a+b) \\ a/(a+b) \end{bmatrix}$.

D2 (a) Since column sums of T are all 1, $\sum_{k=1}^{n} t_{kj} = 1$. Then

$$\sum_{k=1}^{n} (T\vec{x})_k = \sum_{k=1}^{n}\left(\sum_{j=1}^{n} t_{kj}x_j\right) = \sum_{j=1}^{n}\left(\sum_{k=1}^{n} t_{kj}\right)x_j = \sum_{j=1}^{n} x_j$$

as required.

(b) If $T\vec{v} = \lambda\vec{v}$, then $\sum_{k=1}^{n}(T\vec{v})_k = \lambda\sum_{k=1}^{n} v_k$. However, by part (a), $\sum_{k=1}^{n}(T\vec{v})_k = \sum_{k=1}^{n} v_k$. Thus,

$$\lambda\sum_{k=1}^{n} v_k = \sum_{k=1}^{n}(T\vec{v})_k = \sum_{k=1}^{n} v_k$$

Since $\lambda \neq 1$, we must have $\sum_{k=1}^{n} v_k = 0$.

6.4 Diagonalization and Differential Equations

Practice Problems

A1 (a) The solution will be of the form $\begin{bmatrix} y \\ z \end{bmatrix} = ce^{\lambda t}\begin{bmatrix} a \\ b \end{bmatrix}$, so substitute this into the original system and use the fact that $\frac{d}{dt}ce^{\lambda t}\begin{bmatrix} a \\ b \end{bmatrix} = \lambda ce^{\lambda t}\begin{bmatrix} a \\ b \end{bmatrix}$ to get

$$\lambda ce^{\lambda t}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & -4 \end{bmatrix} ce^{\lambda t}\begin{bmatrix} a \\ b \end{bmatrix}$$

Cancelling the common factor $ce^{\lambda t}$, we see that $\begin{bmatrix} a \\ b \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 3 & 2 \\ 4 & -4 \end{bmatrix}$ with eigenvalue λ. Thus, we need to find the eigenvalues of $A = \begin{bmatrix} 3 & 2 \\ 4 & -4 \end{bmatrix}$. We have

$$C(\lambda) = \begin{vmatrix} 3-\lambda & 2 \\ 4 & -4-\lambda \end{vmatrix} = \lambda^2 + \lambda - 20 = (\lambda+5)(\lambda-4)$$

Hence, the eigenvalues are $\lambda_1 = -5$ and $\lambda_2 = 4$. For $\lambda_1 = -5$

$$A - (-5)I = \begin{bmatrix} 8 & 2 \\ 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/4 \\ 0 & 0 \end{bmatrix}$$

Hence, an eigenvector corresponding to λ_1 is $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$. So, one solution to the system is $e^{-5t}\begin{bmatrix} -1 \\ 4 \end{bmatrix}$. For $\lambda_2 = 4$,

$$A - 4I = \begin{bmatrix} -1 & 2 \\ 4 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

Hence, an eigenvector corresponding to λ_2 is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. So, one solution to the system is $e^{4t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Since the system is a linear homogeneous system, the general solution will be an arbitrary linear combination of the two solutions. The general solution is therefore

$$\begin{bmatrix} y \\ z \end{bmatrix} = ae^{-5t}\begin{bmatrix} -1 \\ 4 \end{bmatrix} + be^{4t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad a, b \in \mathbb{R}$$

(b) The solution will be of the form $\begin{bmatrix} y \\ z \end{bmatrix} = ce^{\lambda t}\begin{bmatrix} a \\ b \end{bmatrix}$, so substitute this into the original system and use the fact that $\frac{d}{dt}ce^{\lambda t}\begin{bmatrix} a \\ b \end{bmatrix} = \lambda ce^{\lambda t}\begin{bmatrix} a \\ b \end{bmatrix}$ to get

$$\lambda ce^{\lambda t}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0.2 & 0.7 \\ 0.1 & -0.4 \end{bmatrix} ce^{\lambda t}\begin{bmatrix} a \\ b \end{bmatrix}$$

Cancelling the common factor $ce^{\lambda t}$, we see that $\begin{bmatrix} a \\ b \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 0.2 & 0.7 \\ 0.1 & -0.4 \end{bmatrix}$ with eigenvalue λ. Thus, we need to find the eigenvalues of $A = \begin{bmatrix} 0.2 & 0.7 \\ 0.1 & -0.4 \end{bmatrix}$. We have

$$C(\lambda) = \begin{vmatrix} 0.2-\lambda & 0.7 \\ 0.1 & -0.4-\lambda \end{vmatrix} = \lambda^2 + 0.2\lambda - 0.15 = (\lambda + 0.5)(\lambda - 0.3)$$

Hence, the eigenvalues are $\lambda_1 = -0.5$ and $\lambda_2 = 0.3$. For $\lambda_1 = -0.5$

$$A - (-0.5)I = \begin{bmatrix} 0.7 & 0.7 \\ 0.1 & 0.1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Hence, an eigenvector corresponding to λ_1 is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. So, one solution to the system is $e^{-0.5t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. For $\lambda_2 = 0.3$,

$$A - 0.3I = \begin{bmatrix} -0.1 & 0.7 \\ 0.1 & -0.7 \end{bmatrix} \sim \begin{bmatrix} 1 & -7 \\ 0 & 0 \end{bmatrix}$$

Hence, an eigenvector corresponding to λ_2 is $\begin{bmatrix} 7 \\ 1 \end{bmatrix}$. So, one solution to the system is $e^{0.3t}\begin{bmatrix} 7 \\ 1 \end{bmatrix}$.

Since the system is a linear homogeneous system, the general solution will be an arbitrary linear combination of the two solutions. The general solution is therefore

$$\begin{bmatrix} y \\ z \end{bmatrix} = ae^{-0.5t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + be^{0.3t}\begin{bmatrix} 7 \\ 1 \end{bmatrix}, \quad a, b \in \mathbb{R}$$

Homework Problems

B1 (a) $ae^{-0.8t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + be^{-0.4t}\begin{bmatrix} 3 \\ 1 \end{bmatrix}, a, b \in \mathbb{R}$.

(b) $ae^{-9t}\begin{bmatrix} 1 \\ -2 \end{bmatrix} + be^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, a, b \in \mathbb{R}$.

(c) $a\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + be^{-4t}\begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} + ce^{-2t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, a, b, c \in \mathbb{R}$.

Chapter 6 Quiz

Problems

E1 (a) We have

$$\begin{bmatrix} 5 & -16 & -4 \\ 2 & -7 & -2 \\ -2 & 8 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

So, $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ is not an eigenvector of A.

(b) We have

$$\begin{bmatrix} 5 & -16 & -4 \\ 2 & -7 & -2 \\ -2 & 8 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

So, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector with eigenvalue 1.

(c) We have

$$\begin{bmatrix} 5 & -16 & -4 \\ 2 & -7 & -2 \\ -2 & 8 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$$

So, $\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector with eigenvalue 1.

(d) We have

$$\begin{bmatrix} 5 & -16 & -4 \\ 2 & -7 & -2 \\ -2 & 8 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

So, $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ is an eigenvector with eigenvalue -1.

E2 We first need to find the eigenvalues of A by factoring the characteristic polynomial. We have

$$C(\lambda) = \begin{vmatrix} -3-\lambda & 1 & 0 \\ 13 & -7-\lambda & -8 \\ -11 & 5 & 4-\lambda \end{vmatrix} = -\lambda(\lambda+4)(\lambda+2)$$

Hence, the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = -4$, and $\lambda_3 = -2$. Since there are three distinct eigenvalues, the matrix is diagonalizable by Corollary 6.2.4.
For $\lambda_1 = 0$ we get

$$A - 0I = \begin{bmatrix} -3 & 1 & 0 \\ 13 & -7 & -8 \\ -11 & 5 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}1\\3\\-1\end{bmatrix}\right\}$.

For $\lambda_2 = -4$ we get

$$A-(-4)I = \begin{bmatrix}1 & 1 & 0\\13 & -3 & -8\\-11 & 5 & 8\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -1/2\\0 & 1 & 1/2\\0 & 0 & 0\end{bmatrix}$$

Thus, a basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}\right\}$.

For $\lambda_3 = -2$ we get

$$A-(-2)I = \begin{bmatrix}-1 & 1 & 0\\13 & -5 & -8\\-11 & 5 & 6\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -1\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix}$$

Thus, a basis for the eigenspace of λ_3 is $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$.

We take the columns of P to be a basis for $\mathbb{R}^3$ of eigenvectors of A. Hence, we take $P = \begin{bmatrix}1 & 1 & 1\\3 & -1 & 1\\-1 & 2 & 1\end{bmatrix}$. The corresponding matrix D is the diagonal matrix whose diagonal entries are the eigenvalues corresponding columnwise to the eigenvectors in P. Hence, $D = \begin{bmatrix}0 & 0 & 0\\0 & -4 & 0\\0 & 0 & -2\end{bmatrix}$.

E3 We have

$$C(\lambda) = \begin{vmatrix}4-\lambda & 2 & 2\\-1 & 1-\lambda & -1\\1 & 1 & 3-\lambda\end{vmatrix} = -(\lambda-2)^2(\lambda-4)$$

The eigenvalues are $\lambda_1 = 2$ with algebraic multiplicity 2 and $\lambda_2 = 4$ with algebraic multiplicity 1.
For $\lambda_1 = 2$ we get

$$A-2I = \begin{bmatrix}2 & 2 & 2\\-1 & -1 & -1\\1 & 1 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 1 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

Thus, a basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$ so the geometric multiplicity is 2.

For $\lambda_2 = 4$ we get

$$A-4I = \begin{bmatrix}0 & 2 & 2\\-1 & -3 & -1\\1 & 1 & -1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -2\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}$$

A basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix}2\\-1\\1\end{bmatrix}\right\}$ so the geometric multiplicity is 1.

Therefore, by Corollary 6.2.3, A is diagonalizable.

E4 Since A is invertible, 0 is not an eigenvalue of A (see Problem 6.2 D8). Then, if $A\vec{x} = \lambda\vec{x}$ we get $\vec{x} = \lambda A^{-1}\vec{x}$, so $A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}$.

E5 The condition $\det A = 0$ can be rewritten $\det(A - 0I) = 0$, it follows that 0 is an eigenvalue of A. Similarly, -2 and 3 are eigenvalues, so A has three distinct real eigenvalues. Hence, each eigenvalue has algebraic and geometric multiplicity 1.

(a) The solution space $A\vec{x} = \vec{0}$ is one-dimensional, since it is the space of eigenvectors corresponding to the eigenvalue 0.

(b) 2 cannot be an eigenvalue, because we already have three eigenvalues for the 3×3 matrix A. Hence, there are no vectors that satisfy $A\vec{x} = 2\vec{x}$, so the solution space is zero dimensional in this case.

(c) The range of the mapping is determined by A is the subspace spanned by the eigenvectors corresponding to the non-zero eigenvalues. Hence, the rank of A is two.

Alternately, since $A\vec{x} = \vec{0}$ is one-dimensional, we could apply the Rank-Nullity Theorem, to get that

$$\operatorname{rank}(A) = n - \operatorname{nullity}(A) = 3 - 1 = 2$$

E6 Since the columns in A sum to 1 it is a Markov matrix. If $\lambda = 1$, then

$$A - \lambda I = \begin{bmatrix} -0.1 & 0.1 & 0 \\ 0 & -0.2 & 0.1 \\ 0.1 & 0.1 & -0.1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -0.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, an eigenvector corresponding to λ_1 is $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. The components in the state vector must sum to 1, so the invariant state corresponding to λ_1 is

$$\frac{1}{1+1+2}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/2 \end{bmatrix}$$

which is invariant under the transformation with matrix A.

E7 The solution will be of the form $\begin{bmatrix} y \\ z \end{bmatrix} = ce^{\lambda t}\begin{bmatrix} a \\ b \end{bmatrix}$, so substitute this into the original system and use the fact that $\frac{d}{dt}ce^{\lambda t}\begin{bmatrix} a \\ b \end{bmatrix} = \lambda ce^{\lambda t}\begin{bmatrix} a \\ b \end{bmatrix}$ to get

$$\lambda ce^{\lambda t}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.2 \end{bmatrix} ce^{\lambda t}\begin{bmatrix} a \\ b \end{bmatrix}$$

Cancelling the common factor $ce^{\lambda t}$, we see that $\begin{bmatrix} a \\ b \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.2 \end{bmatrix}$ with eigenvalue λ. Thus, we need to find the eigenvalues of $A = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.2 \end{bmatrix}$. We have

$$C(\lambda) = \begin{vmatrix} 0.1 - \lambda & 0.2 \\ 0.3 & 0.2 - \lambda \end{vmatrix} = \lambda^2 - 0.3\lambda - 0.04 = (\lambda + 0.1)(\lambda - 0.4)$$

Hence, the eigenvalues are $\lambda_1 = -0.1$ and $\lambda_2 = 0.4$. For $\lambda_1 = -0.1$

$$A - (-0.1)I = \begin{bmatrix} 0.2 & 0.2 \\ 0.3 & 0.3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Hence, an eigenvector corresponding to λ_1 is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. So, one solution to the system is $e^{-0.1t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. For $\lambda_2 = 0.4$,

$$A - 0.4I = \begin{bmatrix} -0.3 & 0.2 \\ 0.3 & -0.2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2/3 \\ 0 & 0 \end{bmatrix}$$

Hence, an eigenvector corresponding to λ_2 is $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. So, one solution to the system is $e^{0.4t}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.
Since the system is a linear homogeneous system, the general solution will be an arbitrary linear combination of the two solutions. The general solution is therefore

$$\begin{bmatrix} y \\ z \end{bmatrix} = ae^{-0.1t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + be^{0.4t}\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad a, b \in \mathbb{R}$$

Chapter 6 Further Problems

Problems

F1 (a) Suppose that $\vec{v}$ is any eigenvector of A with eigenvalue λ. Since the algebraic multiplicity of λ is 1, the geometric multiplicity is also 1, and hence the eigenspace of λ is Span$\{\vec{v}\}$.

$$AB\vec{v} = BA\vec{v} = B(\lambda\vec{v}) = \lambda B\vec{v}$$

Thus, $B\vec{v}$ is an eigenvector of A with eigenvalue λ. Thus, $B\vec{v} \in$ Span$\{\vec{v}\}$, so $B\vec{v} = \mu\vec{v}$ for some μ. Therefore, $\vec{v}$ is an eigenvector of B.

(b) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Then $AB = BA$ and every non-zero vector in $\mathbb{R}^2$ is an eigenvector of A with eigenvalue 1, but $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, for example, is not an eigenvector of B.

F2 We have

$$B(AB - \lambda I) = BAB - \lambda B = (BA - \lambda I)B$$

so

$$\det\Big(B(AB - \lambda I)\Big) = \det\Big((BA - \lambda I)B\Big)$$

But, the determinant of a product is the product of the determinants, so

$$\det B \det(AB - \lambda I) = \det(BA - \lambda I) \det B$$

Since $\det B \neq 0$, we get

$$\det(AB - \lambda I) = \det(BA - \lambda I)$$

Thus AB and BA have the same characteristic polynomial, hence the same eigenvalues.

F3 Since the eigenvalues are distinct, the corresponding eigenvectors form a basis, so

$$\vec{x} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n$$

Note that

$$(A - \lambda_i I)(A - \lambda_j I) = A^2 - \lambda_i A - \lambda_j A + \lambda_i\lambda_j I = (A - \lambda_j I)(A - \lambda_i I)$$

so these factors commute. Note also that

$$(A - \lambda_i I)\vec{v}_i = \vec{0}$$

for every i. Hence,

$$(A - \lambda_1 I)(A - \lambda_2 I)\cdots(A - \lambda_n I)\vec{x} = \vec{0}$$

Since this is true for every $\vec{x} \in \mathbb{R}^n$,

$$(A - \lambda_1 I)(A - \lambda_2 I)\cdots(A - \lambda_n I) = O_{n,n}$$

The characteristic polynomial of A is

$$(-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) = (-1)^n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

Thus, if we define

$$C(X) = (-1)^n(X - \lambda_1 I)(X - \lambda_2 I)\cdots(X - \lambda_n I) = (-1)^n X^n + c_{n-1}X^{n-1} + \cdots + c_1 X + c_0$$

for any $n \times n$ matrix X, we get that

$$(-1)^n A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I = O_{n,n}$$

F4 By Problem F3, we have

$$\begin{aligned} O_{n,n} &= (-1)^n A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I \\ -c_0 I &= A((-1)^n A^{n-1} + c_{n-1}A^{n-2} + \cdots + c_1 I) \end{aligned}$$

Since A is invertible, $\det A \neq 0$ and thus, by Problem 6.2 D7, the product of the eigenvalues $c_0 \neq 0$. Hence,

$$A^{-1} = -\frac{1}{c_0}((-1)^n A^{n-1} + c_{n-1}A^{n-2} + \cdots + c_1 I)$$

CHAPTER 7 Orthonormal Bases

7.1 Orthonormal Bases and Orthogonal Matrices

Practice Problems

A1 (a) The set is orthogonal since $\begin{bmatrix}1\\2\end{bmatrix}\cdot\begin{bmatrix}2\\-1\end{bmatrix} = 1(2)+2(-1) = 0$. The norm of each vector is $\left\|\begin{bmatrix}1\\2\end{bmatrix}\right\| = \sqrt{1^2+2^2} = \sqrt{5}$ and $\left\|\begin{bmatrix}2\\-1\end{bmatrix}\right\| = \sqrt{2^2+(-1)^2} = \sqrt{5}$. The orthonormal set is then $\frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{5}}\\ \frac{2}{\sqrt{5}}\end{bmatrix}$, $\frac{1}{\sqrt{5}}\begin{bmatrix}2\\-1\end{bmatrix} = \begin{bmatrix}\frac{2}{\sqrt{5}}\\ \frac{-1}{\sqrt{5}}\end{bmatrix}$, and the orthogonal change of coordinate matrix is $P = \begin{bmatrix}1/\sqrt{5} & 2/\sqrt{5}\\ 2/\sqrt{5} & -1/\sqrt{5}\end{bmatrix}$.

(b) Take for instance the first and third vector, and calculate their dot product: $\begin{bmatrix}1\\1\\-1\end{bmatrix}\cdot\begin{bmatrix}2\\-1\\1\end{bmatrix} = 2(1)+1(-1)+(-1)(1) = 4 \neq 0$. These two of the vectors are not orthogonal, consequently the set is not orthogonal.

(c) Calculating the dot products of each pair of vectors, we find

$$\begin{bmatrix}1\\1\\3\end{bmatrix}\cdot\begin{bmatrix}3\\0\\-1\end{bmatrix} = 1(3)+1(0)+3(-1) = 0$$

$$\begin{bmatrix}1\\1\\3\end{bmatrix}\cdot\begin{bmatrix}1\\-10\\3\end{bmatrix} = 1(1)+1(-10)+3(3) = 0$$

$$\begin{bmatrix}3\\0\\-1\end{bmatrix}\cdot\begin{bmatrix}1\\-10\\3\end{bmatrix} = 3(1)+0(-10)+(-1)(3) = 0$$

Hence, the set is orthogonal. The norm of each vector is

$$\left\|\begin{bmatrix}1\\1\\3\end{bmatrix}\right\| = \sqrt{1^2+1^2+3^2} = \sqrt{11}$$
$$\left\|\begin{bmatrix}3\\0\\-1\end{bmatrix}\right\| = \sqrt{3^2+0^2+(-1)^2} = \sqrt{10}$$
$$\left\|\begin{bmatrix}1\\-10\\3\end{bmatrix}\right\| = \sqrt{1^2+(-10)^2+3^2} = \sqrt{110}$$

The orthonormal set is then $\begin{bmatrix}1/\sqrt{11}\\1/\sqrt{11}\\3/\sqrt{11}\end{bmatrix}, \begin{bmatrix}3/\sqrt{10}\\0\\-1/\sqrt{10}\end{bmatrix}, \begin{bmatrix}1/\sqrt{110}\\-10/\sqrt{110}\\3/\sqrt{110}\end{bmatrix}$, and the orthogonal change of coordinate matrix is $P = \begin{bmatrix}1/\sqrt{11} & 3/\sqrt{10} & 1/\sqrt{110}\\1/\sqrt{11} & 0 & -10/\sqrt{110}\\3/\sqrt{11} & -1/\sqrt{10} & 3/\sqrt{110}\end{bmatrix}$.

(d) Take for instance the first and fourth vector and calculate their dot product:

$$\begin{bmatrix}1\\0\\1\\1\end{bmatrix} \cdot \begin{bmatrix}-2\\1\\1\\0\end{bmatrix} = 1(-2)+0(1)+1(1)+1(0) = -1 \neq 0$$

These two of the vectors are not orthogonal, consequently the set is not orthogonal.

A2 Let us denote the vectors of the basis $\mathcal{B}$ by $\vec{v}_1 = \frac{1}{3}\begin{bmatrix}1\\-2\\2\end{bmatrix}, \vec{v}_2 = \frac{1}{3}\begin{bmatrix}2\\2\\1\end{bmatrix}, \vec{v}_3 = \frac{1}{3}\begin{bmatrix}-2\\1\\2\end{bmatrix}$.

(a) Using Theorem 7.1.2, the coordinates of $\vec{w}$ with respect to the basis $\mathcal{B}$ are found by calculating the dot product of $\vec{w}$ with each of the vectors in the basis $\mathcal{B}$:

$$\vec{w}\cdot\vec{v}_1 = \frac{1}{3}[1(4)+(-2)(3)+2(5)] = \frac{8}{3}$$
$$\vec{w}\cdot\vec{v}_2 = \frac{1}{3}(4(2)+3(2)+5(1)) = \frac{19}{3}$$
$$\vec{w}\cdot\vec{v}_3 = \frac{1}{3}(4(-2)+3(1)+5(2)) = \frac{5}{3}$$

So, the $\mathcal{B}$-coordinates of $\vec{w}$ are $[\vec{w}]_{\mathcal{B}} = \begin{bmatrix}8/3\\19/3\\5/3\end{bmatrix}$.

(b) Using Theorem 7.1.2, the coordinates of $\vec{x}$ with respect to the basis $\mathcal{B}$ are found by calculating the dot product of $\vec{x}$ with each of the vectors in the basis $\mathcal{B}$:

$$\begin{aligned}
\vec{x}\cdot\vec{v}_1 &= \frac{1}{3}(3(1)+(-7)(-2)+2(2)) = \frac{21}{3} = 7\\
\vec{x}\cdot\vec{v}_2 &= \frac{1}{3}(3(2)+(-7)(2)+2(1)) = \frac{-6}{3} = -2\\
\vec{x}\cdot\vec{v}_3 &= \frac{1}{3}(3(-2)+(-7)(1)+2(2)) = \frac{-9}{3} = -3
\end{aligned}$$

Hence, the $\mathcal{B}$-coordinates of $\vec{x}$ are $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 7\\ -2\\ -3\end{bmatrix}$.

(c) Using Theorem 7.1.2, the coordinates of $\vec{y}$ with respect to the basis $\mathcal{B}$ are found by calculating the dot product of $\vec{y}$ with each of the vectors in the basis $\mathcal{B}$:

$$\begin{aligned}
\vec{y}\cdot\vec{v}_1 &= \frac{1}{3}(2(1)+(-4)(-2)+6(2)) = \frac{22}{3}\\
\vec{y}\cdot\vec{v}_2 &= \frac{1}{3}(2(2)+(-4)(2)+6(1)) = \frac{2}{3}\\
\vec{y}\cdot\vec{v}_3 &= \frac{1}{3}(2(-2)+(-4)(1)+6(2)) = \frac{4}{3}
\end{aligned}$$

So, the $\mathcal{B}$-coordinates of $\vec{y}$ are $[\vec{y}]_{\mathcal{B}} = \begin{bmatrix} 22/3\\ 2/3\\ 4/3\end{bmatrix}$.

(d) Using Theorem 7.1.2, the coordinates of $\vec{z}$ with respect to the basis $\mathcal{B}$ are found by calculating the dot product of $\vec{z}$ with each of the vectors in the basis $\mathcal{B}$:

$$\begin{aligned}
\vec{z}\cdot\vec{v}_1 &= \frac{1}{3}(6(1)+6(-2)+3(2)) = 0\\
\vec{z}\cdot\vec{v}_2 &= \frac{1}{3}(6(2)+6(2)+3(1)) = 9\\
\vec{z}\cdot\vec{v}_3 &= \frac{1}{3}(6(-2)+6(1)+3(2)) = 0
\end{aligned}$$

Therefore, the $\mathcal{B}$-coordinates of $\vec{z}$ are $[\vec{z}]_{\mathcal{B}} = \begin{bmatrix} 0\\ 9\\ 0\end{bmatrix}$.

A3 Let us denote the vectors of the basis $\mathcal{B}$ by $\vec{v}_1 = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $\vec{v}_2 = \frac{1}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}$, $\vec{v}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}$, $\vec{v}_4 = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}$.

(a) Using Theorem 7.1.2, the coordinates of $\vec{x}$ in the basis $\mathcal{B}$ are found by calculating the dot product of $\vec{x}$ with each of the vectors in the basis $\mathcal{B}$. Hence, we get

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}\vec{x}\cdot\vec{v}_1\\ \vec{x}\cdot\vec{v}_2\\ \vec{x}\cdot\vec{v}_3\\ \vec{x}\cdot\vec{v}_4\end{bmatrix} = \begin{bmatrix}4\\ -5\\ -5/\sqrt{2}\\ -1/\sqrt{2}\end{bmatrix}$$

(b) Using Theorem 7.1.2, the coordinates of $\vec{y}$ in the basis $\mathcal{B}$ are found by calculating the dot product of $\vec{y}$ with each of the vectors in the basis $\mathcal{B}$. Hence, we get

$$[\vec{y}]_{\mathcal{B}} = \begin{bmatrix} \vec{y}\cdot\vec{v}_1 \\ \vec{y}\cdot\vec{v}_2 \\ \vec{y}\cdot\vec{v}_3 \\ \vec{y}\cdot\vec{v}_4 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 3/2 \\ 7/\sqrt{2} \\ 6/\sqrt{2} \end{bmatrix}$$

(c) Using Theorem 7.1.2, the coordinates of $\vec{w}$ in the basis $\mathcal{B}$ are found by calculating the dot product of $\vec{w}$ with each of the vectors in the basis $\mathcal{B}$. Hence, we get

$$[\vec{w}]_{\mathcal{B}} = \begin{bmatrix} \vec{w}\cdot\vec{v}_1 \\ \vec{w}\cdot\vec{v}_2 \\ \vec{w}\cdot\vec{v}_3 \\ \vec{w}\cdot\vec{v}_4 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 1/2 \\ -3/\sqrt{2} \\ 0 \end{bmatrix}$$

(d) Using Theorem 7.1.2, the coordinates of $\vec{z}$ in the basis $\mathcal{B}$ are found by calculating the dot product of $\vec{z}$ with each of the vectors in the basis $\mathcal{B}$. Hence, we get

$$[\vec{z}]_{\mathcal{B}} = \begin{bmatrix} \vec{z}\cdot\vec{v}_1 \\ \vec{z}\cdot\vec{v}_2 \\ \vec{z}\cdot\vec{v}_3 \\ \vec{z}\cdot\vec{v}_4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 3/\sqrt{2} \\ 6/\sqrt{2} \end{bmatrix}$$

A4 (a) $A^TA = \begin{bmatrix} 5/13 & 12/13 \\ 12/13 & -5/13 \end{bmatrix}\begin{bmatrix} 5/13 & 12/13 \\ 12/13 & -5/13 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. So, it is orthogonal.

(b) $A^TA = \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & -3/5 \end{bmatrix}\begin{bmatrix} 3/5 & 4/5 \\ -4/5 & -3/5 \end{bmatrix} = \begin{bmatrix} 1 & 24/25 \\ 24/25 & 1 \end{bmatrix}$. Therefore, it is not orthogonal since the columns of the matrix are not orthogonal.

(c) $A^TA = \begin{bmatrix} 2/5 & 1/5 \\ -1/5 & 2/5 \end{bmatrix}\begin{bmatrix} 2/5 & 1/5 \\ -1/5 & 2/5 \end{bmatrix} = \begin{bmatrix} 3/25 & 0 \\ 0 & 3/25 \end{bmatrix}$. It is not orthogonal as the columns are not unit vectors.

(d) $A^TA = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \end{bmatrix}\begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4/9 \\ 0 & 1 & -4/9 \\ 4/9 & -4/9 & 1 \end{bmatrix}$. Therefore, it is not orthogonal since the third column is not orthogonal to the first or second column.

(e) $A^TA = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix}\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Hence, it is orthogonal.

A5 (a) We have $\vec{g}_3 \cdot \vec{g}_1 = 0$. Consequently, the vectors are orthogonal.

We have $\vec{g}_2 = \vec{g}_3 \times \vec{g}_1 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \times \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 6 \end{bmatrix}$.

(b) The change of coordinate matrix for the change from the basis $\mathcal{B}$ to the standard basis $\mathcal{S}$ is given by $P = \begin{bmatrix} \vec{f}_1 & \vec{f}_2 & \vec{f}_3 \end{bmatrix}$. It remains to calculate the orthonormal vectors $\vec{f}_i$, $i = 1,2,3$ by dividing the components of the vectors $\vec{g}_i$, $i = 1,2,3$ by their respective norm. We have $\|\vec{g}_1\| = \sqrt{(-1)^2 + 2^2 + 2^2} = 3$, $\|\vec{g}_2\| = \sqrt{6^2 + (-3)^2 + 6^2} = 9$, and $\|\vec{g}_3\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3$. So, the vectors $\vec{f}_i$ are given by: $\vec{f}_1 = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$, $\vec{f}_2 = \frac{1}{9}\begin{bmatrix} 6 \\ -3 \\ 6 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, and $\vec{f}_3 = \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$. Using the vectors $\vec{f}_i$, $i = 1,2,3$ together with the definition of P, we get $P = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$.

(c) The $\mathcal{B}$-matrix of L is the same as the matrix which represents a rotation through angle $\pi/4$ about the standard x_3-axis. Hence, it is defined by $[L]_{\mathcal{B}} = \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) & 0 \\ \sin(\pi/4) & \cos(\pi/4) & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Since $\cos(\pi/4) = 1/\sqrt{2} = \sin(\pi/4)$, we get $[L]_{\mathcal{B}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$.

(d) Since we calculated $\vec{g}_2 = \vec{g}_1 \times \vec{g}_3$, the basis $\vec{f}_1, \vec{f}_2, \vec{f}_3$ is orthonormal and right-handed. The standard matrix can be obtained using P, the change of coordinate matrix for the change from the basis $\mathcal{B}$ to the standard basis $\mathcal{S}$: $[L]_{\mathcal{S}} = P[L]_{\mathcal{B}}P^{-1}$. Since $\{\vec{f}_1, \vec{f}_2, \vec{f}_3\}$ is orthonormal, the inverse of P is given by its transposed: $P^{-1} = P^T$. We find:

$$\begin{aligned}
[L]_{\mathcal{S}} &= \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \\
&= \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \frac{1}{3\sqrt{2}}\begin{bmatrix} -3 & 3 & 0 \\ 1 & 1 & 4 \\ 2\sqrt{2} & 2\sqrt{2} & -\sqrt{2} \end{bmatrix} \\
&= \frac{1}{9\sqrt{2}}\begin{bmatrix} 5+4\sqrt{2} & -1+4\sqrt{2} & 8-2\sqrt{2} \\ -7+4\sqrt{2} & 5+4\sqrt{2} & -4-2\sqrt{2} \\ -4-2\sqrt{2} & 8-2\sqrt{2} & 8+\sqrt{2} \end{bmatrix}
\end{aligned}$$

A6 Since $\mathcal{B}$ is orthonormal, the matrix $P = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix}$ is orthogonal. Hence, the rows of P are orthonormal. Thus, we can pick another orthonormal basis to be $\left\{ \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{3} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{3} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{3} \\ -1/\sqrt{2} \end{bmatrix} \right\}$.

Homework Problems

B1 (a) The set is orthogonal. $P = \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ 3/\sqrt{10} & -1/\sqrt{10} \end{bmatrix}$.

(b) The set is orthogonal. $P = \begin{bmatrix} 2/\sqrt{6} & 1/\sqrt{5} & 2/\sqrt{30} \\ -1/\sqrt{6} & 0 & 5/\sqrt{30} \\ 1/\sqrt{6} & -2/\sqrt{5} & 1/\sqrt{30} \end{bmatrix}$.

(c) The set is not orthogonal.

(d) The set is not orthogonal.

B2 (a) $[\vec{w}]_{\mathcal{B}} = \begin{bmatrix} 7/\sqrt{2} \\ -\sqrt{3} \\ 9/\sqrt{6} \end{bmatrix}$ (b) $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -1/\sqrt{2} \\ 3\sqrt{3} \\ -3/\sqrt{6} \end{bmatrix}$

(c) $[\vec{y}]_{\mathcal{B}} = \begin{bmatrix} -\sqrt{2} \\ -5/\sqrt{3} \\ 14/\sqrt{6} \end{bmatrix}$ (d) $[\vec{z}]_{\mathcal{B}} = \begin{bmatrix} 1/\sqrt{2} \\ 7/\sqrt{3} \\ 5/\sqrt{6} \end{bmatrix}$

B3 (a) $[\vec{w}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 5 \\ 3/\sqrt{2} \\ -3/\sqrt{2} \end{bmatrix}$ (b) $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ -\sqrt{2} \\ -8/\sqrt{2} \end{bmatrix}$

(c) $[\vec{y}]_{\mathcal{B}} = \begin{bmatrix} 5/2 \\ 1/2 \\ -7/\sqrt{2} \\ -\sqrt{2} \end{bmatrix}$ (d) $[\vec{z}]_{\mathcal{B}} = \begin{bmatrix} 7/2 \\ -3/2 \\ -6/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

B4 (a) It is not orthogonal. The columns of the matrix are not orthogonal.

(b) It is orthogonal.

(c) It is not orthogonal. The columns are not unit vectors.

(d) It is orthogonal.

(e) It is not orthogonal. The columns of the matrix are not orthogonal.

B5 (a) $\vec{w}_3 = \vec{w}_1 \times \vec{w}_2 = \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}$

(b) $\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} -3/\sqrt{6} \\ -8/\sqrt{5} \\ -21/\sqrt{30} \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 4/\sqrt{6} \\ 9/\sqrt{5} \\ -2/\sqrt{30} \end{bmatrix}$

Conceptual Problems

D1 Let P and Q be orthogonal. Then $P^T P = I$ and $Q^T Q = I$. Hence,

$$(PQ)^T(PQ) = Q^T P^T PQ = Q^T IQ = Q^T Q = I$$

Thus, PQ is orthogonal.

D2 (a) We have $1 = \det I = \det(P^T P) = \det P^T \det P = (\det P)^2$. Hence, $\det P = \pm 1$.

(b) Let $A = \begin{bmatrix} 2 & 0 \\ 2 & 1/2 \end{bmatrix}$. Then $\det A = 1$, but A is certainly not orthogonal.

D3 (a) Suppose that P is orthogonal, so that $P^TP = I$. Then for every $\vec{x} \in \mathbb{R}^n$ we have

$$\|P\vec{x}\|^2 = (P\vec{x}) \cdot (P\vec{x}) = \vec{x}^T P^T P\vec{x} = \vec{x}^T\vec{x} = \|\vec{x}\|^2$$

So $\|P\vec{x}\|^2 = \|\vec{x}\|^2$. (The converse is also true; see Problem F1.)

(b) Suppose that λ is a real eigenvalue of the orthogonal matrix P with eigenvector $\vec{v}$. Then $P\vec{v} = \lambda\vec{v}$. Thus, by part (a), $\|\vec{v}\|\|P\vec{v}\| = |\lambda|\|\vec{v}\|$. So $|\lambda| = 1$, and $\lambda = \pm 1$.

D4 Let $\vec{v}_1^T, \dots, \vec{v}_n^T$ denote the rows of P. Then, the columns of P^T are $\vec{v}_1, \dots, \vec{v}_n$. By the usual rule for matrix multiplication,

$$(PP^T)_{ij} = (\vec{v}_i^T)\vec{v}_j = \vec{v}_i \cdot \vec{v}_j$$

Hence, $PP^T = I$ if and only if $\vec{v}_i \cdot \vec{v}_i = 1$ and $\vec{v}_i \cdot \vec{v}_j = 0$ for all $i \neq j$. But this is true if and only if the rows of P form an orthonormal set.

7.2 Projections and the Gram-Schmidt Procedure

Practice Problems

A1 (a) We definition of projection requires an orthonormal basis for $\mathbb{S} = \operatorname{Span}\mathcal{A}$. Since the two spanning vectors are orthogonal, it is only necessary to normalize them. An orthonormal basis for $\mathbb{S}$ is therefore

$$\mathcal{B} = \{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{10} \\ 2/\sqrt{10} \\ 1/\sqrt{10} \\ 2/\sqrt{10} \end{bmatrix} \right\}$$

It follows that the projection of $\vec{x}$ onto $\mathbb{S}$ is

$$\begin{aligned} \operatorname{proj}_{\mathbb{S}} \vec{x} &= (\vec{x} \cdot \vec{v}_1)\vec{v}_1 + (\vec{x} \cdot \vec{v}_2)\vec{v}_2 \\ &= 0 \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} + \frac{25}{\sqrt{10}} \begin{bmatrix} 1/\sqrt{10} \\ 2/\sqrt{10} \\ 1/\sqrt{10} \\ 2/\sqrt{10} \end{bmatrix} = \begin{bmatrix} 5/2 \\ 5 \\ 5/2 \\ 5 \end{bmatrix} \end{aligned}$$

(b) We definition of projection requires an orthonormal basis for $\mathbb{S} = \operatorname{Span}\mathcal{A}$. Since the two spanning vectors are orthogonal, it is only necessary to normalize them. An orthonormal basis for $\mathbb{S}$ is therefore

$$\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \\ 0 \end{bmatrix} \right\}$$

It follows that the projection of $\vec{x}$ onto $\mathbb{S}$ is

$$\begin{aligned} \operatorname{proj}_{\mathbb{S}} \vec{x} &= (\vec{x} \cdot \vec{v}_1)\vec{v}_1 + (\vec{x} \cdot \vec{v}_2)\vec{v}_2 + (\vec{x} \cdot \vec{v}_3)\vec{v}_3 \\ &= \frac{7}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix} + \frac{9}{\sqrt{2}} \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} + \frac{-3}{\sqrt{2}} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 9/2 \\ 5 \\ 9/2 \end{bmatrix} \end{aligned}$$

(c) We definition of projection requires an orthonormal basis for $\mathbb{S} = \operatorname{Span}\mathcal{A}$. Since the two spanning vectors are orthogonal, it is only necessary to normalize them. An orthonormal basis for $\mathbb{S}$ is therefore

$$\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \right\}$$

It follows that the projection of $\vec{x}$ onto $\mathbb{S}$ is

$$\begin{aligned} \operatorname{proj}_{\mathbb{S}} \vec{x} &= (\vec{x} \cdot \vec{v}_1)\vec{v}_1 + (\vec{x} \cdot \vec{v}_2)\vec{v}_2 + (\vec{x} \cdot \vec{v}_3)\vec{v}_3 \\ &= 8 \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + 3 \begin{bmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + 1 \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 6 \end{bmatrix} \end{aligned}$$

A2 (a) We need to find all vectors $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$ such that

$$0 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = v_1 + 2v_2 - v_3$$

We find the general solution of this homogeneous system is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

Hence, a basis for $\mathbb{S}^\perp$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(b) We need to find all vectors $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$ such that

$$0 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = v_1 + v_2 + v_3$$

$$0 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = -v_1 + v_2 + 3v_3$$

Row reducing the coefficient matrix of the corresponding homogeneous system gives

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

Thus, the general solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = t\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

Hence, a basis for $\mathbb{S}^\perp$ is $\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right\}$.

(c) We need to find all vectors $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \in \mathbb{R}^4$ such that

$$0 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = v_1 + v_3$$

$$0 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \\ 3 \end{bmatrix} = 2v_1 - v_2 + v_3 + 3v_4$$

Row reducing the coefficient matrix of the corresponding homogeneous system gives

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -3 \end{bmatrix}$$

Thus, the general solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = s\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

Hence, a basis for $\mathbb{S}^\perp$ is $\left\{\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}\right\}$.

A3 (a) Denote the given basis by $\mathcal{B} = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\} = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$.

First Step: Let $\vec{v}_1 = \vec{w}_1$ and $\mathbb{S}_1 = \operatorname{Span}\{\vec{v}_1\}$.

Second Step: Let

$$\vec{v}_2 = \operatorname{perp}_{\mathbb{S}_1} \vec{w}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and define $\mathbb{S}_2 = \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$.

Third Step: Let

$$\vec{v}_3 = \text{perp}_{\mathbb{S}_2}\, \vec{w}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{1}{1}\begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{1}{1}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Hence, an orthogonal basis for the subspace spanned $\mathcal{B}$ is $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$.

(b) Denote the given basis by $\mathcal{B} = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\} = \left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$.

First Step: Let $\vec{v}_1 = \vec{w}_1$ and $\mathbb{S}_1 = \text{Span}\{\vec{v}_1\}$.

Second Step: We have

$$\text{perp}_{\mathbb{S}_1}\, \vec{w}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1/3\\1/3\\-2/3\end{bmatrix}$$

Since we can take any scalar multiple of this vector, to make future calculations easier, we pick $\vec{v}_2 = \begin{bmatrix}1\\1\\-2\end{bmatrix}$ and define $\mathbb{S}_2 = \text{Span}\{\vec{v}_1, \vec{v}_2\}$.

Third Step: We have

$$\text{perp}_{\mathbb{S}_2}\, \vec{w}_3 = \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{1}{6}\begin{bmatrix}1\\1\\-2\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\\0\end{bmatrix}$$

So, we pick $\vec{v}_3 = \begin{bmatrix}1\\-1\\0\end{bmatrix}$.

Hence, an orthogonal basis for the subspace spanned $\mathcal{B}$ is $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-2\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}\right\}$.

(c) Denote the given basis by $\mathcal{B} = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\} = \left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}2\\-1\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\1\\1\\1\end{bmatrix}\right\}$.

First Step: Let $\vec{v}_1 = \vec{w}_1$ and $\mathbb{S}_1 = \text{Span}\{\vec{v}_1\}$.

Second Step: Let

$$\vec{v}_2 = \text{perp}_{\mathbb{S}_1}\, \vec{w}_2 = \begin{bmatrix}2\\-1\\1\\2\end{bmatrix} - \frac{4}{2}\begin{bmatrix}1\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\-1\\1\\0\end{bmatrix}$$

and define $\mathbb{S}_2 = \text{Span}\{\vec{v}_1, \vec{v}_2\}$.

Third Step: Observe that $\vec{v}_1 \cdot \vec{w}_3 = 0 = \vec{v}_2 \cdot \vec{w}_3$, so $\vec{w}_3$ is already orthogonal to $\vec{v}_1$ and $\vec{v}_2$. Hence, we can simply take $\vec{v}_3 = \vec{w}_3$.

Hence, an orthogonal basis for the subspace spanned $\mathcal{B}$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(d) Denote the given set by $\mathcal{B} = \{\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

First Step: Let $\vec{v}_1 = \vec{w}_1$ and $\mathbb{S}_1 = \operatorname{Span}\{\vec{v}_1\}$.

Second Step: We have

$$\operatorname{perp}_{\mathbb{S}_1} \vec{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 1 \\ 1/3 \end{bmatrix}$$

Since we can take any scalar multiple of this vector, to make future calculations easier, we pick $\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 1 \end{bmatrix}$ and define $\mathbb{S}_2 = \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$.

Third Step: We have

$$\operatorname{perp}_{\mathbb{S}_2} \vec{w}_3 = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix} - \frac{5}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{-10}{15} \begin{bmatrix} 1 \\ -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence, $\vec{w}_3$ is actually a linear combination of $\vec{v}_1$ and $\vec{v}_2$, so we can omit it.

Third Step: We have

$$\operatorname{perp}_{\mathbb{S}_2} \vec{w}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{15} \begin{bmatrix} 1 \\ -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/5 \\ -2/5 \\ -2/5 \\ 1/5 \end{bmatrix}$$

Hence, we take $\vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ -2 \\ 1 \end{bmatrix}$.

Thus, an orthogonal basis for the subspace spanned $\mathcal{B}$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -2 \\ 1 \end{bmatrix} \right\}$.

A4 We will first use the Gram-Schmidt Procedure to produce an orthogonal basis for the subspace spanned by each set and then we will normalize the vectors to get an orthonormal basis.

(a) Denote the given basis by $\mathcal{B} = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

First Step: Let $\vec{v}_1 = \vec{w}_1$ and $\mathbb{S}_1 = \operatorname{Span}\{\vec{v}_1\}$.

Second Step: Let

$$\vec{v}_2 = \operatorname{perp}_{\mathbb{S}_1} \vec{w}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and define $\mathbb{S}_2 = \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$.

Third Step: We have

$$\operatorname{perp}_{\mathbb{S}_2} \vec{w}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{0}{1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \end{bmatrix}$$

So, we take $\vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

Hence, an orthogonal basis for the subspace spanned $\mathcal{B}$ is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$.

Normalizing each vector we get that an orthonormal basis for the subspace spanned by $\mathcal{B}$ is

$$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} \right\}$$

(b) Denote the given basis by $\mathcal{B} = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

First Step: Let $\vec{v}_1 = \vec{w}_1$ and $\mathbb{S}_1 = \operatorname{Span}\{\vec{v}_1\}$.

Second Step: We have

$$\operatorname{perp}_{\mathbb{S}_1} \vec{w}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{9} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4/3 \\ 2/3 \\ 5/3 \end{bmatrix}$$

Thus, we take $\vec{v}_2 = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$ and define $\mathbb{S}_2 = \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$.

Third Step: We have

$$\operatorname{perp}_{\mathbb{S}_2} \vec{w}_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} - \frac{-5}{9} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} - \frac{-1}{45} \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1/5 \\ -2/5 \\ 0 \end{bmatrix}$$

So, we take $\vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$.

Hence, an orthogonal basis for the subspace spanned $\mathcal{B}$ is $\left\{\begin{bmatrix}2\\1\\-2\end{bmatrix}, \begin{bmatrix}4\\2\\5\end{bmatrix}, \begin{bmatrix}1\\-2\\0\end{bmatrix}\right\}$.

Normalizing each vector we get that an orthonormal basis for the subspace spanned by $\mathcal{B}$ is

$$\left\{\begin{bmatrix}2/3\\1/3\\-2/3\end{bmatrix}, \begin{bmatrix}4/\sqrt{45}\\2/\sqrt{45}\\5/\sqrt{45}\end{bmatrix}, \begin{bmatrix}1/\sqrt{5}\\-2/\sqrt{5}\\0\end{bmatrix}\right\}$$

(c) Denote the given basis by $\mathcal{B} = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\} = \left\{\begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix}\right\}$.

First Step: Let $\vec{v}_1 = \vec{w}_1$ and $\mathbb{S}_1 = \operatorname{Span}\{\vec{v}_1\}$.

Second Step: We have

$$\operatorname{perp}_{\mathbb{S}_1} \vec{w}_2 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1\\1\\0\\1\end{bmatrix} = \begin{bmatrix}-2/3\\1/3\\1\\1/3\end{bmatrix}$$

Thus, we take $\vec{v}_2 = \begin{bmatrix}-2\\1\\3\\1\end{bmatrix}$ and define $\mathbb{S}_2 = \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$.

Third Step: We have

$$\operatorname{perp}_{\mathbb{S}_2} \vec{w}_3 = \begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix} - \frac{-1}{3}\begin{bmatrix}1\\1\\0\\1\end{bmatrix} - \frac{-7}{15}\begin{bmatrix}-2\\1\\3\\1\end{bmatrix} = \begin{bmatrix}2/5\\-1/5\\2/5\\-1/5\end{bmatrix}$$

So, we take $\vec{v}_3 = \begin{bmatrix}2\\-1\\2\\-1\end{bmatrix}$.

Hence, an orthogonal basis for the subspace spanned $\mathcal{B}$ is $\left\{\begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}-2\\1\\3\\1\end{bmatrix}, \begin{bmatrix}2\\-1\\2\\-1\end{bmatrix}\right\}$.

Normalizing each vector we get that an orthonormal basis for the subspace spanned by $\mathcal{B}$ is

$$\left\{\begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\0\\1/\sqrt{3}\end{bmatrix}, \begin{bmatrix}-2/\sqrt{15}\\1/\sqrt{15}\\3/\sqrt{15}\\1/\sqrt{15}\end{bmatrix}, \begin{bmatrix}2/\sqrt{10}\\-1/\sqrt{10}\\2/\sqrt{10}\\-1/\sqrt{10}\end{bmatrix}\right\}$$

(d) Denote the given basis by $\mathcal{B} = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

First Step: Let $\vec{v}_1 = \vec{w}_1$ and $\mathbb{S}_1 = \operatorname{Span}\{\vec{v}_1\}$.

Second Step: We have

$$\operatorname{perp}_{\mathbb{S}_1} \vec{w}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} - \frac{0}{3} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

So, we take $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and define $\mathbb{S}_2 = \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$.

Third Step: We have

$$\operatorname{perp}_{\mathbb{S}_2} \vec{w}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 1 \\ 1/3 \\ 2/3 \\ 0 \end{bmatrix}$$

So, we take $\vec{v}_3 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 2 \\ 0 \end{bmatrix}$.

Hence, an orthogonal basis for the subspace spanned $\mathcal{B}$ is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\}$.

Normalizing each vector we get that an orthonormal basis for the subspace spanned by $\mathcal{B}$ is

$$\left\{ \begin{bmatrix} 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} \\ 0 \\ -1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{15} \\ 3/\sqrt{15} \\ 1/\sqrt{15} \\ 2/\sqrt{15} \\ 0 \end{bmatrix} \right\}$$

A5 Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be an orthonormal basis for $\mathbb{S}$, and let $\{\vec{v}_{k+1}, \dots, \vec{v}_n\}$ be an orthonormal basis for $\mathbb{S}^\perp$. Then, $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthonormal basis for $\mathbb{R}^n$. Thus, any $\vec{x} \in \mathbb{R}^n$ can be written

$$\vec{x} = (\vec{x} \cdot \vec{v}_1)\vec{v}_1 + \cdots + (\vec{x} \cdot \vec{v}_n)\vec{v}_n$$

Then

$$\begin{aligned}\operatorname{perp}_{\mathbb{S}}(\vec{x}) &= \vec{x} - \operatorname{proj}_{\mathbb{S}} \vec{x}\\ &= \vec{x} - [(\vec{x}\cdot\vec{v}_1)\vec{v}_1 + \cdots + (\vec{x}\cdot\vec{v}_k)\vec{v}_k]\\ &= (\vec{x}\cdot\vec{v}_{k+1})\vec{v}_{k+1} + \cdots + (\vec{x}\cdot\vec{v}_n)\vec{v}_n\\ &= \operatorname{proj}_{\mathbb{S}^\perp} \vec{x}\end{aligned}$$

Homework Problems

B1 (a) $\begin{bmatrix}74/21\\-1/7\\-74/21\\83/21\end{bmatrix}$ (b) $\begin{bmatrix}29/6\\19/6\\1/2\\19/6\end{bmatrix}$ (c) $\begin{bmatrix}3\\4\\-1\\5\end{bmatrix}$ (d) $\begin{bmatrix}16/3\\5/3\\-2\\11/3\end{bmatrix}$

B2 (a) $\left\{\begin{bmatrix}0\\1\\-2\end{bmatrix},\begin{bmatrix}1\\0\\-3\end{bmatrix}\right\}$ (b) $\left\{\begin{bmatrix}6\\1\\2\end{bmatrix}\right\}$

(c) The empty set. (d) $\left\{\begin{bmatrix}1\\3\\5\\-12\end{bmatrix}\right\}$

B3 (a) $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}1\\-1\\-1\end{bmatrix},\begin{bmatrix}1\\2\\-1\end{bmatrix}\right\}$ (b) $\left\{\begin{bmatrix}0\\1\\1\end{bmatrix},\begin{bmatrix}2\\-1\\1\end{bmatrix},\begin{bmatrix}1\\1\\-1\end{bmatrix}\right\}$

(c) $\left\{\begin{bmatrix}1\\0\\-1\\1\end{bmatrix},\begin{bmatrix}2\\3\\4\\2\end{bmatrix},\begin{bmatrix}8\\-10\\5\\-3\end{bmatrix}\right\}$ (d) $\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix},\begin{bmatrix}1\\-2\\0\\-1\end{bmatrix},\begin{bmatrix}2\\2\\-3\\-2\end{bmatrix}\right\}$

B4 (a) $\left\{\begin{bmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix},\begin{bmatrix}1/\sqrt{3}\\-1/\sqrt{3}\\-1/\sqrt{3}\end{bmatrix},\begin{bmatrix}1/\sqrt{6}\\2/\sqrt{6}\\-1/\sqrt{6}\end{bmatrix}\right\}$ (b) $\left\{\begin{bmatrix}1/\sqrt{11}\\1/\sqrt{11}\\3/\sqrt{11}\end{bmatrix},\begin{bmatrix}3/\sqrt{22}\\3/\sqrt{22}\\-2/\sqrt{22}\end{bmatrix},\begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\\0\end{bmatrix}\right\}$

(c) $\left\{\begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\0\\1/\sqrt{3}\end{bmatrix},\begin{bmatrix}0\\-1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix},\begin{bmatrix}5/\sqrt{39}\\-2/\sqrt{39}\\1/\sqrt{39}\\-3/\sqrt{39}\end{bmatrix}\right\}$ (d) $\left\{\begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\0\\1/\sqrt{3}\\0\end{bmatrix},\begin{bmatrix}-1/2\\0\\1/2\\1/2\\1/2\end{bmatrix},\begin{bmatrix}1/\sqrt{24}\\-2/\sqrt{24}\\-3/\sqrt{24}\\1/\sqrt{24}\\3/\sqrt{24}\end{bmatrix}\right\}$

B5 (a) $\mathcal{B} = \left\{\begin{bmatrix}1/\sqrt{6}\\2/\sqrt{6}\\1/\sqrt{6}\\0\end{bmatrix},\begin{bmatrix}1/\sqrt{12}\\-1/\sqrt{12}\\1/\sqrt{12}\\3/\sqrt{12}\end{bmatrix}\right\}$ (b) $\operatorname{perp}_{\mathbb{S}} \vec{x} = \begin{bmatrix}1/2\\1/2\\-3/2\\1/2\end{bmatrix}$

(c) $C = \left\{\begin{bmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\\0\end{bmatrix},\begin{bmatrix}-1/2\\1/2\\-1/2\\1/2\end{bmatrix}\right\}$ (d) $\operatorname{proj}_{\mathbb{S}^\perp} \vec{x} = \begin{bmatrix}1/2\\1/2\\-3/2\\1/2\end{bmatrix}$

Conceptual Problems

D1 If $\vec{x} \in \mathbb{S} \cap \mathbb{S}^\perp$, then $\vec{x} \in \mathbb{S}$ and $\vec{x} \in \mathbb{S}^\perp$. Since every vector is $\mathbb{S}^\perp$ is orthogonal to every vector in $\mathbb{S}$, $\vec{x} \cdot \vec{x} = 0$. Hence, $\vec{x} = \vec{0}$.

D2 Let $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ be a basis for $\mathbb{S}$. $\mathbb{S}^\perp$ is the solution space of the system $\vec{v}_1 \cdot \vec{x} = 0, \vec{v}_2 \cdot \vec{x} = 0, \dots, \vec{v}_k \cdot \vec{x} = 0$. Since $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is linearly independent, the matrix $\begin{bmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_k^T \end{bmatrix}$ has rank k. Therefore, by the Rank Theorem, the solution space $\mathbb{S}^\perp$ has dimension $n - k$.

D3 If $\vec{s} \in \mathbb{S}$, then $\vec{s} \cdot \vec{x} = 0$ for all $\vec{x} \in \mathbb{S}^\perp$ by definition of $\mathbb{S}^\perp$. But, the definition of $(\mathbb{S}^\perp)^\perp$ is $(\mathbb{S}^\perp)^\perp = \{\vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{s} = 0 \text{ for all } \vec{s} \in \mathbb{S}^\perp\}$. Hence $\vec{s} \in (\mathbb{S}^\perp)^\perp$. So $\mathbb{S} \subseteq (\mathbb{S}^\perp)^\perp$ Also, by Problem D2, $\dim(\mathbb{S}^\perp)^\perp = n - \dim \mathbb{S}^\perp = n - (n - \dim \mathbb{S}) = \dim \mathbb{S}$. Hence $\mathbb{S} = (\mathbb{S}^\perp)^\perp$.

D4 Since every vector in $\mathbb{S}^\perp$ is orthogonal to every vector in $\mathbb{S}$, $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_n\}$ is an orthonormal set in $\mathbb{R}^n$, and hence linearly independent. A linearly independent set of n vectors in $\mathbb{R}^n$ is necessarily a basis for $\mathbb{R}^n$.

D5 If $\vec{x} \in \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}, \vec{v}_k + t_1\vec{v}_1 + \cdots + t_{k-1}\vec{v}_{k-1}\}$, then

$$\begin{aligned} \vec{x} &= c_1\vec{v}_1 + \cdots + c_{k-1}\vec{v}_{k-1} + c_k(\vec{v}_k + t_1\vec{v}_1 + \cdots + t_{k-1}\vec{v}_{k-1}) \\ &= (c_1 + c_k t_1)\vec{v}_1 + \cdots + (c_{k-1} + c_k t_{k-1})\vec{v}_{k-1} + c_k\vec{v}_k \in \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\} \end{aligned}$$

On the other hand, if $\vec{x} \in \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_k\}$, then

$$\begin{aligned} \vec{x} &= c_1\vec{v}_1 + \cdots + c_{k-1}\vec{v}_{k-1} + c_k\vec{v}_k \\ &= c_1\vec{v}_1 + \cdots + c_{k-1}\vec{v}_{k-1} + c_k(-t_1\vec{v}_1 - \cdots - t_{k-1}\vec{v}_{k-1} + \vec{v}_k + t_1\vec{v}_1 + \cdots + t_{k-1}\vec{v}_{k-1}) \\ &= (c_1 - c_k t_1)\vec{v}_1 + \cdots + (c_{k-1} - c_k t_{k-1})\vec{v}_{k-1} + c_k(\vec{v}_k + t_1\vec{v}_1 + \cdots + t_{k-1}\vec{v}_{k-1}) \end{aligned}$$

Hence, $\vec{x} \in \operatorname{Span}\{\vec{v}_1, \dots, \vec{v}_{k-1}, \vec{v}_k + t_1\vec{v}_1 + \cdots + t_{k-1}\vec{v}_{k-1}\}$.

D6 Since $\mathcal{B}$ is a basis of $\mathbb{S}$, for any $\vec{s} \in \mathbb{S}$ we can write we can write

$$\vec{x} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$$

Then for any $1 \le i \le k$ we get

$$\vec{x} \cdot \vec{v}_i = (c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) \cdot \vec{v}_i = c_i\vec{v}_i \cdot \vec{v}_i = c_i\|\vec{v}_i\|^2$$

Thus, since $\|\vec{v}_i\| \neq 0$ we get that

$$c_i = \frac{\vec{x} \cdot \vec{v}_i}{\|\vec{v}_i\|^2}$$

D7 Using the fact that $\vec{x}^T\vec{y} = \vec{x} \cdot \vec{y}$, we get that

$$\begin{aligned} (\vec{v}_1\vec{v}_1^T + \cdots + \vec{v}_k\vec{v}_k^T)\vec{x} &= \vec{v}_1\vec{v}_1^T\vec{x} + \cdots + \vec{v}_k\vec{v}_k^T\vec{x} \\ &= \vec{v}_1(\vec{v}_1 \cdot \vec{x}) + \cdots + \vec{v}_k(\vec{v}_k \cdot \vec{x}) \\ &= (\vec{x} \cdot \vec{v}_1)\vec{v}_1 + \cdots + (\vec{x} \cdot \vec{v}_k)\vec{v}_k \\ &= \operatorname{proj}_{\mathbb{S}} \vec{x} \\ &= [\operatorname{proj}_{\mathbb{S}}]\vec{x} \end{aligned}$$

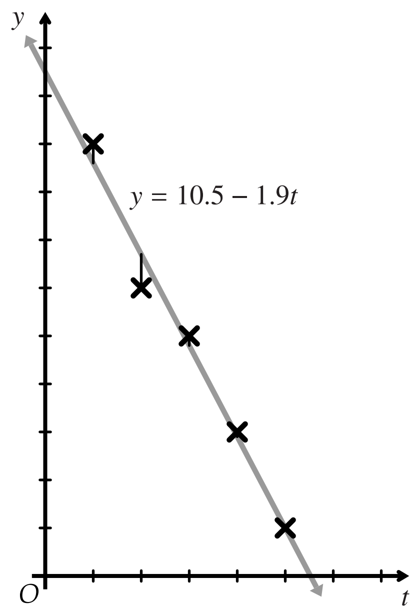
for all $\vec{x} \in \mathbb{R}^n$. Thus, $[\operatorname{proj}_{\mathbb{S}}] = \vec{v}_1\vec{v}_1^T + \cdots + \vec{v}_k\vec{v}_k^T$.

7.3 Method of Least Squares

Practice Problems

A1 (a) Since we want an equation of the form $y = a + bt$ we let $X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix}$.

We also let $\vec{y} = \begin{bmatrix} 9 \\ 6 \\ 5 \\ 3 \\ 1 \end{bmatrix}$. Then, the required a and b satisfy



$$X^T X \begin{bmatrix} a \\ b \end{bmatrix} = X^T \vec{y}$$

Since $X^T X = \begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix}$ is invertible we get

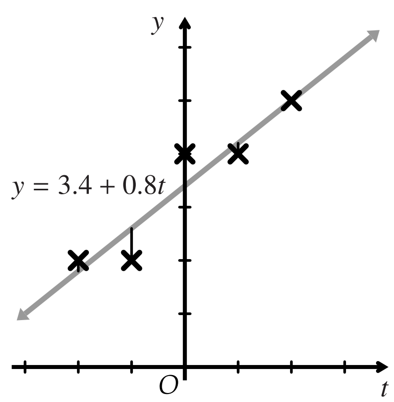
$$\begin{aligned} \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 5 & 15 \\ 15 & 55 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 9 \\ 6 \\ 5 \\ 3 \\ 1 \end{bmatrix} \\ &= \frac{1}{10} \begin{bmatrix} 11 & -3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 24 \\ 53 \end{bmatrix} \\ &= \begin{bmatrix} 10.5 \\ -1.9 \end{bmatrix} \end{aligned}$$

Thus, the line of best fit is $y = 10.5 - 1.9t$.

(b) Since we want an equation of the form $y = a + bt$ we let $X = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$. We also let $\vec{y} = \begin{bmatrix} 2 \\ 2 \\ 4 \\ 4 \\ 5 \end{bmatrix}$.

Then, the required a and b satisfy $X^T X \begin{bmatrix} a \\ b \end{bmatrix} = X^T \vec{y}$

Since $X^T X = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$ is invertible we get



$$\begin{aligned} \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 4 \\ 4 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} 1/5 & 0 \\ 0 & 1/10 \end{bmatrix} \begin{bmatrix} 17 \\ 8 \end{bmatrix} \\ &= \begin{bmatrix} 3.4 \\ 0.8 \end{bmatrix} \end{aligned}$$

Thus, the line of best fit is $y = 3.4 + 0.8t$.

A2 (a) Since we want an equation of the form $y = at + bt^2$ we let

$$X = \begin{bmatrix} -1 & (-1)^2 \\ 0 & 0^2 \\ 1 & 1^2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

We also let $\vec{y} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$. Then, the required a and b satisfy

$$X^T X \begin{bmatrix} a \\ b \end{bmatrix} = X^T \vec{y}$$

Since $X^T X = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is invertible we get

$$\begin{aligned} \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} -3/2 \\ 5/2 \end{bmatrix} \end{aligned}$$

Thus, the equation of best fit is $y = -\frac{3}{2}t + \frac{5}{2}t^2$.

(b) Since we want an equation of the form $y = a + bt^2$ we let

$$X = \begin{bmatrix} 1 & (-2)^2 \\ 1 & (-1)^2 \\ 1 & 0^2 \\ 1 & 1^2 \\ 1 & 2^2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 4 \end{bmatrix}$$

We also let $\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ -2 \end{bmatrix}$. Then, the required a and b satisfy

$$X^T X \begin{bmatrix} a \\ b \end{bmatrix} = X^T \vec{y}$$

Since $X^TX = \begin{bmatrix} 5 & 10 \\ 10 & 34 \end{bmatrix}$ is invertible we get

$$\begin{aligned}
\begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 5 & 10 \\ 10 & 34 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 17/35 & -1/7 \\ -1/7 & 1/14 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 17/7 \\ -5/7 \end{bmatrix}
\end{aligned}$$

Thus, the equation of best fit is $y = \frac{17}{7} - \frac{5}{7}t^2$.

A3 (a) Write the augmented matrix $\left[A \mid \vec{b}\right]$ and row reduce:

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 2 & -3 & 6 \\ 1 & -12 & -4 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -7 & -4 \\ 0 & 0 & -1 \end{array}\right]$$

Thus, the system is inconsistent. The $\vec{x}$ that minimizes $\|A\vec{x} - \vec{b}\|$ must satisfy $A^TA\vec{x} = A^T\vec{b}$. Solving this for $\vec{x}$ gives

$$\begin{aligned}
\vec{x} &= (A^TA)^{-1}A^T\vec{b} \\
&= \begin{bmatrix} 6 & -16 \\ -16 & 157 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 & 1 \\ 2 & -3 & -12 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \\ -4 \end{bmatrix} \\
&= \frac{1}{686} \begin{bmatrix} 157 & 16 \\ 16 & 6 \end{bmatrix} \begin{bmatrix} 13 \\ 40 \end{bmatrix} = \begin{bmatrix} 383/98 \\ 32/49 \end{bmatrix}
\end{aligned}$$

Therefore, $\vec{x} = \begin{bmatrix} 383/98 \\ 32/49 \end{bmatrix}$ minimizes $\|A\vec{x} - \vec{b}\|$.

(b) Write the augmented matrix $\left[A \mid \vec{b}\right]$ and row reduce:

$$\left[\begin{array}{cc|c} 2 & 3 & -4 \\ 3 & -2 & 4 \\ 1 & -6 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 3 & -4 \\ 0 & -13/2 & 10 \\ 0 & 0 & -33/13 \end{array}\right]$$

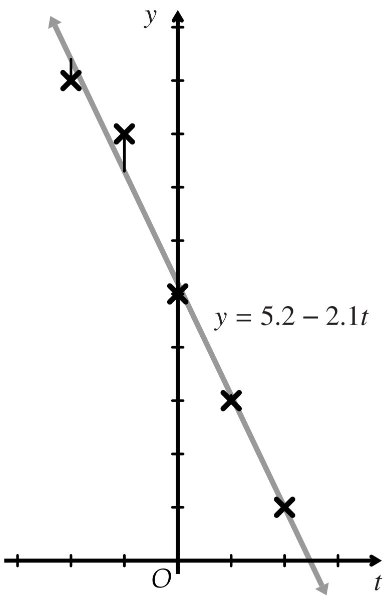
Thus, the system is inconsistent. The $\vec{x}$ that minimizes $\|A\vec{x} - \vec{b}\|$ must satisfy $A^TA\vec{x} = A^T\vec{b}$. Solving this for $\vec{x}$ gives

$$\begin{aligned}
\vec{x} &= (A^TA)^{-1}A^T\vec{b} \\
&= \begin{bmatrix} 14 & -6 \\ -6 & 49 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 3 & 1 \\ 3 & -2 & -6 \end{bmatrix} \begin{bmatrix} -4 \\ 4 \\ 7 \end{bmatrix} \\
&= \frac{1}{650} \begin{bmatrix} 49 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} 11 \\ -62 \end{bmatrix} = \begin{bmatrix} 167/650 \\ -401/325 \end{bmatrix}
\end{aligned}$$

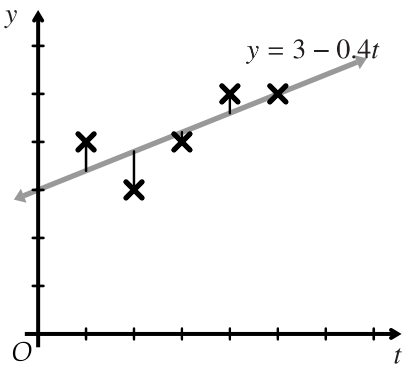
Therefore, $\vec{x} = \begin{bmatrix} 167/650 \\ -401/325 \end{bmatrix}$ minimizes $\|A\vec{x} - \vec{b}\|$.

Homework Problems

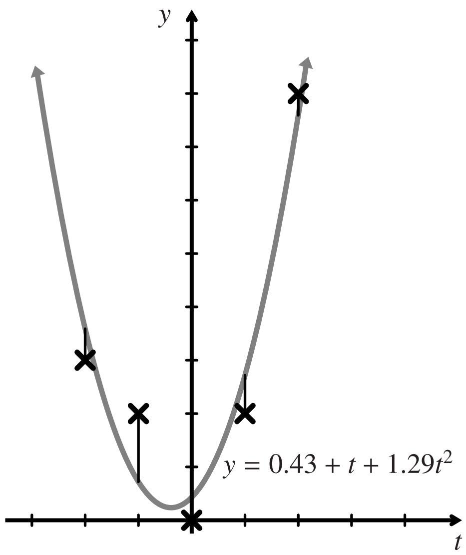
B1 (a)



(b)



B2

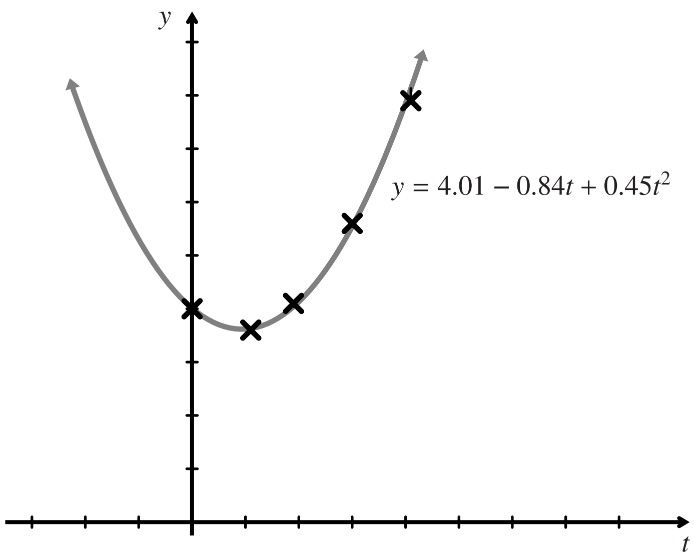


B3 (a) $y = \frac{5}{2}t - \frac{3}{2}t^2$ (b) $y = -\frac{7}{5} + \frac{1}{12}t + \frac{5}{12}t^3$

B4 (a) $\vec{x} = \begin{bmatrix} 227/90 \\ -113/90 \end{bmatrix}$ (b) $\vec{x} = \begin{bmatrix} 35/6 \\ 5/2 \end{bmatrix}$

Computer Problems

C1



Conceptual Problems

D1 By direct calculation we have

$$X^TX = \begin{bmatrix} \vec{1}^T \\ \vec{t}^T \\ (\vec{t}^2)^T \end{bmatrix} \begin{bmatrix} \vec{1} & \vec{t} & \vec{t}^2 \end{bmatrix} = \begin{bmatrix} n & \sum_{i=1}^n t_i & \sum_{i=1}^n t_i^2 \\ \sum_{i=1}^n t_i & \sum_{i=1}^n t_i^2 & \sum_{i=1}^n t_i^3 \\ \sum_{i=1}^n t_i^2 & \sum_{i=1}^n t_i^3 & \sum_{i=1}^n t_i^4 \end{bmatrix}$$

D2 (a) Suppose that $m+1$ of the t_i are distinct and consider a linear combination of the columns of X.

$$c_0 \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + c_1 \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix} + \cdots + c_m \begin{bmatrix} t_1^m \\ t_2^m \\ \vdots \\ t_n^m \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Now, let $p(t) = c_0 + c_1 t + \cdots + c_m t^m$. Equating coefficients we see that $t_1, t_2, \dots, t_n$ are all roots of $p(t)$. Then $p(t)$ is a degree m polynomial with at least $m+1$ distinct roots, hence $p(t)$ must be the zero polynomial. Consequently, $c_i = 0$ for $0 \leq i \leq n$. Thus, the columns of X are linearly independent.

(b) To show this implies that X^TX is invertible we will use the Invertible Matrix Theorem. In particular, we will show that the only solution to $X^TX\vec{v} = \vec{0}$ is $\vec{v} = \vec{0}$.

Assume $X^TX\vec{v} = \vec{0}$. Then

$$\|X\vec{v}\|^2 = (X\vec{v})^TX\vec{v} = \vec{v}^TX^TX\vec{v} = \vec{v}^T\vec{0} = 0$$

Hence, $X\vec{v} = \vec{0}$. This implies that $\vec{v} = \vec{0}$ since the columns of X are linearly independent. Thus X^TX is invertible as required.

7.4 Inner Product Spaces

Practice Problems

A1 (a) $\langle x - 2x^2, 1 + 3x\rangle = 0(1) + -1(4) + (-6)(7) = -46$

(b) $\langle 2 - x + 3x^2, 4 - 3x^2\rangle = 2(4) + 4(1) + 12(-8) = -84$

(c) $\|3 - 2x + x^2\| = \sqrt{3^2 + 2^2 + 3^2} = \sqrt{22}$

(d) $\|9 + 9x + 9x^2\| = |9|\|1 + x + x^2\| = 9\sqrt{1^2 + 3^2 + 7^2} = 9\sqrt{59}$

A2 To show that $\langle\, , \rangle$ defines on an inner product on P_2 we need to show that it satisfies all three properties in the definition of an inner product. On the other hand, to prove it is not an inner product, we just need to find one example of where it does not satisfy one of the three properties. Just as we saw for subspaces, it is helpful to develop an intuition about whether the function should define an inner product or not.

(a) We know that P_2 has dimension 3, so the fact that $\langle\, , \rangle$ only has 2 terms in it makes us think this is not an inner product. To prove this we observe that

$$\langle a + bx + cx^2, a + bx + cx^2\rangle = a^2 + (a + b + c)^2$$

Hence, we see that we can make this expression equal 0 by taking $a = 0$ and $b = -c$. For example, $\langle x - x^2, x - x^2\rangle = 0$. Thus, this function is not positive definite, so it is not an inner product.

(b) The fact that all the terms are in absolute values means that this function can never take negative values. Hence, we should realize that this function cannot be bilinear. Indeed, $\langle -p, q\rangle \neq -\langle p, q\rangle$, so it is not an inner product.

(c) This function has three terms and looks quite a bit like the inner products for P_2 we have seen before. So, we suspect that it is an inner product. We have

$$\langle a + bx + cx^2, a + bx + cx^2\rangle = (a - b + c)^2 + 2a^2 + (a + b + c)^2 \geq 0$$

and $\langle a + bx + cx^2, a + bx + cx^2\rangle = 0$ if and only if $a - b + c = 0$, $a = 0$, and $a + b + c = 0$. Solving this homogeneous system we find that $a = b = c = 0$. Hence, the function is positive definite. Also for any $p, q \in P_2$ we have

$$\begin{aligned}\langle p, q\rangle &= p(-1)q(-1) + 2p(0)q(0) + p(1)q(1)\\ &= q(-1)p(-1) + 2q(0)p(0) + q(1)p(1) = \langle q, p\rangle\end{aligned}$$

Hence, it is symmetric. Finally, for any $p, q, r \in P_2$ and $s, t \in \mathbb{R}$

$$\begin{aligned}\langle p, tq + sr\rangle =& p(-1)[tq(-1) + sr(-1)] + 2p(0)[tq(0) + sr(0)]+\\ &+ p(1)[tq(1) + sr(1)]\\ =& t[p(-1)q(-1) + 2p(0)q(0) + p(1)q(1)]+\\ &+ s[p(-1)q(-1) + 2p(0)q(0) + p(1)q(1)]\\ =& t\langle p, q\rangle + s\langle p, r\rangle\end{aligned}$$

So, it is also bilinear. Therefore, it is an inner product on P_2.

(d) Observe that the terms being multiplied by each other do not have to be the same, so we suspect this is not positive definite. In particular, we have $\langle x, x\rangle = -2$, so it is not an inner product.

A3 To calculate the inner products in this question, we use what was learned from Exercise 7.4.1.

(a) Denote the given basis by $\mathcal{B} = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\} = \left\{\begin{bmatrix} 1 & 0 \\ -1 & 1\end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1\end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 1 & -1\end{bmatrix}\right\}$.

First Step: Let $\vec{v}_1 = \vec{w}_1$ and $\mathbb{S}_1 = \operatorname{Span}\{\vec{v}_1\}$.

Second Step: We have

$$\operatorname{perp}_{\mathbb{S}_1} \vec{w}_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1\end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 & 0 \\ -1 & 1\end{bmatrix} = \begin{bmatrix} 1/3 & 1 \\ 2/3 & 1/3\end{bmatrix}$$

Thus, we take $\vec{v}_2 = \begin{bmatrix} 1 & 3 \\ 2 & 1\end{bmatrix}$ and define $\mathbb{S}_2 = \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$.

Third Step: We have

$$\operatorname{perp}_{\mathbb{S}_2} \vec{w}_3 = \begin{bmatrix} 2 & 0 \\ 1 & -1\end{bmatrix} - \frac{0}{3}\begin{bmatrix} 1 & 0 \\ -1 & 1\end{bmatrix} - \frac{3}{15}\begin{bmatrix} 1 & 3 \\ 2 & 1\end{bmatrix} = \begin{bmatrix} 9/5 & -3/5 \\ 3/5 & -6/5\end{bmatrix}$$

So, we take $\vec{v}_3 = \begin{bmatrix} 3 & -1 \\ 1 & -2\end{bmatrix}$.

Hence, an orthogonal basis for the subspace spanned $\mathcal{B}$ is

$$\left\{\begin{bmatrix} 1 & 0 \\ -1 & 1\end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 2 & 1\end{bmatrix}, \begin{bmatrix} 3 & -1 \\ 1 & -2\end{bmatrix}\right\}$$

Normalizing each vector we get that an orthonormal basis for the subspace spanned by $\mathcal{B}$ is

$$\left\{\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 \\ -1 & 1\end{bmatrix}, \frac{1}{\sqrt{15}}\begin{bmatrix} 1 & 3 \\ 2 & 1\end{bmatrix}, \frac{1}{\sqrt{15}}\begin{bmatrix} 3 & -1 \\ 1 & -2\end{bmatrix}\right\}$$

Then

$$\operatorname{proj}_{\mathbb{S}}\begin{bmatrix} 4 & 3 \\ -2 & 1\end{bmatrix} = \frac{7}{3}\begin{bmatrix} 1 & 0 \\ -1 & 1\end{bmatrix} + \frac{10}{15}\begin{bmatrix} 1 & 3 \\ 2 & 1\end{bmatrix} + \frac{5}{15}\begin{bmatrix} 3 & -1 \\ 1 & -2\end{bmatrix} = \begin{bmatrix} 4 & 5/3 \\ -2/3 & 7/3\end{bmatrix}$$

(b) Denote the given basis by $\mathcal{B} = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\} = \left\{\begin{bmatrix} 1 & 1 \\ 0 & 1\end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 1\end{bmatrix}, \begin{bmatrix} 2 & 0 \\ -1 & 1\end{bmatrix}\right\}$.

First Step: Let $\vec{v}_1 = \vec{w}_1$ and $\mathbb{S}_1 = \operatorname{Span}\{\vec{v}_1\}$.

Second Step: Observe that $\vec{w}_2$ is already orthogonal to $\vec{v}_1$, so we take $\vec{v}_2 = \begin{bmatrix} 0 & -1 \\ 1 & 1\end{bmatrix}$ and define $\mathbb{S}_2 = \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$.

Third Step: Let

$$\vec{v}_3 = \operatorname{perp}_{\mathbb{S}_2} \vec{w}_3 = \begin{bmatrix} 2 & 0 \\ -1 & 1\end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 & 1 \\ 0 & 1\end{bmatrix} - \frac{0}{3}\begin{bmatrix} 0 & -1 \\ 1 & 1\end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 0\end{bmatrix}$$

Hence, an orthogonal basis for the subspace spanned $\mathcal{B}$ is

$$\left\{\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & -1\\1 & 1\end{bmatrix}, \begin{bmatrix}1 & -1\\-1 & 0\end{bmatrix}\right\}$$

Normalizing each vector we get that an orthonormal basis for the subspace spanned by $\mathcal{B}$ is

$$\left\{\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}0 & -1\\1 & 1\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}1 & -1\\-1 & 0\end{bmatrix}\right\}$$

Then

$$\text{proj}_{\mathbb{S}}\begin{bmatrix}4 & 3\\-2 & 1\end{bmatrix} = \frac{8}{3}\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix} + \frac{-4}{3}\begin{bmatrix}0 & -1\\1 & 1\end{bmatrix} + \frac{3}{3}\begin{bmatrix}1 & -1\\-1 & 0\end{bmatrix} = \begin{bmatrix}11/3 & 3\\-7/3 & 4/3\end{bmatrix}$$

A4 (a) Denote the given basis by $\mathcal{B} = \{\vec{w}_1, \vec{w}_2, \vec{w}_3\} = \left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\\-2\end{bmatrix}\right\}$.

First Step: Let $\vec{v}_1 = \vec{w}_1$ and $\mathbb{S}_1 = \text{Span}\{\vec{v}_1\}$.

Second Step: We have

$$\text{perp}_{\mathbb{S}_1} \vec{w}_2 = \begin{bmatrix}-1\\1\\0\end{bmatrix} - \frac{-1}{3}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}-2/3\\4/3\\0\end{bmatrix}$$

Thus, we take $\vec{v}_2 = \begin{bmatrix}-1\\2\\0\end{bmatrix}$ and define $\mathbb{S}_2 = \text{Span}\{\vec{v}_1, \vec{v}_2\}$.

Third Step: We have

$$\text{perp}_{\mathbb{S}_2} \vec{w}_3 = \begin{bmatrix}-1\\1\\-2\end{bmatrix} - \frac{-1}{3}\begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{4}{6}\begin{bmatrix}-1\\2\\0\end{bmatrix} = \begin{bmatrix}0\\0\\-2\end{bmatrix}$$

Thus, we take $\vec{v}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$.

Hence, an orthogonal basis for $\mathbb{S}$ is $\mathcal{B} = \left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\2\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

(b) Normalizing the vectors, we find that an orthonormal basis for $\mathbb{S}$ is

$$\left\{\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\0\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\2\\0\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}0\\0\\1\end{bmatrix}\right\} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$$

Hence,

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}\langle \vec{x}, \vec{u}_1\rangle\\ \langle \vec{x}, \vec{u}_2\rangle\\ \langle \vec{x}, \vec{u}_3\rangle\end{bmatrix} = \begin{bmatrix}2/\sqrt{3}\\-2/\sqrt{6}\\0\end{bmatrix}$$

A5 We have

$$\langle \mathbf{v}_1 + \cdots + \mathbf{v}_k, \mathbf{v}_1 + \cdots + \mathbf{v}_k \rangle = \langle \mathbf{v}_1, \mathbf{v}_1 \rangle + \cdots + \langle \mathbf{v}_1, \mathbf{v}_k \rangle + \langle \mathbf{v}_2, \mathbf{v}_1 \rangle + \langle \mathbf{v}_2, \mathbf{v}_2 \rangle + \\ + \cdots + \langle \mathbf{v}_2, \mathbf{v}_k \rangle + \cdots + \langle \mathbf{v}_k, \mathbf{v}_1 \rangle + \cdots + \langle \mathbf{v}_k, \mathbf{v}_k \rangle$$

Since $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is orthogonal, we have $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ if $i \neq j$. Hence,

$$\|\mathbf{v}_1 + \cdots + \mathbf{v}_k\|^2 = \langle \mathbf{v}_1, \mathbf{v}_1 \rangle + \cdots + \langle \mathbf{v}_k, \mathbf{v}_k \rangle = \|\mathbf{v}_1\|^2 + \cdots + \|\mathbf{v}_k\|^2$$

Homework Problems

B1 (a) $\langle 1 - 3x^2, 1 + x + 2x^2 \rangle = -128$ (b) $\langle 3 - x, -2 - x - x^2 \rangle = -22$
(c) $\|1 - 5x + 2x^2\| = \sqrt{6}$ (d) $\|73x + 73x^2\| = 73\sqrt{40}$

B2 (a) It does not define an inner product since $\langle x - x^3, x - x^3 \rangle = 0$.

(b) It does define an inner product.

(c) It does not define an inner product since if $p(x) = (x+1)(x-1)(x-3)$ and $q(x) = x(x-2)(x-3)$, then $\langle p, q \rangle = 0$, but $\langle q, p \rangle = -36$.

B3 (a) $\left\{1, \frac{2}{3} + x - x^2\right\}$, $\quad \text{proj}_{\mathbb{S}}(1 + x + x^2) = 2 + \frac{1}{2}x - \frac{1}{2}x^2$

(b) $\left\{1 + x^2, 4 + 9x - 5x^2\right\}$, $\quad \text{proj}_{\mathbb{S}}(1 + x + x^2) = \frac{7}{5} + \frac{9}{10}x + \frac{1}{2}x^2$

B4 (a) $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(b) $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1/4 \\ -1/4 \\ 0 \end{bmatrix}$.

Conceptual Problems

D1 (a) By the bilinearity of $\langle\ ,\ \rangle$,

$$\begin{aligned} \langle \vec{x}, \vec{y} \rangle &= \langle x_1\vec{e}_1 + x_2\vec{e}_2, y_1\vec{e}_1 + y_2\vec{e}_2 \rangle \\ &= x_1\langle \vec{e}_1, y_1\vec{e}_1 + y_2\vec{e}_2 \rangle + x_2\langle \vec{e}_2, y_1\vec{e}_1 + y_2\vec{e}_2 \rangle \\ &= x_1y_1\langle \vec{e}_1, \vec{e}_1 \rangle + x_1y_2\langle \vec{e}_1, \vec{e}_2 \rangle + x_2y_1\langle \vec{e}_2, \vec{e}_1 \rangle + x_2y_2\langle \vec{e}_2, \vec{e}_2 \rangle \end{aligned}$$

(b) Since the inner product is symmetric,

$$g_{ji} = \langle \vec{e}_j, \vec{e}_i \rangle = \langle \vec{e}_i, \vec{e}_j \rangle = g_{ij}$$

Thus,

$$\begin{aligned} \langle \vec{x}, \vec{y} \rangle &= g_{11}x_1y_1 + g_{12}x_1y_2 + g_{21}x_2y_1 + g_{22}x_2y_2 \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \end{aligned}$$

(c) Let $\vec{v}_1 = \frac{1}{g_{11}}\vec{e}_1$. Then, $\langle \vec{v}_1, \vec{v}_1 \rangle = 1$. Let

$$\vec{w}_2 = \operatorname{perp}_{\vec{v}_1} \vec{e}_2 = \vec{e}_2 - \frac{g_{12}}{g_{11}^2}\vec{e}_1$$

Finally, let $\vec{v}_2 = \vec{w}_2/\|\vec{w}_2\|$.

(d) By construction $\langle \vec{v}_1, \vec{v}_1 \rangle = 1 = \langle \vec{v}_2, \vec{v}_2 \rangle$ and $\langle \vec{v}_1, \vec{v}_2 \rangle = 0$. So,

$$\tilde{G} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then

$$\begin{aligned} \langle \vec{x}, \vec{y} \rangle &= \langle \tilde{x}_1\vec{v}_1 + \tilde{x}_2\vec{v}_2, \tilde{y}_1\vec{v}_1 + \tilde{y}_2\vec{v}_2 \rangle \\ &= \tilde{x}_1\tilde{y}_1\langle \vec{v}_1, \vec{v}_1 \rangle + \tilde{x}_2\tilde{y}_2\langle \vec{v}_1, \vec{v}_2 \rangle + \tilde{x}_2\tilde{y}_1\langle \vec{v}_2, \vec{v}_1 \rangle + \tilde{x}_2\tilde{y}_2\langle \vec{v}_2, \vec{v}_2 \rangle \\ &= \tilde{x}_1\tilde{y}_1 + \tilde{x}_2\tilde{y}_2 \end{aligned}$$

D2 (a) Because the inner product is symmetric,

$$(G)_{ji} = \langle \vec{v}_j, \vec{v}_i \rangle = \langle \vec{v}_i, \vec{v}_j \rangle = (G)_{ij}$$

(b) By the bilinearity of $\langle\ ,\ \rangle$

$$\begin{aligned} \langle \vec{x}, \vec{y} \rangle &= \langle x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3, y_1\vec{v}_1 + y_2\vec{v}_2 + y_3\vec{v}_3 \rangle \\ &= x_1y_1\langle \vec{v}_1, \vec{v}_1 \rangle + x_1y_2\langle \vec{v}_1, \vec{v}_2 \rangle + \cdots + x_3y_3\langle \vec{v}_3, \vec{v}_3 \rangle \\ &= \big(x_1(G)_{11} + x_2(G)_{21} + x_3(G)_{31}\big)y_1 + \big(x_1(G)_{12} + x_2(G)_{22} + x_3(G)_{32}\big)y_2 + \\ &\quad + \big(x_1(G)_{13} + x_2(G)_{23} + x_3(G)_{33}\big)y_3 \\ &= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} G \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \end{aligned}$$

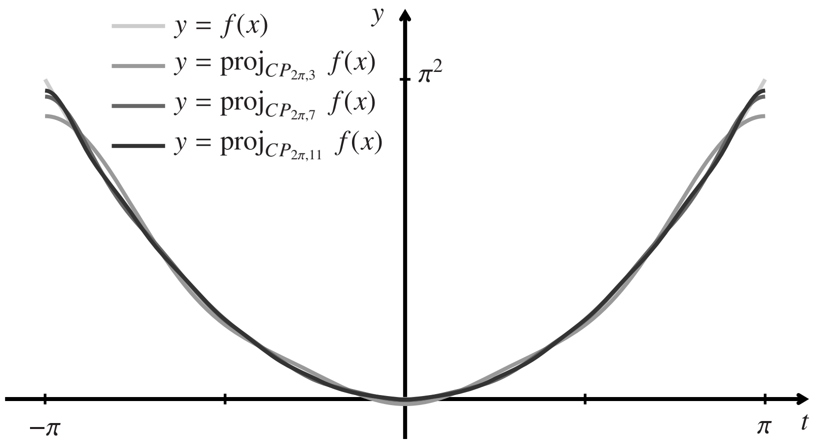
(c) We have

$$G = \begin{bmatrix} <1,1> & <1,x> & <1,x^2> \\ <x,1> & <x,x> & <x,x^2> \\ <x^2,1> & <x^2,x> & <x^2,x^2> \end{bmatrix} = \begin{bmatrix} 3 & 3 & 5 \\ 3 & 5 & 9 \\ 5 & 9 & 17 \end{bmatrix}$$

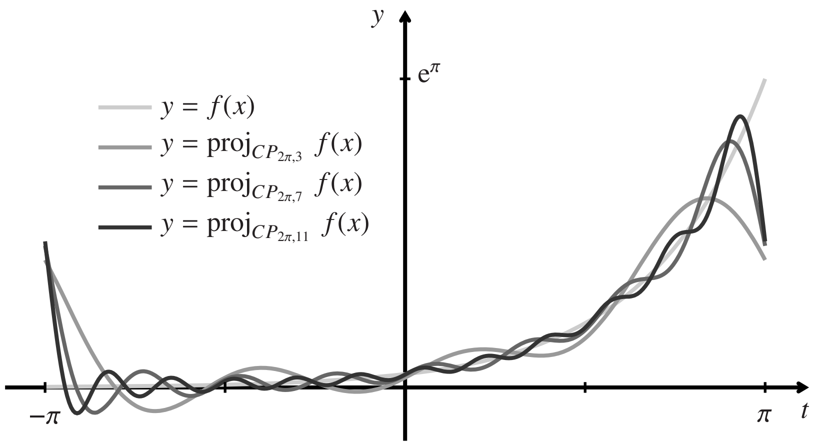
7.5 Fourier Series

Computer Problems

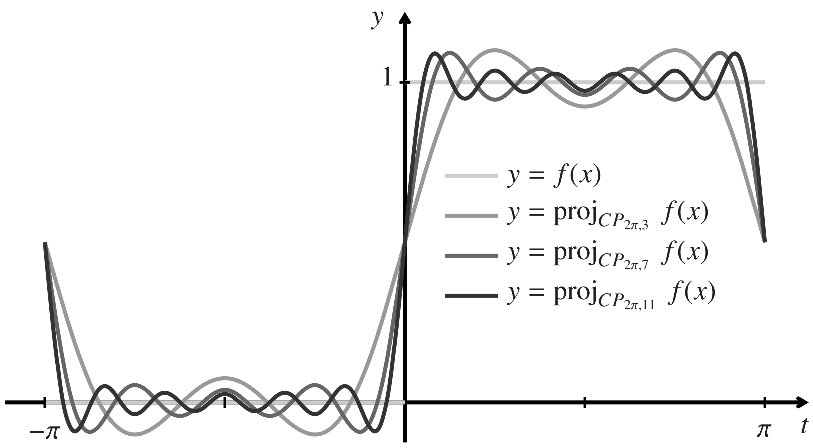
C1 (a)



(b)



(c)



Chapter 7 Quiz

Problems

E1 (a) Neither. Observe that $\left(\frac{1}{3}\begin{bmatrix}1\\0\\1\\1\end{bmatrix}\right)\cdot\left(\frac{1}{2}\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}\right) = -\frac{1}{6}$. Hence, the first and third vector are not orthogonal, so the set is not orthogonal or orthonormal.

(b) Neither. Observe that $\left(\frac{1}{\sqrt{3}}\begin{bmatrix}1\\0\\1\\1\end{bmatrix}\right)\cdot\left(\frac{1}{\sqrt{5}}\begin{bmatrix}0\\0\\1\\-2\end{bmatrix}\right) = -\frac{1}{\sqrt{15}}$. So, the vectors are not orthogonal and hence the set is not orthogonal or orthonormal.

(c) We have

$$\frac{1}{\sqrt{3}}\begin{bmatrix}1\\0\\1\\1\end{bmatrix}\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\\0\end{bmatrix} = 0$$

$$\frac{1}{\sqrt{3}}\begin{bmatrix}1\\0\\1\\1\end{bmatrix}\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}0\\-1\\-1\\1\end{bmatrix} = 0$$

$$\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\\0\end{bmatrix}\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}0\\-1\\-1\\1\end{bmatrix} = 0$$

Hence, the set is orthogonal. But, we also have

$$\frac{1}{\sqrt{3}}\begin{bmatrix}1\\0\\1\\1\end{bmatrix}\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}1\\0\\1\\1\end{bmatrix} = 1$$

$$\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\\0\end{bmatrix}\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\\0\end{bmatrix} = 1$$

$$\frac{1}{\sqrt{3}}\begin{bmatrix}0\\-1\\-1\\1\end{bmatrix}\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}0\\-1\\-1\\1\end{bmatrix} = 1$$

Consequently, the set is orthonormal.

E2 Let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ denote the vectors in $\mathcal{B}$. We have

$$\vec{v}_1 \cdot \vec{x} = 2, \qquad \vec{v}_2 \cdot \vec{x} = \frac{9}{\sqrt{3}}, \qquad \vec{v}_3 \cdot \vec{x} = \frac{6}{\sqrt{3}}$$

Hence, $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 9/\sqrt{3} \\ 6/\sqrt{3} \end{bmatrix}$.

E3 (a) We have

$$1 = \det I = \det(P^T P) = (\det P^T)(\det P) = (\det P)^2$$

Thus, $\det P = \pm 1$.

(b) We have $P^T P = I$ and $R^T R = I$. Hence,

$$(PR)^T(PR) = R^T P^T PR = R^T IR = R^T R = I$$

Thus, PR is orthogonal.

E4 (a) Denote the vectors in the spanning set for $\mathbb{S}$ by $\vec{z}_1, \vec{z}_2, \vec{z}_3$. Let $\vec{w}_1 = \vec{z}_1$. Then,

$$\vec{w}_2 = \operatorname{perp}_{\mathbb{S}_1} \vec{z}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{w}_3 = \operatorname{perp}_{\mathbb{S}_2} \vec{z}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Normalizing $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ gives us the orthonormal basis
$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix} \right\}$.

(b) By the Approximation Theorem the closest point in $\mathbb{S}$ to $\vec{x}$ is $\operatorname{proj}_{\mathbb{S}} \vec{x}$. We find that

$$\operatorname{proj}_{\mathbb{S}} \vec{x} = \vec{x}$$

Hence, $\vec{x}$ is already in $\mathbb{S}$.

E5 (a) Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$. Since $\det A = 0$, it follows that

$$\langle A, A \rangle = \det(AA) = (\det A)(\det A) = 0$$

Hence, it is not an inner product.

(b) We verify that $\langle\, , \rangle$ satisfies the three properties of the inner product.

$$\langle A, A\rangle = a_{11}^2 + 2a_{12}^2 + 2a_{21}^2 + a_{22}^2 \geq 0$$

and equals zero if and only if $A = O_{2,2}$.

$$\begin{aligned}\langle A, B\rangle =& a_{11}b_{11} + 2a_{12}b_{12} + 2a_{21}b_{21} + a_{22}b_{22}\\ =& b_{11}a_{11} + 2b_{12}a_{12} + 2b_{21}a_{21} + b_{22}a_{22} = \langle B, A\rangle\\ \langle A, sB + tC\rangle =& a_{11}(sb_{11} + tc_{11}) + 2a_{12}(sb_{12} + tc_{12}) +\\ &+ 2a_{21}(sb_{21} + tc_{21}) + a_{22}(sb_{22} + tc_{22})\\ =& s(a_{11}b_{11} + 2a_{12}b_{12} + 2a_{21}b_{21} + a_{22}b_{22}) +\\ &+ t(a_{11}c_{11} + 2a_{12}c_{12} + 2a_{21}c_{21} + a_{22}c_{22})\\ =& s\langle A, B\rangle + t\langle A, C\rangle\end{aligned}$$

Thus, it is an inner product.

Chapter 7 Further Problems

Problems

F1 (a) For every $\vec{x} \in \mathbb{R}^3$ we have $\|L(\vec{x})\| = \|\vec{x}\|$, so $L(\vec{x}) \cdot L(\vec{x}) = \|\vec{x}\|^2$ for every $\vec{x} \in \mathbb{R}^3$ and and $\|L(\vec{x}+\vec{y})\|^2 = \|\vec{x}+\vec{y}\|^2$ for every $\vec{x} + \vec{y} \in \mathbb{R}^3$. Hence,

$$\begin{aligned}(L(\vec{x} + \vec{y})) \cdot (L(\vec{x} + \vec{y})) &= (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y})\\ (L(\vec{x}) + L(\vec{y})) \cdot (L(\vec{x}) + L(\vec{y})) &= \vec{x} \cdot \vec{x} + 2\vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y}\\ L(\vec{x}) \cdot L(\vec{x}) + 2L(\vec{x}) \cdot L(\vec{y}) + L(\vec{y}) \cdot L(\vec{y}) &= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2\\ \|\vec{x}\|^2 + 2L(\vec{x}) \cdot L(\vec{y}) + \|\vec{y}\|^2 &= \|\vec{x}\|^2 + 2\vec{x} \cdot \vec{y} + \|\vec{y}\|^2\\ L(\vec{x}) \cdot L(\vec{y}) &= \vec{x} \cdot \vec{y}\end{aligned}$$

as required.

(b) If $[L]$ is orthogonal, then $[L]^T[L] = I$. As in Problem 7.1.D3, it follows that

$$([L]\vec{x}) \cdot ([L]\vec{x}) = \vec{x}^T[L]^T[L]\vec{x} = \vec{x} \cdot \vec{x}$$

so L is an isometry.

Conversely, suppose that L is an isometry. Let $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be the standard basis for $\mathbb{R}^3$. Since L preserves dot products, $\{L(\vec{e}_1), L(\vec{e}_2), L(\vec{e}_3)\}$ is an orthonormal set. Thus, $P = \begin{bmatrix} L(\vec{e}_1) & L(\vec{e}_2) & L(\vec{e}_3)\end{bmatrix}$ is orthogonal.

(c) Let L be an isometry of $\mathbb{R}^3$. The characteristic polynomial of $[L]$ is of degree three, so has either 1 or 3 real roots (since complex roots come in complex conjugate pairs).

(d) We first note that we must not assume that an eigenvalue of algebraic multiplicity greater than 1 has geometric multiplicity greater than 1.

For simplicity, denote the standard matrix of L by A, and suppose that 1 is an eigenvalue with eigenvector $\vec{u}$. Let $\vec{v}$ and $\vec{w}$ be vectors such that $\{\vec{u}, \vec{v}, \vec{w}\}$ is an orthonormal basis for $\mathbb{R}^3$. Then $P = \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$ is an orthogonal matrix. We calculate

$$\begin{aligned} P^T AP &= \begin{bmatrix} \vec{u}^T \\ \vec{v}^T \\ \vec{w}^T \end{bmatrix} A \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} \\ &= \begin{bmatrix} \vec{u}^T A\vec{u} & \vec{u}^T A\vec{v} & \vec{u}^T A\vec{w} \\ \vec{v}^T A\vec{u} & \vec{v}^T A\vec{v} & \vec{v}^T A\vec{w} \\ \vec{w}^T A\vec{u} & \vec{w}^T A\vec{v} & \vec{w}^T \vec{w} \end{bmatrix} \\ &= \begin{bmatrix} \vec{u}^T \vec{u} & \vec{u}^T A\vec{v} & \vec{u}^T A\vec{w} \\ \vec{v}^T \vec{u} & \vec{v}^T A\vec{v} & \vec{v}^T A\vec{w} \\ \vec{w}^T \vec{u} & \vec{w}^T A\vec{v} & \vec{w}^T \vec{w} \end{bmatrix} \end{aligned}$$

Since $\{\vec{u}, \vec{v}, \vec{w}\}$ is orthonormal the first column is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Since $P^T AP$ is orthogonal, this implies that the first row must be $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ Thus,

$$P^T AP = \begin{bmatrix} 1 & 0_{12} \\ 0_{21} & A^* \end{bmatrix}$$

where A^* is orthogonal because its columns are orthonormal.

Consider $\det(A - \lambda I)$. Expanding the determinant along the first column gives

$$\det(A - \lambda I) = (1 - \lambda) \det(A^* - \lambda I)$$

as required. The case where one eigenvalue is -1 is similar.

(e) The possible forms of a 2×2 orthogonal matrix were determined in Problem F5 of Chapter 3. It follows that A^* is either the matrix of a rotation or the matrix of a reflection. Note that the possibilities include $A^* = I$ (rotation through angle 0), $A^* = -I$ (rotation through angle π, or a composition of two reflections), and $A^* = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (a reflection). Thus, with one eigenvalue of A being 1, we have that A describes a rotation, a reflection, or a composition of reflections.

In the case where the first eigenvalue of A is -1, we get that A is the matrix of a composition of a reflection with any of the mappings in the previous case.

F2 If $A^2 = I$ and $A = A^T$, then $A^T A = A^2 = I$, so A is orthogonal.
Suppose that $A^2 = I$ and $A^T A = I$, then $A^T = A^T A^2 = (A^T A)A = A$, so A is symmetric.
If $A = A^T$ and $A^T A = I$, then $A^2 = A^T A = I$, so A is an involution.

F3 The vector $\vec{x}$ is in $(\mathbb{S} + \mathbb{T})^\perp$ is and only if $\vec{x} \cdot \vec{s} = 0$ for every $\vec{s} \in \mathbb{S}$ and $\vec{x} \cdot \vec{t} = 0$ for every $\vec{t} \in \mathbb{T}$. Then $\vec{x} \in \mathbb{S}^\perp$ and $\vec{x} \in \mathbb{T}^\perp$, so $\vec{x} \in \mathbb{S}^\perp \cap \mathbb{T}^\perp$.
Conversely, if $\vec{x} \in \mathbb{S}^\perp \cap \mathbb{T}^\perp$, then $\vec{x} \cdot \vec{s} = 0$ for every $\vec{s} \in \mathbb{S}$ and $\vec{x} \cdot \vec{t} = 0$ for every $\vec{t} \in \mathbb{T}$. Thus, $\vec{x} \cdot (\vec{s} + \vec{t}) = 0$ for every $\vec{s} \in \mathbb{S}$ and $\vec{t} \in \mathbb{T}$. Thus, $\vec{x} \in (\mathbb{S} + \mathbb{T})^\perp$.

F4 Let $\{\vec{u}_1, \dots, \vec{u}_k\}$ be an orthonormal basis for $\mathbb{S}_j$, and let $\{\vec{u}_1, \dots, \vec{u}_\ell\}$ be an orthonormal basis for $\mathbb{S}_{j+1}$. Finally, let $\{\vec{u}_1, \dots, \vec{u}_n\}$ be a basis for $\mathbb{V}$. Then $\vec{v} \in \mathbb{V}$ can be written

$$\vec{v} = c_1\vec{u}_1 + \cdots + c_n\vec{u}_n$$

It follows that

$$\begin{aligned} \|\vec{v} - \mathrm{proj}_{\mathbb{S}_j}\,\vec{v}\| &= \|\vec{x} - (c_1\vec{u}_1 + \cdots + c_k\vec{u}_k)\| \\ &= \|c_{k+1}\vec{u}_{k+1} + \cdots + c_n\vec{u}_n\| \\ &= \sqrt{c_{k+1}^2 + \cdots + c_\ell^2 + \cdots + c_n^2} \end{aligned}$$

On the other hand,

$$\begin{aligned} \|\vec{v} - \mathrm{proj}_{\mathbb{S}_{j+1}}\| &= \|\vec{x} - (c_1\vec{u}_1 + \cdots + c_\ell\vec{u}_\ell)\| \\ &= \|c_{\ell+1}\vec{u}_{\ell+1} + \cdots + c_n\vec{u}_n\| \\ &= \sqrt{c_{\ell+1}^2 + \cdots + c_n^2} \end{aligned}$$

Thus $\|\vec{v} - \mathrm{proj}_{\mathbb{S}_{j+1}}\,\vec{v}\| \leq \|\vec{v} - \mathrm{proj}_{\mathbb{S}_j}\,\vec{v}\|$.

F5 Since A is invertible, its columns $\{\vec{a}_1, \dots, \vec{a}_n\}$ form a basis for $\mathbb{R}^n$. Perform the Gram-Schmidt procedure on the basis, thus producing an orthonormal basis $\{\vec{q}_1, \dots, \vec{q}_n\}$. It is a feature of this procedure that for each i,

$$\mathrm{Span}\{\vec{a}_1, \dots, \vec{a}_i\} = \mathrm{Span}\{\vec{q}_1, \dots, \vec{q}_i\}$$

Hence, we can determine coefficients r_{ij} such that

$$\vec{a}_j = r_{1j}\vec{q}_1 + \cdots + r_{jj}\vec{q}_j + 0\vec{q}_{j+1} + \cdots + 0\vec{q}_n$$

Thus we obtain

$$\begin{bmatrix}\vec{a}_1 & \cdots & \vec{a}_n\end{bmatrix} = \begin{bmatrix}\vec{q}_1 & \cdots & \vec{q}_n\end{bmatrix}\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \ddots & \vdots \\ 0 & \ddots & \ddots & r_{(n-1)n} \\ 0 & \cdots & 0 & r_{nn} \end{bmatrix}$$

That is, $A = QR$ where Q is orthogonal and R is upper triangular.

CHAPTER 8 Symmetric Matrices and Quadratic Forms

8.1 Diagonalization of Symmetric Matrices

Practice Problems

A1 (a) A is symmetric.

(b) B is symmetric.

(c) C is not symmetric since $c_{12} \neq c_{21}$.

(d) D is symmetric.

A2 Alternate correct answers are possible.

(a) We have

$$C(\lambda) = \begin{vmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$$

The eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -1$.
For $\lambda_1 = 3$ we get

$$A - 3I = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

A basis for the eigenspace of λ_1 is $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.
For $\lambda_2 = -1$ we get

$$A - (-1)I = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

A basis for the eigenspace of λ_2 is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.
Observe that, as predicted by Theorem 8.1.2, the eigenvectors in the different eigenspaces are orthogonal. Hence, we get that $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ forms an orthogonal basis for $\mathbb{R}^2$ of eigenvectors of A. To orthogonally diagonalize A we need to form an orthogonal matrix P whose columns are eigenvectors of A. It is very important to remember that the columns of an orthogonal matrix must form an *orthonormal* basis for $\mathbb{R}^n$.

Thus, we must normalize the vectors in the orthogonal basis for $\mathbb{R}^2$. We get $\left\{\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\right\}$. Hence, the matrix $P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ orthogonally diagonalizes A to $P^TAP = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} = D$.

(b) We have

$$C(\lambda) = \begin{vmatrix} 5-\lambda & 3 \\ 3 & -3-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 24 = (\lambda+4)(\lambda-6)$$

The eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = 6$.

For $\lambda_1 = -4$ we get

$$A - (-4)I = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/3 \\ 0 & 0 \end{bmatrix}$$

A basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right\}$.

For $\lambda_2 = 6$ we get

$$A - 6I = \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

A basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}$.

We normalize the basis vectors for the eigenspaces to get the orthonormal basis for $\mathbb{R}^2$ of eigenvectors of A $\left\{\begin{bmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}, \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}\right\}$. Hence, $P = \begin{bmatrix} -1/\sqrt{10} & 3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}$ orthogonally diagonalizes A to $P^TAP = \begin{bmatrix} -4 & 0 \\ 0 & 6 \end{bmatrix}$.

(c) We have

$$C(\lambda) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -(\lambda-2)(\lambda+1)^2$$

The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = -1$.

For $\lambda_1 = 2$ we get

$$A - 2I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$.

For $\lambda_2 = -1$ we get

$$A - (-1)I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$. These vectors are not orthogonal to each other. Since we require an orthonormal basis of eigenvectors of A, we need to find an orthonormal basis for the eigenspace of λ_2. We can do this by applying the Gram-Schmidt procedure to this set.

Pick $\vec{v}_1 = \begin{bmatrix}-1\\1\\0\end{bmatrix}$. Then $\mathbb{S}_1 = \operatorname{Span}\{\vec{v}_1\}$ and

$$\vec{v}_2 = \operatorname{perp}_{\mathbb{S}_1}\begin{bmatrix}-1\\0\\1\end{bmatrix} = \begin{bmatrix}-1\\0\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-1\\1\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}-1\\-1\\2\end{bmatrix}$$

Then, $\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\-1\\2\end{bmatrix}\right\}$ is an orthogonal basis for the eigenspace of λ_2.

We normalize the basis vectors for the eigenspaces to get the orthonormal basis for $\mathbb{R}^3$ of eigenvectors of A $\left\{\begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}, \begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}, \begin{bmatrix}-1/\sqrt{6}\\-1/\sqrt{6}\\2/\sqrt{6}\end{bmatrix}\right\}$. Hence, $P = \begin{bmatrix}1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6}\\1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6}\\1/\sqrt{3} & 0 & 2/\sqrt{6}\end{bmatrix}$ orthogonally diagonalizes A to $P^TAP = \begin{bmatrix}2 & 0 & 0\\0 & -1 & 0\\0 & 0 & -1\end{bmatrix}$.

(d) We have

$$C(\lambda) = \begin{vmatrix}1-\lambda & 0 & -2\\0 & -1-\lambda & -2\\-2 & -2 & -\lambda\end{vmatrix} = -\lambda(\lambda-3)(\lambda+3)$$

The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 3$, and $\lambda_3 = -3$.

For $\lambda_1 = 0$ we get

$$A - 0I = \begin{bmatrix}1 & 0 & -2\\0 & -1 & -2\\-2 & -2 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -2\\0 & 1 & 2\\0 & 0 & 0\end{bmatrix}$$

A basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}2\\-2\\1\end{bmatrix}\right\}$.

For $\lambda_2 = 3$ we get

$$A - 3I = \begin{bmatrix}-2 & 0 & -2\\0 & -4 & -2\\-2 & -2 & -3\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 1\\0 & 1 & 1/2\\0 & 0 & 0\end{bmatrix}$$

A basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix}2\\1\\-2\end{bmatrix}\right\}$.

For $\lambda_3 = -3$ we get

$$A - (-3)I = \begin{bmatrix}4 & 0 & -2\\0 & 2 & -2\\-2 & -2 & 3\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -1/2\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix}$$

A basis for the eigenspace of λ_3 is $\left\{\begin{bmatrix}1\\2\\2\end{bmatrix}\right\}$.

We normalize the basis vectors for the eigenspaces to get the orthonormal basis for $\mathbb{R}^2$ of eigenvectors of A $\left\{\begin{bmatrix}2/3\\-2/3\\1/3\end{bmatrix}, \begin{bmatrix}2/3\\1/3\\-2/3\end{bmatrix}, \begin{bmatrix}1/3\\2/3\\2/3\end{bmatrix}\right\}$. Hence, $P = \begin{bmatrix}2/3 & 2/3 & 1/3\\-2/3 & 1/3 & 2/3\\1/3 & -2/3 & 2/3\end{bmatrix}$ orthogonally diagonalizes A to $P^TAP = \begin{bmatrix}0 & 0 & 0\\0 & 3 & 0\\0 & 0 & -3\end{bmatrix}$.

(e) We have

$$C(\lambda) = \begin{vmatrix}1-\lambda & 8 & 4\\8 & 1-\lambda & -4\\4 & -4 & 7-\lambda\end{vmatrix} = -(\lambda-9)^2(\lambda+9)$$

The eigenvalues are $\lambda_1 = 9$ and $\lambda_2 = -9$.

For $\lambda_1 = 9$ we get

$$A - 9I = \begin{bmatrix}-8 & 8 & 4\\8 & -8 & -4\\4 & -4 & -2\end{bmatrix} \sim \begin{bmatrix}1 & -1 & -1/2\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

A basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$. These vectors are not orthogonal to each other. Since we require an orthonormal basis of eigenvectors of A, we need to find an orthonormal basis for the eigenspace of λ_2. We can do this by applying the Gram-Schmidt procedure to this set.

Pick $\vec{v}_1 = \begin{bmatrix}1\\0\\2\end{bmatrix}$. Then $\mathbb{S}_1 = \operatorname{Span}\{\vec{v}_1\}$ and

$$\vec{v}_2 = \operatorname{perp}_{\mathbb{S}_1}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{1}{5}\begin{bmatrix}1\\0\\2\end{bmatrix} = \begin{bmatrix}4/5\\1\\-2/5\end{bmatrix}$$

Then, $\left\{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}4\\5\\-2\end{bmatrix}\right\}$ is an orthogonal basis for the eigenspace of λ_1.

For $\lambda_2 = -9$ we get

$$A - (-9)I = \begin{bmatrix}10 & 8 & 4\\8 & 10 & -4\\4 & -4 & 16\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 2\\0 & 1 & -2\\0 & 0 & 0\end{bmatrix}$$

A basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix}-2\\2\\1\end{bmatrix}\right\}$.

We normalize the basis vectors for the eigenspaces to get the orthonormal basis for $\mathbb{R}^3$ of eigenvectors of A $\left\{\begin{bmatrix}1/\sqrt{5}\\0\\2/\sqrt{5}\end{bmatrix}, \begin{bmatrix}4/\sqrt{45}\\5/\sqrt{45}\\-2/\sqrt{45}\end{bmatrix}, \begin{bmatrix}-2/3\\2/3\\1/3\end{bmatrix}\right\}$. Hence, $P = \begin{bmatrix}1/\sqrt{5} & 4/\sqrt{45} & -2/3\\0 & 5/\sqrt{45} & 2/3\\2/\sqrt{5} & -2/\sqrt{45} & 1/3\end{bmatrix}$ orthogonally diagonalizes A to $P^TAP = \begin{bmatrix}9 & 0 & 0\\0 & 9 & 0\\0 & 0 & -9\end{bmatrix}$.

Homework Problems

B1 (a) A is symmetric.

(b) B is not symmetric since $b_{12} \neq b_{21}$,

(c) C is not symmetric since $c_{31} \neq c_{13}$.

(d) D is not symmetric since $d_{12} \neq d_{21}$.

(e) E is symmetric.

B2 Alternate correct answers are possible.

(a) $P = \begin{bmatrix}1/\sqrt{5} & 2/\sqrt{5}\\-2/\sqrt{5} & 1/\sqrt{5}\end{bmatrix}$, $D = \begin{bmatrix}1 & 0\\0 & 6\end{bmatrix}$

(b) $P = \begin{bmatrix}1/\sqrt{5} & 2/\sqrt{5}\\-2/\sqrt{5} & 1/\sqrt{5}\end{bmatrix}$, $D = \begin{bmatrix}0 & 0\\0 & 5\end{bmatrix}$

(c) $P = \begin{bmatrix}-1/\sqrt{10} & 3/\sqrt{10}\\3/\sqrt{10} & 1/\sqrt{10}\end{bmatrix}$, $D = \begin{bmatrix}-2 & 0\\0 & 8\end{bmatrix}$

(d) $P = \begin{bmatrix}-1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6}\\1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6}\\0 & 1/\sqrt{3} & 2/\sqrt{6}\end{bmatrix}$, $D = \begin{bmatrix}-1 & 0 & 0\\0 & 4 & 0\\0 & 0 & 1\end{bmatrix}$

(e) $P = \begin{bmatrix}1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6}\\-1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6}\\1/\sqrt{3} & 0 & 2/\sqrt{6}\end{bmatrix}$, $D = \begin{bmatrix}-2 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

(f) $P = \begin{bmatrix}1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6}\\-1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6}\\1/\sqrt{3} & 0 & 2/\sqrt{6}\end{bmatrix}$, $D = \begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 3\end{bmatrix}$

(g) $P = \begin{bmatrix}-1/\sqrt{5} & 4/\sqrt{45} & -2/3\\2/\sqrt{5} & 2/\sqrt{45} & -1/3\\0 & 5/\sqrt{45} & 2/3\end{bmatrix}$, $D = \begin{bmatrix}-3 & 0 & 0\\0 & -3 & 0\\0 & 0 & 6\end{bmatrix}$

(h) $P = \begin{bmatrix}-1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3}\\-2/\sqrt{6} & 0 & 1/\sqrt{3}\\1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3}\end{bmatrix}$, $D = \begin{bmatrix}3 & 0 & 0\\0 & -3 & 0\\0 & 0 & -3\end{bmatrix}$

(i) $P = \begin{bmatrix} -2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -6 \end{bmatrix}$

Computer Problems

C1 (a) The eigenvalues are $\lambda_1 = 9$, $\lambda_2 = 9$, and $\lambda_3 = -9$. An orthonormal basis of eigenvectors is

$$\left\{ \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{bmatrix}, \begin{bmatrix} 4/\sqrt{45} \\ 5/\sqrt{45} \\ -2/\sqrt{45} \end{bmatrix}, \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix} \right\}$$

(b) The eigenvalues are $\lambda_1 = 5.111028$, $\lambda_2 = 0.299752$, and $\lambda_3 = -2.210779$. An orthonormal basis of eigenvectors is $\left\{ \begin{bmatrix} 0.889027 \\ 0.447058 \\ 0.098841 \end{bmatrix}, \begin{bmatrix} 0.456793 \\ -0.880742 \\ -0.125033 \end{bmatrix}, \begin{bmatrix} -0.311567 \\ -0.156308 \\ 0.9872168 \end{bmatrix} \right\}$.

(c) The eigenvalues are $\lambda_1 = 1.4$, $\lambda_2 = 0.8$, $\lambda_3 = -1$, and $\lambda_4 = -0.6$. An orthonormal basis of eigenvectors is $\left\{ \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \right\}$.

C2 The eigenvalues of $S(t)$ are found to be

t	λ_1	λ_2	λ_3
-0.1	2.2244	1.7974	0.8781
-0.05	2.1120	1.8932	0.9448
0	2	2	1
0.05	1.8885	2.1163	1.0452
0.1	1.7779	2.2403	1.0819

Conceptual Problems

D1 We have that $A^T = A$ and $B^T = B$. Using properties of the transpose we get:

(a) $(A + B)^T = A^T + B^T = A + B$. Hence, $A + B$ is symmetric.

(b) $(A^TA)^T = A^T(A^T)^T = A^TA$. Hence, A^TA is symmetric.

(c) $(AB)^T = B^TA^T = BA$. So, AB is not symmetric unless $AB = BA$.

(d) $(A^2)^T = (AA)^T = A^TA^T = AA = A^2$. Hence, A^2 is symmetric.

D2 We are assuming that there exists an orthogonal matrix P and diagonal matrix D such that $P^TAP = D$. Hence, $A = PDP^T$ and

$$A^T = (PDP^T)^T = (P^T)^TD^TP^T = PDP^T = A$$

Thus, A is symmetric.

D3 We are assuming that there exists an orthogonal matrix P and diagonal matrix D such that $P^TAP = D$. If A is invertible, then $0 \neq \det A = \det D$, so D is also invertible. We get that

$$D^{-1} = (P^TAP)^{-1} = P^{-1}A^{-1}(P^T)^{-1} = P^TA^{-1}P$$

since P is orthogonal. Thus, A^{-1} is orthogonally diagonalized by P to D^{-1}.

8.2 Quadratic Forms

Practice Problems

A1 (a) $Q(x_1, x_2) = x_1^2 + 6x_1x_2 - x_2^2$

(b) $Q(x_1, x_2, x_3) = x_1^2 - 2x_2^2 + 6x_2x_3 - x_3^2$

(c) $Q(x_1, x_2, x_3) = -2x_1^2 + 2x_1x_2 + 2x_1x_3 + x_2^2 - 2x_2x_3$

A2 (a) The corresponding symmetric matrix is $A = \begin{bmatrix} 1 & -3/2 \\ -3/2 & 1 \end{bmatrix}$. We have

$$C(\lambda) = \begin{vmatrix} 1-\lambda & -3/2 \\ -3/2 & 1-\lambda \end{vmatrix} = (\lambda - 5/2)(\lambda + 1/2)$$

The eigenvalues are $\lambda_1 = 5/2$ and $\lambda_2 = -1/2$.
For $\lambda_1 = 5/2$ we get

$$A - \frac{5}{2}I = \begin{bmatrix} -3/2 & -3/2 \\ -3/2 & -3/2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Hence, a basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$.
For $\lambda_2 = -1/2$ we get

$$A - \left(-\frac{1}{2}\right)I = \begin{bmatrix} 3/2 & -3/2 \\ -3/2 & 3/2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Hence, a basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

Therefore, A is orthogonally diagonalized by $P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ to $D = \begin{bmatrix} 5/2 & 0 \\ 0 & -1/2 \end{bmatrix}$. Let $\vec{x} = P\vec{y}$. Then we get

$$Q(\vec{x}) = \vec{x}^T A\vec{x} = \vec{y}^T D\vec{y} = \frac{5}{2}y_1^2 - \frac{1}{2}y_2^2$$

Since A has positive and negative eigenvalues we get that $Q(\vec{x})$ is indefinite by Theorem 8.2.2.

(b) The corresponding symmetric matrix is $A = \begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}$.

$$C(\lambda) = \begin{vmatrix} 5-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix} = (\lambda - 1)(\lambda - 6)$$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 6$.
For $\lambda_1 = 1$ we get

$$A - I = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$$

Hence, a basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$.

For $\lambda_2 = 6$ we get

$$A - 6I = \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Hence, a basis for the eigenspace of λ_2 is $\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$.

Therefore, A is orthogonally diagonalized by $P = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$ to $D = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$. Let $\vec{x} = P\vec{y}$. Then we get

$$Q(\vec{x}) = \vec{x}^T A\vec{x} = \vec{y}^T D\vec{y} = y_1^2 + 6y_2^2$$

Since A has all positive eigenvalues we get that $Q(\vec{x})$ is positive definite by Theorem 8.2.2.

(c) The corresponding symmetric matrix is $A = \begin{bmatrix} -2 & 6 \\ 6 & 7 \end{bmatrix}$.

$$C(\lambda) = \begin{vmatrix} -2-\lambda & 6 \\ 6 & 7-\lambda \end{vmatrix} = (\lambda - 10)(\lambda + 5)$$

The eigenvalues are $\lambda_1 = 10$ and $\lambda_2 = -5$.
For $\lambda_1 = 10$ we get

$$A - 10I = \begin{bmatrix} -12 & 6 \\ 6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$$

Hence, a basis for the eigenspace of λ_1 is $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.
For $\lambda_2 = -5$ we get

$$A - (-5)I = \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Hence, a basis for the eigenspace of λ_2 is $\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$.

Therefore, A is orthogonally diagonalized by $P = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$ to $D = \begin{bmatrix} 10 & 0 \\ 0 & -5 \end{bmatrix}$. Let $\vec{x} = P\vec{y}$. Then we get

$$Q(\vec{x}) = \vec{x}^T A\vec{x} = \vec{y}^T D\vec{y} = 10y_1^2 - 5y_2^2$$

Since A has positive and negative eigenvalues we get that $Q(\vec{x})$ is indefinite by Theorem 8.2.2.

(d) The corresponding symmetric matrix is $A = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 1 & 3 \\ 3 & 3 & -3 \end{bmatrix}$.

$$C(\lambda) = \begin{vmatrix} 1-\lambda & -1 & 3 \\ -1 & 1-\lambda & 3 \\ 3 & 3 & -3-\lambda \end{vmatrix} = -(\lambda - 2)(\lambda - 3)(\lambda + 6)$$

The eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 3$, and $\lambda_3 = -6$.

For $\lambda_1 = 2$ we get

$$A - 2I = \begin{bmatrix} -1 & -1 & 3 \\ -1 & -1 & 3 \\ 3 & 3 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for the eigenspace of λ_1 is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

For $\lambda_2 = 3$ we get

$$A - 3I = \begin{bmatrix} -2 & -1 & 3 \\ -1 & -2 & 3 \\ 3 & 3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for the eigenspace of λ_2 is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

For $\lambda_3 = -6$ we get

$$A - (-6)I = \begin{bmatrix} 7 & -1 & 3 \\ -1 & 7 & 3 \\ 3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for the eigenspace of λ_3 is $\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}$.

Therefore, A is orthogonally diagonalized by $P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \end{bmatrix}$ to $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -6 \end{bmatrix}$. Let $\vec{x} = P\vec{y}$. Then we get

$$Q(\vec{x}) = \vec{x}^T A\vec{x} = \vec{y}^T D\vec{y} = 2y_1^2 + 3y_2^2 - 6y_3^2$$

Since A has positive and negative eigenvalues we get that $Q(\vec{x})$ is indefinite by Theorem 8.2.2.

(e) The corresponding symmetric matrix is $A = \begin{bmatrix} -4 & 1 & 0 \\ 1 & -5 & -1 \\ 0 & -1 & -4 \end{bmatrix}$.

$$C(\lambda) = \begin{vmatrix} -4-\lambda & 1 & 0 \\ 1 & -5-\lambda & -1 \\ 0 & -1 & -4-\lambda \end{vmatrix} = -(\lambda + 3)(\lambda + 6)(\lambda + 4)$$

The eigenvalues are $\lambda_1 = -3$, $\lambda_2 = -6$, and $\lambda_3 = -4$.

For $\lambda_1 = -3$ we get

$$A - (-3)I = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -2 & -1 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for the eigenspace of λ_1 is $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

For $\lambda_2 = -6$ we get

$$A - (-6)I = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for the eigenspace of λ_2 is $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$.

For $\lambda_3 = -4$ we get

$$A - (-4)I = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, a basis for the eigenspace of λ_3 is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Therefore, A is orthogonally diagonalized by $P = \begin{bmatrix} -1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ -1/\sqrt{3} & 2/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix}$ to $D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -4 \end{bmatrix}$. Let $\vec{x} = P\vec{y}$. Then we get

$$Q(\vec{x}) = \vec{x}^T A\vec{x} = \vec{y}^T D\vec{y} = -3y_1^2 - 6y_2^2 - 4y_3^2$$

Since A has all negative eigenvalues we get that $Q(\vec{x})$ is negative definite by Theorem 8.2.2.

A3 (a) We have

$$C(\lambda) = \begin{vmatrix} 4-\lambda & -2 \\ -2 & 4-\lambda \end{vmatrix} = (\lambda - 2)(\lambda - 6)$$

Hence, the eigenvalues are 2 and 6. Since both of the eigenvalues are positive we get that A is positive definite by Theorem 8.2.2.

(b) Since A is diagonal, the eigenvalues are the diagonal entries of A. Hence, the eigenvalues are 1, 2, and 3, so A is positive definite by Theorem 8.2.2.

(c) We have

$$C(\lambda) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -2-\lambda & 6 \\ 0 & 6 & 7-\lambda \end{vmatrix} = -(\lambda - 1)(\lambda - 10)(\lambda + 5)$$

So, the eigenvalues are 1, 10, and -5. Since A has positive and negative eigenvalues we get that A is indefinite by Theorem 8.2.2.

(d) We have

$$C(\lambda) = \begin{vmatrix} -3-\lambda & 1 & -1 \\ 1 & -3-\lambda & 1 \\ -1 & 1 & -3-\lambda \end{vmatrix} = -(\lambda + 5)(\lambda + 2)^2$$

So, the eigenvalues are -5, and -2. Since all eigenvalues of A are negative, so we get that A is negative definite by Theorem 8.2.2.

(e) We have

$$C(\lambda) = \begin{vmatrix} 7-\lambda & 2 & -1 \\ 2 & 10-\lambda & -2 \\ -1 & -2 & 7-\lambda \end{vmatrix} = -(\lambda-12)(\lambda-6)^2$$

So, the eigenvalues are 12, and 6. Since all eigenvalues of A are positive, so we get that A is positive definite by Theorem 8.2.2.

(f) We have

$$C(\lambda) = \begin{vmatrix} -4-\lambda & -5 & 5 \\ -5 & 2-\lambda & 1 \\ 5 & 1 & 2-\lambda \end{vmatrix} = -(\lambda+9)(\lambda-3)(\lambda-6)$$

So, the eigenvalues are -9, 3, and 6. Since A has positive and negative eigenvalues we get that A is indefinite by Theorem 8.2.2.

Homework Problems

B1 (a) $2x_1^2 + 8x_1x_2 + 3x_2^2$

(b) $-x_1^2 + 2x_1x_2 + 4x_1x_3 + 3x_2^2 + 2x_2x_3 - 2x_3^2$

(c) $4x_1x_2 + 6x_1x_3 - 2x_2x_3$

B2 (a) $A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}$; $Q(\vec{x}) = 3y_1^2 + 8y_2^2$, $P = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$; $Q(\vec{x})$ is positive definite.

(b) $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$; $Q(\vec{x}) = 5y_1^2 - y_2^2$, $P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$; $Q(\vec{x})$ is indefinite.

(c) $A = \begin{bmatrix} 1 & 3/2 \\ 3/2 & 1 \end{bmatrix}$; $Q(\vec{x}) = \frac{5}{2}y_1^2 - \frac{1}{2}y_2^2$, $P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$; $Q(\vec{x})$ is indefinite.

(d) $A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 5 & 1 \\ -1 & 1 & 3 \end{bmatrix}$; $Q(\vec{x}) = 2y_1^2 + 3y_2^2 + 6y_3^2$, $P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix}$; $Q(\vec{x})$ is positive definite.

(e) $A = \begin{bmatrix} 2 & -2 & 3 \\ -2 & 2 & 3 \\ 3 & 3 & -3 \end{bmatrix}$; $Q(\vec{x}) = 4y_1^2 + 3y_2^2 - 6y_3^2$, $P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \end{bmatrix}$; $Q(\vec{x})$ is indefinite.

B3 (a) Positive definite.

(b) Indefinite.

(c) Negative definite.

(d) Indefinite.

(e) Negative definite.

Computer Problems

C1 (a) Negative definite.

(b) Indefinite.

(c) Positive definite.

Conceptual Problems

D1 By Theorem 1 there exists an orthogonal matrix P such that

$$Q(\vec{x}) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

where $\vec{x} = P\vec{y}$ and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. Clearly, $Q(\vec{x}) < 0$ for all $\vec{y} \neq \vec{0}$ if and only if the eigenvalues are all negative. Moreover, since P is orthogonal, it is invertible, hence $\vec{x} = \vec{0}$ if and only if $\vec{y} = \vec{0}$ since $\vec{x} = P\vec{y}$. Thus we have shown that $Q(\vec{x})$ is negative definite if and only if all eigenvalues of A are negative.

D2 By Theorem 1 there exists an orthogonal matrix P such that

$$Q(\vec{x}) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

where $\vec{x} = P\vec{y}$ and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. Clearly if some eigenvalues are positive and some are negative, then $Q(\vec{x}) > 0$ for some $\vec{y}$ and $Q(\vec{x}) < 0$ for some $\vec{y}$. On the other hand, if $Q(\vec{x})$ takes both positive and negative values, then we must have both positive and negatives values of λ.

D3 (a) By Theorem 1 there exists an orthogonal matrix P such that

$$Q(\vec{x}) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

where $\vec{x} = P\vec{y}$ and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. Clearly, $Q(\vec{x}) \geq 0$ for all $\vec{y} \neq \vec{0}$ if and only if the eigenvalues are all non-negative.

(b) Let λ be an eigenvalue of A^TA with unit eigenvector $\vec{v}$. Then $A^TA\vec{v} = \lambda\vec{v}$. Hence,

$$\|A\vec{v}\|^2 = (A\vec{v})^T A\vec{v} = \vec{v}^T A^T A\vec{v} = \vec{v}^T(\lambda\vec{v}) = \lambda(\vec{v}^T\vec{v}) = \lambda\|\vec{v}\| = \lambda$$

Thus, since $\|A\vec{v}\| \geq 0$ we have that $\lambda \geq 0$.

D4 (a) Since A is positive definite, we have that

$$a_{ii} = \vec{e}_i^T A\vec{e}_i > 0$$

(b) If A is positive definite, then all of its eigenvalues are positive. Moreover, since A is symmetric it is similar to a diagonal matrix D whose diagonal entries are the eigenvalues of A. But then, $\det A = \det D > 0$. Hence A is invertible.

(c) Let λ be an eigenvalue of A^{-1} with eigenvector $\vec{v}$. Then $A^{-1}\vec{v} = \lambda\vec{v}$. Multiplying both sides by A gives

$$A\lambda\vec{v} = AA^{-1}\vec{v} = \vec{v}$$

Thus, $\lambda\vec{v}$ is an eigenvector of A with corresponding eigenvector $\frac{1}{\lambda}$. Thus, $\lambda > 0$ since all the eigenvalues of A are positive.

(d) Pick any $\vec{y} \in \mathbb{R}^n$, $\vec{y} \neq \vec{0}$. Then,

$$\vec{y}^T P^T A P \vec{y} = (P\vec{y})^T A (P\vec{y}) > 0$$

since $\vec{x}^T A \vec{x} > 0$ for all $\vec{x} \neq \vec{0}$ as A is positive definite. Hence, $P^T AP$ is positive definite.

D5 (a) We have

$$\begin{aligned}(A^+)^T &= \frac{1}{2}(A + A^T)^T = \frac{1}{2}(A^T + A) = A^+\\(A^-)^T &= \frac{1}{2}(A - A^T)^T = \frac{1}{2}(A^T - A) = -A^-\end{aligned}$$

So A^+ is symmetric and A^- is skew-symmetric.

$$A^+ + A^- = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = A$$

(b) Since $(A^-)^T = -A^-$ we have that $(A^-)_{ii} = -(A^-)_{ii}$ and so $(A^-)_{ii} = 0$.

(c) We have $(A^+)_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$ and $(A^-)_{ij} = \frac{1}{2}(a_{ij} - a_{ji})$.

(d) We have

$$\vec{x}^T A \vec{x} = \vec{x}^T (A^+ + A^-)\vec{x} = \vec{x}^T A^+ \vec{x} + \vec{x}^T A^- \vec{x}$$

Since $(A^-)^T = -A^-$, the real number $\vec{x}^T A^- \vec{x}$ satisfies

$$\vec{x}^T A^- \vec{x} = (\vec{x}^T A^- \vec{x})^T = \vec{x}^T (A^-)^T \vec{x} = -\vec{x}^T A^- \vec{x}$$

So, $\vec{x}^T A^- \vec{x} = 0$. Thus,

$$\vec{x}^T A \vec{x} = \vec{x}^T A^+ \vec{x}$$

D6 (a) By the bilinearity of the inner product,

$$\begin{aligned}\langle \vec{x}, \vec{y}\rangle &= \langle x_1\vec{e}_1 + x_2\vec{e}_2 + \cdots + x_n\vec{e}_n, y_1\vec{e}_1 + y_2\vec{e}_2 + \cdots + y_n\vec{e}_n\rangle\\&= \sum_{i=1}^n x_i\langle \vec{e}_i, y_1\vec{e}_1 + y_2\vec{e}_2 + \cdots + y_n\vec{e}_n\rangle\\&= \sum_{i=1}^n \sum_{j=1}^n x_i y_j \langle \vec{e}_i, \vec{e}_j\rangle\end{aligned}$$

(b) We have

$$\begin{aligned}\langle \vec{x}, \vec{y}\rangle &= \sum_{i=1}^n \sum_{j=1}^n x_i y_j \langle \vec{e}_i, \vec{e}_j\rangle = \sum_{i=1}^n \sum_{j=1}^n x_i y_j g_{ij}\\&= \sum_{i=1}^n \sum_{j=1}^n x_i g_{ij} y_j = \langle \vec{x}, G\vec{y}\rangle = \vec{x}^T G \vec{y}\end{aligned}$$

(c) Since $\langle \vec{e}_i, \vec{e}_j\rangle = \langle \vec{e}_j, \vec{e}_i\rangle$, $g_{ij} = g_{ji}$ and so G is symmetric. Since $\langle \vec{x}, \vec{x}\rangle = \vec{x}^T G \vec{x} > 0$ for all non-zero $\vec{x}$, G is positive definite and has positive eigenvalues.

(d) Since G is symmetric, there exists an orthogonal matrix $P = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$ such that $P^T GP = D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ where the diagonal entries of D are the eigenvalues $\lambda_1, \ldots, \lambda_n$ of G. Then, $G = PDP^T$. Write $\tilde{\vec{x}} = P^T\vec{x}$ so that $\tilde{\vec{x}}^T = \vec{x}^T P$. It follows that

$$\begin{aligned}\langle \vec{x}, \vec{y}\rangle &= \vec{x}^T G\vec{y} \\ &= \vec{x}^T PDP^T\vec{y} \\ &= \tilde{\vec{x}}^T D\tilde{\vec{y}} \\ &= \lambda_1\tilde{x}_1\tilde{y}_1 + \lambda_2\tilde{x}_2\tilde{y}_2 + \cdots + \lambda_n\tilde{x}_n\tilde{y}_n\end{aligned}$$

(e) Note that since $\vec{x}^T G\vec{x} = \langle \vec{x}, \vec{x}\rangle > 0$ for all non-zero $\vec{x}$, G is positive definite and hence its eigenvalues are all positive. Thus, $\langle \vec{v}_i, \vec{v}_i\rangle = \lambda_i > 0$ and $\langle \vec{v}_i, \vec{v}_j\rangle = 0$. Now, let

$$\vec{w}_i = \frac{1}{\|\vec{v}_i\|}\vec{v}_i = \frac{1}{\sqrt{\lambda_i}}\vec{v}_i$$

so that $\langle \vec{w}_i, \vec{w}_i\rangle = 1$ and $\langle \vec{w}_i, \vec{w}_j\rangle = 0$. Thus it follows that $\{\vec{w}_1, \ldots, \vec{w}_n\}$ is an orthonormal basis with respect to $\langle\ ,\ \rangle$. Note that since $\vec{w}_i = \frac{1}{\sqrt{\lambda_i}}\vec{v}_i$, it also follows that $\vec{x}_i^* = \sqrt{\lambda_i}\tilde{\vec{x}}_i$. Hence,

$$< \vec{x}, \vec{y} >= x_1^* y_1^* + \cdots + x_n^* y_n^*$$

8.3 Graphs of Quadratic Forms

Practice Problems

A1 The quadratic form $Q(\vec{x}) = 2x_1^2 + 4x_1x_2 - x_2^2$ corresponds to the symmetric matrix $A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$, so the characteristic polynomial is

$$C(\lambda) = \begin{vmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = (\lambda - 3)(\lambda + 2)$$

Thus, the eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = -2$. Thus, by an orthogonal change of coordinates, the equation can be brought into the diagonal form

$$3y_1^2 - 2y_2^2 = 6$$

This is an equation of a hyperbola, and we can sketch the graph in the y_1y_2-plane. We observe that the y_1-intercepts are $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$, and there are no intercepts on the y_2-axis. The asymptotes of the hyperbola are determined by the equation $3y_1^2 - 2y_2^2 = 0$. By factoring, we determine that the asymptotes are lines with equations $y_2 = \pm\frac{\sqrt{3}}{\sqrt{2}}y_1$. With this information, we obtain the graph below.

To draw the graph of $2x_1^2 + 4x_1x_2 - x_2^2 = 6$ relative to the original x_1- and x_2-axis we need to find a basis for $\mathbb{R}^2$ of eigenvectors of A.

For $\lambda_1 = 3$,

$$A - 3I = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace is $\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$.

For $\lambda_2 = -2$,

$$A - (-2)I = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

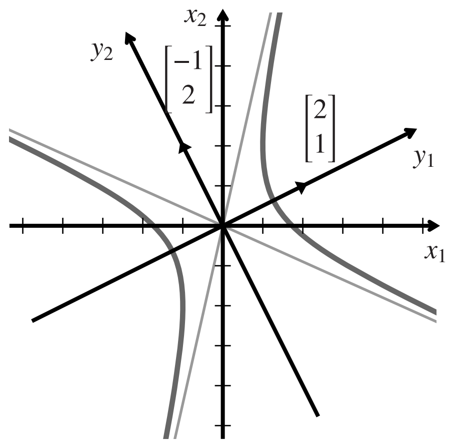
Thus, a basis for the eigenspace is $\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$.

Next, we convert the equations of the asymptotes from the y_1y_2-plane to the x_1x_2-plane by using the change of variables

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = P^T\vec{x} = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1/\sqrt{5} + x_2/\sqrt{5} \\ -x_1/\sqrt{5} + 2x_2/\sqrt{5} \end{bmatrix}$$

Substituting these into the equations of the asymptotes gives

$$-\frac{1}{\sqrt{5}}x_1 + \frac{2}{\sqrt{5}}x_2 = \pm\frac{\sqrt{3}}{\sqrt{2}}\left(\frac{2}{\sqrt{5}}x_1 + \frac{1}{\sqrt{5}}x_2\right)$$
$$-\sqrt{2}x_1 + 2\sqrt{2}x_2 = \pm 2\sqrt{3}x_1 \pm \sqrt{3}x_2$$
$$(2\sqrt{2} \mp \sqrt{3})x_2 = (\sqrt{2} \pm 2\sqrt{3})x_1$$
$$x_2 = \frac{\sqrt{2} \pm 2\sqrt{3}}{2\sqrt{2} \mp \sqrt{3}}x_1$$



Hence, the asymptotes are $x_2 \approx 4.45x_1$ and $x_2 \approx -0.445x_1$.

Now to sketch the graph of $2x_1^2 + 4x_1x_2 - x_2^2 = 6$ in the x_1x_2-plane, we draw the new y_1-axis in the direction of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and draw the new y_2-axis in the direction of $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Then, relative to these new axes, sketch the graph of the hyperbola $3y_1^2 - 2y_2^2 = 6$. The graph is also the graph of the original equation $2x_1^2 + 4x_1x_2 - x_2^2 = 6$. See the graph to the right:

A2 The quadratic form $Q(\vec{x}) = 2x_1^2 + 6x_1x_2 + 10x_2^2$ corresponds to the symmetric matrix $A = \begin{bmatrix} 2 & 3 \\ 3 & 10 \end{bmatrix}$, so the characteristic polynomial is

$$C(\lambda) = \begin{vmatrix} 2-\lambda & 3 \\ 3 & 10-\lambda \end{vmatrix} = (\lambda - 11)(\lambda - 1)$$

Thus, the eigenvalues of A are $\lambda_1 = 11$ and $\lambda_2 = 1$. Thus, by an orthogonal change of coordinates, the equation can be brought into the diagonal form

$$11y_1^2 + y_2^2 = 11$$

This is an equation of an ellipse and we can sketch the graph in the y_1y_2-plane. We observe that the y_1-intercepts are $(1, 0)$ and $(-1, 0)$, and the y_2-intercepts are $(0, \sqrt{11})$ and $(0, -\sqrt{11})$.

To draw the graph of $2x_1^2 + 6x_1x_2 + 10x_2^2 = 11$ relative to the original x_1- and x_2-axis we need to find a basis for $\mathbb{R}^2$ of eigenvectors of A.

For $\lambda_1 = 11$,

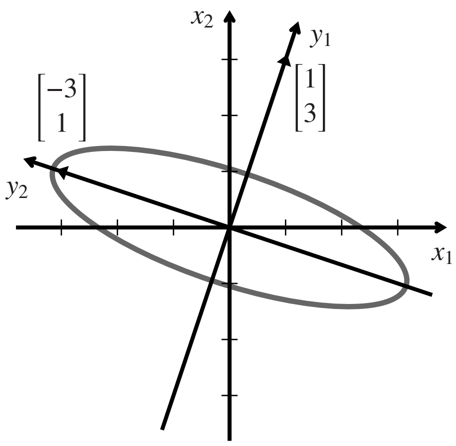
$$A - 11I = \begin{bmatrix} -9 & 3 \\ 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace is $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$.

For $\lambda_2 = 1$,

$$A - I = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace is $\left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\}$.



Now to sketch the graph of $2x_1^2 + 6x_1x_2 + 10x_2^2 = 11$ in the x_1x_2-plane, we draw the new y_1-axis in the direction of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and draw the new y_2-axis in the direction of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$. Then, relative to these new axes, sketch the graph of the ellipse $11y_1^2 + y_2^2 = 1$. The graph is also the graph of the original equation $2x_1^2 + 6x_1x_2 + 10x_2^2 = 11$. See the graph to the right:

A3 The quadratic form $Q(\vec{x}) = 4x_1^2 - 6x_1x_2 + 4x_2^2$ corresponds to the symmetric matrix $A = \begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix}$, so the characteristic polynomial is

$$C(\lambda) = \begin{vmatrix} 4-\lambda & -3 \\ -3 & 4-\lambda \end{vmatrix} = (\lambda - 7)(\lambda - 1)$$

Thus, the eigenvalues of A are $\lambda_1 = 7$ and $\lambda_2 = 1$. Thus, by an orthogonal change of coordinates, the equation can be brought into the diagonal form

$$7y_1^2 + y_2^2 = 12$$

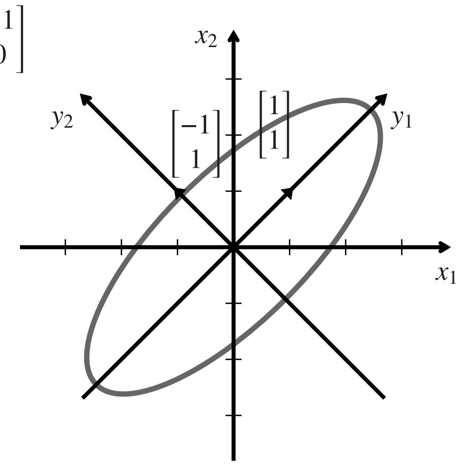
This is an equation of an ellipse and we can sketch the graph in the y_1y_2-plane. We observe that the y_1-intercepts are $(\sqrt{12/7}, 0)$ and $(-\sqrt{12/7}, 0)$, and the y_2-intercepts are $(0, \sqrt{12})$ and $(0, -\sqrt{12})$.
To draw the graph of $4x_1^2 - 6x_1x_2 + 4x_2^2 = 12$ relative to the original x_1- and x_2-axis we need to find a basis for $\mathbb{R}^2$ of eigenvectors of A.
For $\lambda_1 = 7$,

$$A - 7I = \begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace is $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.
For $\lambda_2 = 1$,

$$A - I = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.



Now to sketch the graph of $4x_1^2 - 6x_1x_2 + 4x_2^2 = 12$ in the x_1x_2-plane, we draw the new y_1-axis in the direction of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and draw the new y_2-axis in the direction of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then, relative to these new axes, sketch the graph of the ellipse $7y_1^2 + y_2^2 = 1$. The graph is also the graph of the original equation $4x_1^2 - 6x_1x_2 + 4x_2^2 = 12$. See the graph to the right:

A4 The quadratic form $Q(\vec{x}) = 5x_1^2 + 6x_1x_2 - 3x_2^2$ corresponds to the symmetric matrix $A = \begin{bmatrix} 5 & 3 \\ 3 & -3 \end{bmatrix}$, so the characteristic polynomial is

$$C(\lambda) = \begin{vmatrix} 5-\lambda & 3 \\ 3 & -3-\lambda \end{vmatrix} = (\lambda - 6)(\lambda + 4)$$

Thus, the eigenvalues of A are $\lambda_1 = 6$ and $\lambda_2 = -4$. Thus, by an orthogonal change of coordinates, the equation can be brought into the diagonal form

$$6y_1^2 - 4y_2^2 = 15$$

This is an equation of a hyperbola, and we can sketch the graph in the y_1y_2-plane. We observe that the y_1-intercepts are $(\sqrt{5/2}, 0)$ and $(-\sqrt{5/2}, 0)$, and there are no intercepts on the y_2-axis. The asymptotes of the hyperbola are determined by the equation $6y_1^2 - 4y_2^2 = 0$. By factoring, we determine that the asymptotes are lines with equations $y_2 = \pm\frac{\sqrt{3}}{\sqrt{2}}y_1$. With this information, we obtain the graph below.

To draw the graph of $5x_1^2 + 6x_1x_2 - 3x_2^2 = 15$ relative to the original x_1- and x_2-axis we need to find a basis for $\mathbb{R}^2$ of eigenvectors of A.

For $\lambda_1 = 6$,

$$A - 6I = \begin{bmatrix} -1 & 3 \\ 3 & -9 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace is $\left\{\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\}$.

For $\lambda_2 = -4$,

$$A - (-4)I = \begin{bmatrix} 9 & 3 \\ 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/3 \\ 0 & 0 \end{bmatrix}$$

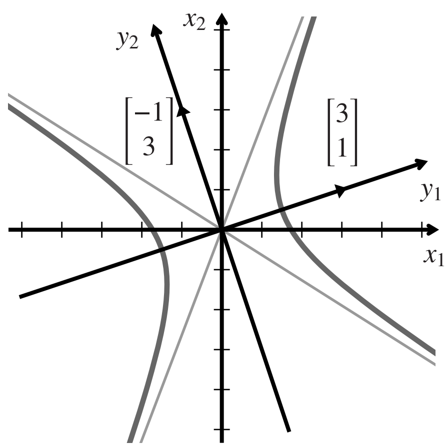
Thus, a basis for the eigenspace is $\left\{\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right\}$.

Next, we convert the equations of the asymptotes from the y_1y_2-plane to the x_1x_2-plane by using the change of variables

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = P^T\vec{x} = \begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ -1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1/\sqrt{10} + x_2/\sqrt{10} \\ -x_1/\sqrt{10} + 3x_2/\sqrt{10} \end{bmatrix}$$

Substituting these into the equations of the asymptotes gives

$$\begin{aligned} -\frac{1}{\sqrt{10}}x_1 + \frac{3}{\sqrt{10}}x_2 &= \pm\frac{\sqrt{3}}{\sqrt{2}}\left(\frac{3}{\sqrt{10}}x_1 + \frac{1}{\sqrt{10}}x_2\right) \\ -\sqrt{2}x_1 + 3\sqrt{2}x_2 &= \pm 3\sqrt{3}x_1 \pm \sqrt{3}x_2 \\ (3\sqrt{2} \mp \sqrt{3})x_2 &= (\sqrt{2} \pm 3\sqrt{3})x_1 \\ x_2 &= \frac{\sqrt{2} \pm 3\sqrt{3}}{3\sqrt{2} \mp \sqrt{3}}x_1 \end{aligned}$$



Hence, the asymptotes are $x_2 \approx 2.633x_1$ and $x_2 \approx -0.633x_1$.

Now to sketch the graph of $5x_1^2 + 6x_1x_2 - 3x_2^2 = 15$ in the x_1x_2-plane, we draw the new y_1-axis in the direction of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and draw the new y_2-axis in the direction of $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Then, relative to these new axes, sketch the graph of the hyperbola $6y_1^2 - 4y_2^2 = 15$. The graph is also the graph of the original equation $5x_1^2 + 6x_1x_2 - 3x_2^2 = 15$. See the graph to the right:

A5 (a) We have

$$C(\lambda) = \begin{vmatrix} 4-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = \lambda(\lambda - 5)$$

Thus, the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 5$. Thus, $Q(\vec{x}) = \vec{x}^T A\vec{x}$ has diagonal form $Q(\vec{x}) = 5y_2^2$. Therefore, the graph of $\vec{x}^T A\vec{x} = 1$ is a rotation of the graph of $5y_2^2 = 1$ which is the graph of the two parallel lines $y_2 = \pm\frac{1}{\sqrt{5}}$. The graph of $\vec{x}^T A\vec{x} = -1$ is a rotation of the graph of $5y_2^2 = -1$ which is empty.

(b) We have

$$C(\lambda) = \begin{vmatrix} 5-\lambda & 3 \\ 3 & -3-\lambda \end{vmatrix} = (\lambda - 6)(\lambda + 4)$$

Thus, the eigenvalues are $\lambda_1 = 6$ and $\lambda_2 = -4$. Thus, $Q(\vec{x}) = \vec{x}^T A\vec{x}$ has diagonal form $Q(\vec{x}) = 6y_1^2 - 4y_2^2$. Therefore, the graph of $\vec{x}^T A\vec{x} = 1$ is a rotation of the graph of $6y_1^2 - 4y_2^2 = 1$ which is a hyperbola opening in the y_1-direction. The graph of $\vec{x}^T A\vec{x} = -1$ is a rotation of the graph of $6y_1^2 - 4y_2^2 = -1$ which is a hyperbola opening in the y_2-direction.

(c) We have

$$C(\lambda) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -(\lambda - 2)(\lambda + 1)^2$$

Thus, the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = -1$, and $\lambda_3 = -1$. Thus, $Q(\vec{x}) = \vec{x}^T A\vec{x}$ has diagonal form $Q(\vec{x}) = 2y_1^2 - y_2^2 - y_3^2$. Therefore, the graph of $\vec{x}^T A\vec{x} = 1$ is a rotation of the graph of $2y_1^2 - y_2^2 - y_3^2 = 1$ which is a hyperboloid of two sheets. The graph of $\vec{x}^T A\vec{x} = -1$ is a rotation of the graph of $2y_1^2 - y_2^2 - y_3^2 = -1$ which is a hyperboloid of one sheet.

(d) We have

$$C(\lambda) = \begin{vmatrix} 1-\lambda & 0 & -2 \\ 0 & -1-\lambda & -2 \\ -2 & -2 & -\lambda \end{vmatrix} = -\lambda(\lambda - 3)(\lambda + 3)$$

Thus, the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 3$, and $\lambda_3 = -3$. Thus, $Q(\vec{x}) = \vec{x}^T A\vec{x}$ has diagonal form $Q(\vec{x}) = 3y_2^2 - 3y_3^2$. Therefore, the graph of $\vec{x}^T A\vec{x} = 1$ is a rotation of the graph of $3y_2^2 - 3y_3^2 = 1$ which is a hyperbolic cylinder. The graph of $\vec{x}^T A\vec{x} = -1$ is a rotation of the graph of $3y_2^2 - 3y_3^2 = -1$ which is a hyperbolic cylinder.

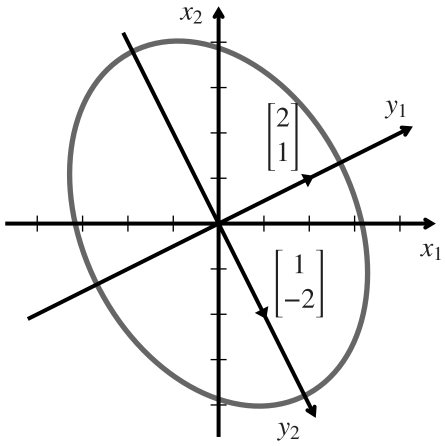
(e) We have

$$C(\lambda) = \begin{vmatrix} 1-\lambda & 8 & 4 \\ 8 & 1-\lambda & -4 \\ 4 & -4 & 7-\lambda \end{vmatrix} = -(\lambda - 9)^2(\lambda + 9)$$

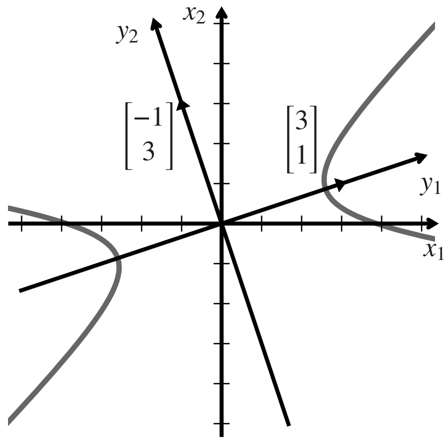
Thus, the eigenvalues are $\lambda_1 = 9$, $\lambda_2 = 9$, and $\lambda_3 = -9$. Thus, $Q(\vec{x}) = \vec{x}^T A\vec{x}$ has diagonal form $Q(\vec{x}) = 9y_1^2 + 9y_2^2 - 9y_3^2$. Therefore, the graph of $\vec{x}^T A\vec{x} = 1$ is a rotation of the graph of $9y_1^2 + 9y_2^2 - 9y_3^2 = 1$ which is a hyperboloid of one sheet. The graph of $\vec{x}^T A\vec{x} = -1$ is a rotation of the graph of $9y_1^2 + 9y_2^2 - 9y_3^2 = -1$ which is a hyperboloid of two sheets.

Homework Problems

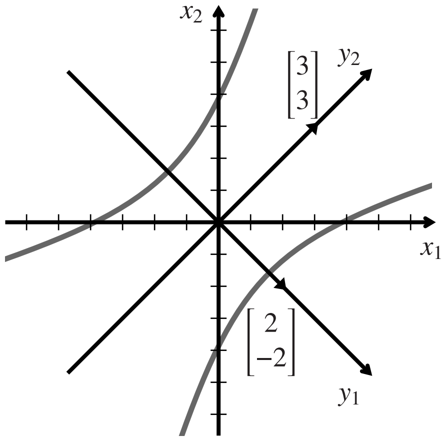
B1



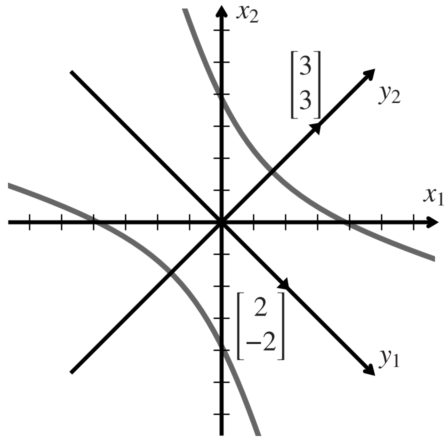
B2



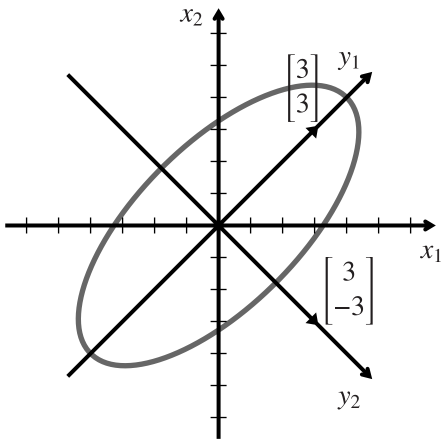
B3



B4



B5



B6 (a) The graph of $\vec{x}^T A\vec{x} = 1$ is a hyperbola. The graph of $\vec{x}^T A\vec{x} = 0$ is intersecting lines. The graph of $\vec{x}^T A\vec{x} = -1$ is a hyperbola.

(b) The graph of $\vec{x}^T A\vec{x} = 1$ is a hyperboloid of one sheet. The graph of $\vec{x}^T A\vec{x} = 0$ is a cone. The graph of

$\vec{x}^T A\vec{x} = -1$ is a hyperboloid of two sheets.

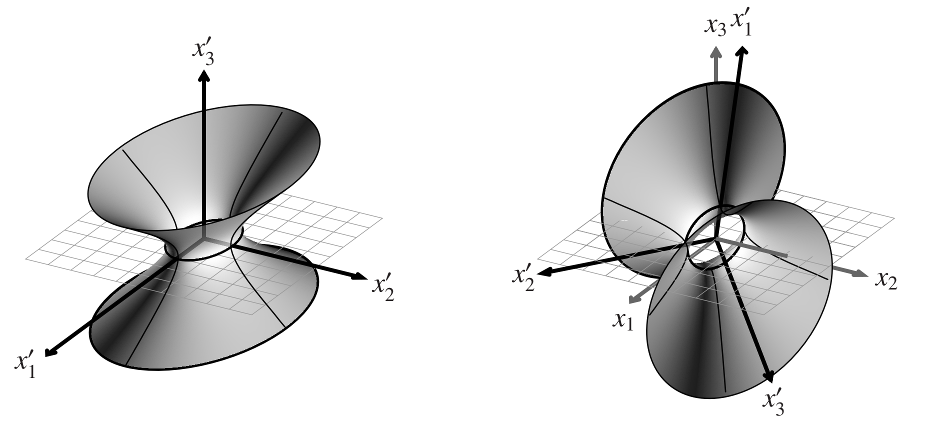
(c) The graph of $\vec{x}^T A\vec{x} = 1$ is an elliptic cylinder. The graph of $\vec{x}^T A\vec{x} = 0$ degenerates to the x_3-axis. The graph of $\vec{x}^T A\vec{x} = -1$ is the empty set.

(d) The graph of $\vec{x}^T A\vec{x} = 1$ is a hyperbolic cylinder. The graph of $\vec{x}^T A\vec{x} = 0$ degenerates to a pair of planes intersecting along the x_2-axis. The graph of $\vec{x}^T A\vec{x} = -1$ is a hyperbolic cylinder.

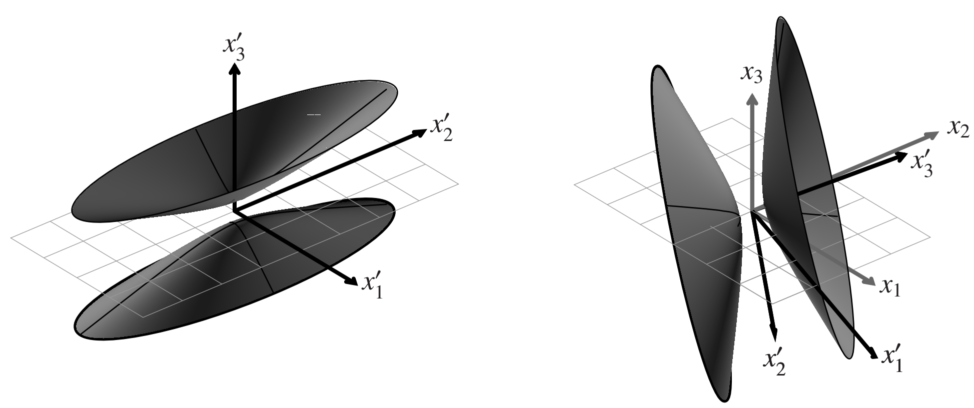
(e) The graph of $\vec{x}^T A\vec{x} = 1$ is an ellipsoid. The graph of $\vec{x}^T A\vec{x} = 0$ is just the origin. The graph of $\vec{x}^T A\vec{x} = -1$ is the empty set.

Computer Problems

C1 (a) The graph is a hyperboloid of one sheet.



(b) The graph is a hyperboloid of two sheets.



8.4 Applications of Quadratic Forms

Conceptual Problems

D1 Let $D = \text{diag}(\epsilon_1, \epsilon_2, \epsilon_3)$. Then $P^T(\beta E)P = D$. It follows that

$$P^T(I + \beta E)P = (P^T I + P^T \beta E)P = P^T P + P^T \beta E P = I + D$$

and $I + D = \text{diag}(1 + \epsilon_1, 1 + \epsilon_2, 1 + \epsilon_3)$ as required.

Chapter 8 Quiz

Problems

E1 We have

$$C(\lambda) = \begin{vmatrix} 2-\lambda & -3 & 2 \\ -3 & 3-\lambda & 3 \\ 2 & 3 & 2-\lambda \end{vmatrix} = -(\lambda - 6)(\lambda + 3)(\lambda - 4)$$

The eigenvalues are $\lambda_1 = 6$, $\lambda_2 = -3$, and $\lambda_3 = 4$.
For $\lambda_1 = 6$ we get

$$A - 6I = \begin{bmatrix} -4 & -3 & 2 \\ -3 & -3 & 3 \\ 2 & 3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for the eigenspace of λ_1 is $\left\{ \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \right\}$.
For $\lambda_2 = -3$ we get

$$A - (-3)I = \begin{bmatrix} 5 & -3 & 2 \\ -3 & 6 & 3 \\ 2 & 3 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for the eigenspace of λ_2 is $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$.
For $\lambda_3 = 4$ we get

$$A - 4I = \begin{bmatrix} -2 & -3 & 2 \\ -3 & -1 & 3 \\ 2 & 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for the eigenspace of λ_3 is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

We normalize the basis vectors for the eigenspaces to get the orthonormal basis for $\mathbb{R}^2$ of eigenvectors of A $\left\{\begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}\right\}$. Hence, $P = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix}$ orthogonally diagonalizes A to $P^TAP = \begin{bmatrix} 6 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

E2 (a) We have

i. The corresponding symmetric matrix is $A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$.

ii. The characteristic polynomial is

$$C(\lambda) = \begin{vmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = (\lambda-3)(\lambda-7)$$

Thus, the eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 7$.
For $\lambda_1 = 3$,

$$A - 3I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

A basis for the eigenspace is $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$.
For $\lambda_2 = 7$,

$$A - 7I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

A basis for the eigenspace is $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

So, after normalizing the vectors, we find that $P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ orthogonally diagonalizes A. Hence, the change of variables $\vec{x} = P\vec{y}$ brings $Q(\vec{x})$ into the form $Q(\vec{x}) = 3y_1^2 + 7y_2^2$.

iii. Since the eigenvalues of A are all positive, $Q(\vec{x})$ is positive definite.

iv. The graph of $Q(\vec{x}) = 1$ is a rotation of the graph of $3y_1^2 + 7y_2^2 = 1$, so it is an ellipse. The graph of $Q(\vec{x}) = 0$ is a rotation of $3y_1^2 + 7y_2^2 = 0$, which is a single point at the origin.

(b) We have

i. The corresponding symmetric matrix is $A = \begin{bmatrix} 2 & -3 & -3 \\ -3 & -3 & 2 \\ -3 & 2 & -3 \end{bmatrix}$.

ii. The characteristic polynomial is

$$C(\lambda) = \begin{vmatrix} 2-\lambda & -3 & -3 \\ -3 & -3-\lambda & 2 \\ -3 & 2 & -3-\lambda \end{vmatrix} = -(\lambda+5)(\lambda-5)(\lambda+4)$$

Thus, the eigenvalues of A are $\lambda_1 = -5$, $\lambda_2 = 5$, and $\lambda_3 = -4$.

For $\lambda_1 = -5$,

$$A - (-5)I = \begin{bmatrix} 7 & -3 & -3 \\ -3 & 2 & 2 \\ -3 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for the eigenspace is $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

For $\lambda_2 = 5$,

$$A - 5I = \begin{bmatrix} -3 & -3 & -3 \\ -3 & -8 & 2 \\ -3 & 2 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for the eigenspace is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right\}$.

For $\lambda_3 = -4$,

$$A - (-4)I = \begin{bmatrix} 6 & -3 & -3 \\ -3 & 1 & 2 \\ -3 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for the eigenspace is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

So, after normalizing the vectors, we find that $P = \begin{bmatrix} 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$ orthogonally diagonalizes A. Hence, the change of variables $\vec{x} = P\vec{y}$ brings $Q(\vec{x})$ into the form $Q(\vec{x}) = -5y_1^2 + 5y_2^2 - 4y_3^2$.

iii. Since A has positive and negative eigenvalues $Q(\vec{x})$ is indefinite.

iv. The graph of $Q(\vec{x}) = 1$ is a rotation of the graph of $-5y_1^2 + 5y_2^2 - 4y_3^2 = 1$, so it is a hyperboloid of two sheets. The graph of $Q(\vec{x}) = 0$ is a rotation of $-5y_1^2 + 5y_2^2 - 4y_3^2 = 0$, which is a cone.

E3 The quadratic form $Q(\vec{x}) = 5x_1^2 - 2x_1x_2 + 5x_2^2$ corresponds to the symmetric matrix $A = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$.

The characteristic polynomial is

$$C(\lambda) = \begin{vmatrix} 5-\lambda & -1 \\ -1 & 5-\lambda \end{vmatrix} = (\lambda - 6)(\lambda - 4)$$

Thus, the eigenvalues of A are $\lambda_1 = 6$ and $\lambda_2 = 4$.

For $\lambda_1 = 6$,

$$A - 6I = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

A basis for the eigenspace is $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

For $\lambda_2 = 4$,

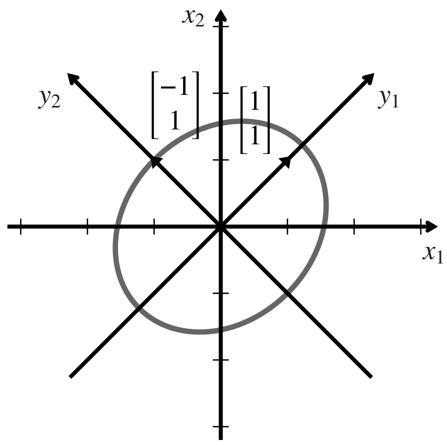
$$A - 4I = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

A basis for the eigenspace is $\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$.

So, after normalizing the vectors, we find that $P = \begin{bmatrix}-1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$ orthogonally diagonalizes A. Hence, the change of variables $\vec{x} = P\vec{y}$ brings $Q(\vec{x})$ into the form $Q(\vec{x}) = 6y_1^2 + 4y_2^2$.

Now, the graph of $6y_1^2 + 4y_2^2 = 12$ is an ellipse in the y_1y_2-plane. We observe that the y_1-intercepts are $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$, and the y_2-intercepts are $(0, \sqrt{3})$ and $(0, -\sqrt{3})$.

Now to sketch the graph of $5x_1^2 - 2x_1x_2 + 5x_2^2 = 12$ in the x_1x_2-plane, we draw the new y_1-axis in the direction of $\begin{bmatrix}-1\\1\end{bmatrix}$ and draw the new y_2-axis in the direction of $\begin{bmatrix}1\\1\end{bmatrix}$. Then, relative to these new axes, sketch the graph of the ellipse $6y_1^2 + 4y_2^2 = 1$. The graph is also the graph of the original equation $5x_1^2 - 2x_1x_2 + 5x_2^2 = 12$. See the graph below:



E4 Since A is positive definite, we have that

$$\langle \vec{x}, \vec{x} \rangle = \vec{x}^T A \vec{x} \geq 0$$

and $\langle \vec{x}, \vec{x} \rangle = 0$ if and only if $\vec{x} = \vec{0}$.

Since A is symmetric, we have that

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T A \vec{y} = (\vec{x}^T A \vec{y})^T = \vec{y}^T A^T \vec{x} = \vec{y}^T A \vec{x} = \langle \vec{y}, \vec{x} \rangle$$

For any $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$ we have

$$\begin{aligned}\langle \vec{x}, s\vec{y} + t\vec{z} \rangle &= \vec{x}^T A(s\vec{y} + t\vec{z}) = \vec{x}^T A(s\vec{y}) + \vec{x}^T A(t\vec{z})\\ &= s\vec{x}^T A\vec{y} + t\vec{x}^T A\vec{z} = s\langle \vec{x}, \vec{y} \rangle + t\langle \vec{x}, \vec{z} \rangle\end{aligned}$$

Thus, $\langle \vec{x}, \vec{y} \rangle$ is an inner product on $\mathbb{R}^n$.

E5 Since A is a 4×4 symmetric matrix, there exists an orthogonal matrix P that diagonalizes A. Since the only eigenvalue of A is 3, we must have $P^T AP = 3I$. Then multiply on the left by P and on the right by P^T and we get

$$A = P(3I)P^T = 3PP^T = 3I$$

Chapter 8 Further Problems

Problems

F1 Given $A = QR$, with Q orthogonal, let $A_1 = RQ$. Then $Q^T A = R$, so $A_1 = (Q^T A)Q$, and A_1 is orthogonally similar to A.

F2 Since A is symmetric, there is an orthogonal matrix Q such that $Q^T AQ = D = \text{diag}(\lambda_1, \dots, \lambda_n)$, where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A. Moreover, since A is positive semidefinite, all of the eigenvalues of A are non-negative. Define $C = \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$ and let $B = QCQ^T$. Then,

$$B^2 = (QCQ^T)(QCQ^T) = QC^2Q^T = QDQ^T = A$$

and

$$B^T = (QCQ^T)^T = QC^TQ^T = QCQ^T = B$$

Moreover, since B is similar to C, we have that the eigenvalues of B are $\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}$ and hence B is also positive semidefinite.

F3 (a) We have $(A^T A)^T = A^T(A^T)^T = A^T A$, so $A^T A$ is symmetric.

Let λ be any eigenvalue of $A^T A$ with corresponding unit eigenvector $\vec{x}$. Since, $A^T A$ is symmetric, we have that λ is real. Also,

$$\langle A\vec{x}, A\vec{x}\rangle = (A\vec{x})^T(A\vec{x}) = \vec{x}^T A^T A\vec{x} = \vec{x}^T(\lambda\vec{x}) = \lambda\vec{x}^T\vec{x} = \lambda$$

Thus, $\lambda \geq 0$.

(b) If A is invertible, we can repeat our work in (a) to get $\lambda = \langle A\vec{x}, A\vec{x}\rangle$. But, since $\vec{x}$ is an eigenvector, we have that $\vec{x} \neq \vec{0}$, so $A\vec{x} \neq \vec{0}$ since A is invertible. Thus, $\lambda > 0$.

F4 (a) By Problem F3, $A^T A$ is positive definite. Let U be the symmetric positive square root of $A^T A$, as defined in Problem F2, so $U^2 = A^T A$. Let $Q = AU^{-1}$. Then, because U is symmetric,

$$Q^T Q = (AU^{-1})^T AU^{-1} = (U^T)^{-1} A^T AU^{-1} = U^{-1}U^2U^{-1} = I$$

and Q is an orthogonal matrix. Since $Q = AU^{-1}$, $A = QU$, as required.

(b) $V^T = (QUQ^T)^T = QU^TQ^T = QUQ^T = V$, so V is symmetric. Also, $VQ = (QUQ^T)Q = QUI = A$. Finally, $AA^T = VQQ^TV^T = VIV = V^2$.

(c) Write $A = QU$, as in part (a). Then the symmetric positive definite matrix U can be diagonalized, and the diagonal matrix can then be factored so that with respect to a suitable orthonormal basis, the matrix is orthogonally similar to

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

and we see that it is the composition of three stretches along mutually orthogonal principal axes. This map is orientation-preserving, and we have assume that A is the matrix of an orientation-preserving mapping, so that $Q = AU^{-1}$ is an orientation-preserving mapping. Thus, by Problem F1 of Chapter 7 Review, Q describes a rotation.

CHAPTER 9 Complex Vector Spaces

9.1 Complex Numbers

Practice Problems

A1 (a) $(2+5i)+(3+2i) = 5+7i$

(b) $(2-7i)+(-5+3i) = -3-4i$

(c) $(-3+5i)-(4+3i) = -7+2i$

(d) $(-5-6i)-(9-11i) = -14+5i$

A2 (a) $(1+3i)(3-2i) = 3-2i+9i+6 = 9+7i$

(b) $(-2-4i)(3-i) = -6+2i-12i-4 = -10-10i$

(c) $(1-6i)(-4+i) = -4+i+24i+6 = 2+25i$

(d) $(-1-i)(1-i) = -1+i-i-1 = -2$

A3 (a) $\overline{3-5i} = 3+5i$

(b) $\overline{2+7i} = 2-7i$

(c) $\overline{3} = 3$

(d) $\overline{-4i} = 4i$

A4 (a) We have $\mathrm{Re}(z) = 3$ and $\mathrm{Im}(z) = -6$.

(b) We have $(2+5i)(1-3i) = 2-6i+5i+15 = 17-i$, so $\mathrm{Re}(z) = 17$ and $\mathrm{Im}(z) = -1$.

(c) We have $\frac{4}{6-i} \cdot \frac{6+i}{6+i} = \frac{24+4i}{36+1} = \frac{24}{37} + \frac{4}{37}i$. Hence, $\mathrm{Re}(z) = 24/37$ and $\mathrm{Im}(z) = 4/37$.

(d) We have $\frac{-1}{i} \cdot \frac{i}{i} = \frac{-i}{-1} = i$. Thus, $\mathrm{Re}(z) = 0$ and $\mathrm{Im}(z) = 1$.

A5 (a) We have $\frac{1}{2+3i} \cdot \frac{2-3i}{2-3i} = \frac{2-3i}{4+9} = \frac{2}{13} - \frac{3}{13}i$.

(b) We have $\frac{3}{2-7i} \cdot \frac{2+7i}{2+7i} = \frac{6+21i}{4+49} = \frac{6}{53} + \frac{21}{53}i$.

(c) We have $\frac{2-5i}{3+2i} \cdot \frac{3-2i}{3-2i} = \frac{6-10-4i-15i}{9+4} = -\frac{4}{13} - \frac{19}{13}i$.

(d) We have $\frac{1+6i}{4-i} \cdot \frac{4+i}{4+i} = \frac{4-6+i+24i}{16+1} = -\frac{2}{17} + \frac{25}{17}i$.

A6 (a) We have $|z_1| = \sqrt{1^2 + 1^2} = \sqrt{2}$ and any argument θ of z_1 satisfies

$$1 = \sqrt{2}\cos\theta \quad \text{and} \quad 1 = \sqrt{2}\sin\theta$$

Hence, $\cos\theta = \frac{1}{\sqrt{2}}$ and $\sin\theta = \frac{1}{\sqrt{2}}$. Thus, $\theta = \frac{\pi}{4} + 2\pi k$, $k \in \mathbb{Z}$. So, a polar form of z_1 is

$$z_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

We have $|z_2| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$ and any argument θ of z_2 satisfies

$$1 = 2\cos\theta \quad \text{and} \quad \sqrt{3} = 2\sin\theta$$

Hence, $\cos\theta = \frac{1}{2}$ and $\sin\theta = \frac{\sqrt{3}}{2}$. Thus, $\theta = \frac{\pi}{3} + 2\pi k$, $k \in \mathbb{Z}$. So, a polar form of z_2 is

$$z_2 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

Thus,

$$\begin{aligned}
z_1 z_2 &= 2\sqrt{2}\left(\cos\left(\frac{\pi}{4} + \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right)\right) \\
&= 2\sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right) \\
\frac{z_1}{z_2} &= \frac{\sqrt{2}}{2}\left(\cos\left(\frac{\pi}{4} - \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{4} - \frac{\pi}{3}\right)\right) \\
&= \frac{\sqrt{2}}{2}\left(\cos\frac{-\pi}{12} + i\sin\frac{-\pi}{12}\right)
\end{aligned}$$

(b) We have $|z_1| = \sqrt{(-\sqrt{3})^2 + (-1)^2} = 2$ and any argument θ of z_1 satisfies

$$-\sqrt{3} = 2\cos\theta \quad \text{and} \quad -1 = 2\sin\theta$$

Hence, $\cos\theta = -\frac{\sqrt{3}}{2}$ and $\sin\theta = -\frac{1}{2}$. Thus, $\theta = \frac{7\pi}{6} + 2\pi k$, $k \in \mathbb{Z}$. So, a polar form of z_1 is

$$z_1 = 2\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$$

We have $|z_2| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$ and any argument θ of z_2 satisfies

$$1 = \sqrt{2}\cos\theta \quad \text{and} \quad -1 = \sqrt{2}\sin\theta$$

Hence, $\cos\theta = \frac{1}{\sqrt{2}}$ and $\sin\theta = -\frac{1}{\sqrt{2}}$. Thus, $\theta = -\frac{\pi}{4} + 2\pi k$, $k \in \mathbb{Z}$. So, a polar form of z_2 is

$$z_2 = \sqrt{2}\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right)$$

Thus,

$$\begin{aligned}
z_1z_2 &= 2\sqrt{2}\left(\cos\left(\frac{7\pi}{6}-\frac{\pi}{4}\right)+i\sin\left(\frac{7\pi}{6}-\frac{\pi}{4}\right)\right)\\
&= 2\sqrt{2}\left(\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}\right)\\
\frac{z_1}{z_2} &= \frac{2}{\sqrt{2}}\left(\cos\left(\frac{7\pi}{6}-\frac{-\pi}{4}\right)+i\sin\left(\frac{7\pi}{6}-\frac{-\pi}{4}\right)\right)\\
&= \frac{2}{\sqrt{2}}\left(\cos\frac{17\pi}{12}+i\sin\frac{17\pi}{12}\right)
\end{aligned}$$

(c) We have $|z_1| = \sqrt{1^2+2^2} = \sqrt{5}$ and any argument θ of z_1 satisfies

$$1 = \sqrt{5}\cos\theta \quad \text{and} \quad 2 = \sqrt{5}\sin\theta$$

Hence, $\cos\theta = \frac{1}{\sqrt{5}}$ and $\sin\theta = \frac{2}{\sqrt{5}}$. So, one value of θ is $\theta_1 = \arctan(2) \approx 1.10715$ rads. Hence, a polar form of z_1 is

$$z_1 = \sqrt{5}\,(\cos\theta_1 + i\sin\theta_1)$$

We have $|z_2| = \sqrt{(-2)^2+(-3)^2} = \sqrt{13}$ and any argument θ of z_2 satisfies

$$-2 = \sqrt{13}\cos\theta \quad \text{and} \quad -3 = \sqrt{13}\sin\theta$$

Hence, $\cos\theta = -\frac{2}{\sqrt{13}}$ and $\sin\theta = -\frac{3}{\sqrt{13}}$. Thus, one value of θ is $\theta_2 = \arctan(3/2)+\pi \approx 0.98279+\pi = 4.124$. So, a polar form of z_2 is

$$z_2 = \sqrt{2}\,(\cos\theta_2 + i\sin\theta_2)$$

Thus,

$$\begin{aligned}
z_1z_2 &= \sqrt{5}\sqrt{13}\,(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2))\\
&= \sqrt{65}\,(\cos 5.23153 + i\sin 5.23153) = 4-7i\\
\frac{z_1}{z_2} &= \frac{\sqrt{5}}{\sqrt{13}}\,(\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2))\\
&= \frac{\sqrt{5}}{\sqrt{13}}\,(\cos -3.01723 + i\sin -3.01723) = -\frac{8}{13}-\frac{1}{13}i
\end{aligned}$$

These answers can be checked using Cartesian form.

(d) We have $|z_1| = \sqrt{(-3)^2+1^2} = \sqrt{10}$ and any argument θ of z_1 satisfies

$$-3 = \sqrt{10}\cos\theta \quad \text{and} \quad 1 = \sqrt{10}\sin\theta$$

Hence, $\cos\theta = -\frac{3}{\sqrt{10}}$ and $\sin\theta = \frac{1}{\sqrt{10}}$. So, one value of θ is $\theta_1 = \arctan(-1/3) \approx 2.81984$ rads. Hence, a polar form of z_1 is

$$z_1 = \sqrt{10}\,(\cos\theta_1 + i\sin\theta_1)$$

We have $|z_2| = \sqrt{6^2+(-1)^2} = \sqrt{37}$ and any argument θ of z_2 satisfies

$$6 = \sqrt{37}\cos\theta \quad \text{and} \quad -1 = \sqrt{37}\sin\theta$$

Hence, $\cos\theta = \frac{6}{\sqrt{37}}$ and $\sin\theta = -\frac{1}{\sqrt{37}}$. Thus, one value of θ is $\theta_2 = \arctan(-1/6) \approx -0.16515$. So, a polar form of z_2 is

$$z_2 = \sqrt{2}\,(\cos\theta_2 + i\sin\theta_2)$$

Thus,

$$\begin{aligned} z_1z_2 &= \sqrt{10}\sqrt{37}\,(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) \\ &= \sqrt{370}\,(\cos 2.65469 + i\sin 2.65469) = -17+9i \\ \frac{z_1}{z_2} &= \frac{\sqrt{10}}{\sqrt{37}}\,(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)) \\ &= \frac{\sqrt{10}}{\sqrt{37}}\,(\cos 2.98499 + i\sin 2.98499) = -\frac{9}{37} + \frac{3}{37}i \end{aligned}$$

These answers can be checked using Cartesian form.

A7 (a) We have $1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, so

$$\begin{aligned} (1+i)^4 &= \left[\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^4 \\ &= (\sqrt{2})^4\left[\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right]^4 \\ &= 4\left(\cos\frac{4\pi}{4} + i\sin\frac{4\pi}{4}\right) \\ &= 4(-1+0i) = -4 \end{aligned}$$

(b) We have $3-3i = 3\sqrt{2}\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right)$, so

$$\begin{aligned} (3-3i)^3 &= \left[3\sqrt{2}\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right)\right]^3 \\ &= (3\sqrt{2})^3\left[\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right)\right]^3 \\ &= 54\sqrt{2}\left(\cos\frac{-3\pi}{4} + i\sin\frac{-3\pi}{4}\right) \\ &= 54\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = -54-54i \end{aligned}$$

(c) We have $-1-\sqrt{3}i = 2\left(\cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3}\right)$, so

$$\begin{aligned}(-1-\sqrt{3}i)^4 &= \left[2\left(\cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3}\right)\right]^4\\ &= 2^4\left[\left(\cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3}\right)\right]^4\\ &= 16\left(\cos\frac{-8\pi}{3} + i\sin\frac{-8\pi}{3}\right)\\ &= 16\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -8 - 8\sqrt{3}i\end{aligned}$$

(d) We have $-2\sqrt{3} + 2i = 4\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$, so

$$\begin{aligned}(-2\sqrt{3}+2i)^5 &= \left[2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)\right]^5\\ &= 4^5\left[\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)\right]^5\\ &= 1024\left(\cos\frac{25\pi}{6} + i\sin\frac{25\pi}{6}\right)\\ &= 1024\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = 512(\sqrt{3}+i)\end{aligned}$$

A8 (a) We have $-1 = 1e^{i(\pi+2\pi k)}$. Thus, the fifth roots are

$$(1)^{1/5}e^{i(\pi+2\pi k)/5}, \quad k = 0, 1, 2, 3, 4$$

Thus, the five fifth roots are

$$\begin{aligned}w_0 &= \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\\ w_1 &= \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\\ w_2 &= \cos\pi + i\sin\pi\\ w_3 &= \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}\\ w_4 &= \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}\end{aligned}$$

(b) We have $-16i = 16e^{i(-\frac{\pi}{2}+2\pi k)}$. Thus, the fourth roots are

$$(16)^{1/4}e^{i(-\frac{\pi}{2}+2\pi k)/4}, \quad k = 0, 1, 2, 3$$

Thus, the four fourth roots are

$$
\begin{aligned}
w_0 &= 2\cos\frac{-\pi}{8} + i2\sin\frac{-\pi}{8}\\
w_1 &= 2\cos\frac{3\pi}{8} + i2\sin\frac{3\pi}{8}\\
w_2 &= 2\cos\frac{7\pi}{8} + i2\sin\frac{7\pi}{8}\\
w_3 &= 2\cos\frac{11\pi}{8} + i2\sin\frac{11\pi}{8}
\end{aligned}
$$

(c) We have $-\sqrt{3} - i = 2e^{i(-\frac{5\pi}{6}+2\pi k)}$. Thus, the third roots are

$$(2)^{1/3}e^{i(-\frac{5\pi}{6}+2\pi k)/3}, \quad k = 0, 1, 2$$

Thus, the three third roots are

$$
\begin{aligned}
w_0 &= 2^{1/3}\cos\frac{-5\pi}{18} + i2^{1/3}\sin\frac{-5\pi}{18}\\
w_1 &= 2^{1/3}\cos\frac{7\pi}{18} + i2^{1/3}\sin\frac{7\pi}{18}\\
w_2 &= 2^{1/3}\cos\frac{19\pi}{18} + i2^{1/3}\sin\frac{19\pi}{18}
\end{aligned}
$$

(d) We have $1 + 4i = \sqrt{17}e^{i(\arctan(4)+2\pi k)}$. Let $\theta = \arctan 4$. Thus, the third roots are

$$(\sqrt{17})^{1/3}e^{i(\theta+2\pi k)/3}, \quad k = 0, 1, 2$$

Thus, the three third roots are

$$
\begin{aligned}
w_0 &= 17^{1/6}\cos\frac{\theta}{3} + i17^{1/6}\sin\frac{\theta}{3}\\
w_1 &= 17^{1/6}\cos\frac{\theta+2\pi}{3} + i17^{1/6}\sin\frac{\theta+2\pi}{3}\\
w_2 &= 17^{1/6}\cos\frac{\theta+4\pi}{3} + i17^{1/6}\sin\frac{\theta+4\pi}{3}
\end{aligned}
$$

Homework Problems

B1 (a) $4 + 9i$ (b) $-4 + 4i$ (c) $-7 + i$ (d) $1 + 7i$

B2 (a) $13 - i$ (b) $-19 - 4i$ (c) $27 + 23i$ (d) -10

B3 (a) $-2i$ (b) 17 (c) $4 + 8i$ (d) $5 - 11i$

B4 (a) $\text{Re}(z) = 4$, $\text{Im}(z) = -7$ (b) $\text{Re}(z) = 12$, $\text{Im}(z) = -5$
(c) $\text{Re}(z) = 20/17$, $\text{Im}(z) = 5/17$ (d) $\text{Re}(z) = -\frac{1}{2}$, $\text{Im}(z) = -\frac{3}{2}$

B5 (a) $\frac{3}{25} - \frac{4}{25}i$ (b) $\frac{3}{17} + \frac{5}{17}i$ (c) $-\frac{1}{2} - \frac{1}{2}i$ (d) $-\frac{16}{41} + \frac{21}{41}i$

B6 (a) $z_1z_2 = 2\sqrt{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$, $\frac{z_1}{z_2} = \sqrt{2}\left(\cos\frac{-13\pi}{12} + i\sin\frac{-13\pi}{12}\right)$

(b) $z_1z_2 = 6\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$, $\frac{z_1}{z_2} = \frac{2}{3\sqrt{2}}\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right)$

(c) $z_1z_2 = 5 - 5i$, $\frac{z_1}{z_2} = -\frac{7}{5} - \frac{1}{5}i$ (This answer can be checked using Cartesian form.)

(d) $z_1z_2 = -7 + 6i$, $\frac{z_1}{z_2} = -\frac{9}{17} + \frac{2}{17}i$ (This answer can be checked using Cartesian form.)

B7 (a) $-8 - 8\sqrt{3}i$ (b) $16 - 16i$ (c) $-8 - 8\sqrt{3}i$ (d) $512(-1 - \sqrt{3}i)$

B8 (a) The roots are $2\left[\cos\left(\frac{2k\pi}{5}\right) + i\sin\left(\frac{2k\pi}{5}\right)\right]$, $0 \le k \le 4$.

(b) The roots are $81^{1/5}\left[\cos\left(\frac{\frac{\pi}{2}+2k\pi}{5}\right) + i\sin\left(\frac{\frac{\pi}{2}+2k\pi}{5}\right)\right]$, $0 \le k \le 4$.

(c) The roots are $2^{1/3}\left[\cos\left(\frac{\frac{5\pi}{6}+2k\pi}{3}\right) + i\sin\left(\frac{\frac{5\pi}{6}+2k\pi}{3}\right)\right]$, $0 \le k \le 2$.

(d) The roots are $17^{1/6}\left[\cos\left(\frac{\theta+2k\pi}{3}\right) + i\sin\left(\frac{\theta+2k\pi}{3}\right)\right]$, $0 \le k \le 2$, where $\theta = \arctan(1/4)$.

Conceptual Problems

D1 Let $z_2 = x_2 + iy_2$.

(3) $\overline{z_1} = -z_1 \Leftrightarrow x_1 + iy_1 = -x_1 + iy_1 \Leftrightarrow x_1 = 0$

(5)

$$\begin{aligned}\overline{(z_1z_2)} &= \overline{(x_1+iy_1)(x_2+iy_2)} = \overline{(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)}\\ &= (x_1x_2 - y_1y_2) - i(x_1y_2 + x_2y_1) = (x_1 - iy_1)(x_2 - iy_2) = \overline{z_1}\,\overline{z_2}\end{aligned}$$

(6) We use induction on n. From (5), $\overline{z_1^2} = \overline{z_1}\,\overline{z_1} = (\overline{z_1})^2$. Assume that $\overline{z_1^k} = (\overline{z_1})^k$. Then, $\overline{z_1^{k+1}} = \overline{z_1 z_1^k} = \overline{z_1}(\overline{z_1})^k = (\overline{z_1})^{k+1}$, as required.

(7) $z_1 + \overline{z_1} = x_1 + iy_1 + x_1 - iy_1 = 2x_1 = 2\operatorname{Re}(z_1)$

(8) $z_1 - \overline{z_1} = x_1 + iy_1 - (x_1 - iy_1) = i2y_1 = i2\operatorname{Im}(z_1)$

(9) $z_1\overline{z_1} = (x_1 + iy_1)(x_1 - iy_1) = x_1^2 + y_1^2$

D2 $|\overline{z}| = r = |z|$. We have $\overline{z} = r(\cos\theta - i\sin\theta) = r(\cos(-\theta) + i(\sin(-\theta)))$. Thus, the argument of $\overline{z}$ is $-\theta$.

D3 (a) $\overline{e^{i\theta}} = \overline{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta = \cos(-\theta) + i(\sin(-\theta)) = e^{-i\theta}$

(b) $e^{i\theta} + e^{-i\theta} = \cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta) = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta = 2\cos\theta$.

(c) $e^{i\theta} - e^{-i\theta} = \cos\theta + i\sin\theta - \cos(-\theta) + i\sin(-\theta) = \cos\theta + i\sin\theta - \cos\theta + i\sin\theta = 2i\sin\theta$.

D4

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}\\ &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}\\ &= \frac{r_1}{r_2}((\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(-\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2))\\ &= \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))\end{aligned}$$

9.2 Systems with Complex Numbers

Practice Problems

A1 (a) Row reducing gives

$$\left[\begin{array}{ccc|c} 1 & i & 1+i & 1-i \\ -2 & 1-2i & -2 & 2i \\ 2i & -2 & -2-3i & -1+3i \end{array}\right] \begin{matrix} \\ R_2+2R_1 \\ R_3-2iR_1 \end{matrix} \sim \left[\begin{array}{ccc|c} 1 & i & 1+i & 1-i \\ 0 & 1 & 2i & 2 \\ 0 & 0 & -5i & -3+i \end{array}\right] \begin{matrix} \\ \\ \frac{i}{5}R_3 \end{matrix} \sim$$

$$\left[\begin{array}{ccc|c} 1 & i & 1+i & 1-i \\ 0 & 1 & 2i & 2 \\ 0 & 0 & 1 & -\frac{1}{5}-\frac{3}{5}i \end{array}\right]$$

The system is consistent with a unique solution. By back-substitution,

$$\begin{aligned} z_3 &= -\frac{1}{5}-\frac{3}{5}i \\ z_2 &= 2-2i\left(-\frac{1}{5}-\frac{3}{5}i\right) = \frac{4}{5}+\frac{2}{5}i \\ z_1 &= 1-i-i\left(\frac{4}{5}+\frac{2}{5}i\right)-(1+i)\left(-\frac{1}{5}-\frac{3}{5}i\right) = 1-i \end{aligned}$$

Hence, the general solution is $\vec{z} = \begin{bmatrix} 1-i \\ \frac{4}{5}+\frac{2}{5}i \\ -\frac{1}{5}-\frac{3}{5}i \end{bmatrix}$.

(b) Row reducing gives

$$\left[\begin{array}{cccc|c} 1 & 1+i & 2 & 1 & 1-i \\ 2 & 2+i & 5 & 2+i & 4-i \\ i & -1+i & 1+2i & 2i & 1 \end{array}\right] \begin{matrix} \\ R_2-2R_1 \\ R_3-iR_1 \end{matrix} \sim \left[\begin{array}{cccc|c} 1 & 1+i & 2 & 1 & 1-i \\ 0 & -i & 1 & i & 2+i \\ 0 & 0 & 1 & i & -i \end{array}\right] \begin{matrix} \\ iR_2 \\ \\ \end{matrix} \sim$$

$$\left[\begin{array}{cccc|c} 1 & 1+i & 2 & 1 & 1-i \\ 0 & 1 & i & -1 & -1+2i \\ 0 & 0 & 1 & i & -i \end{array}\right] \begin{matrix} R_1-(1+i)R_2 \\ \\ \\ \end{matrix} \sim \left[\begin{array}{cccc|c} 1 & 0 & 3-i & 2+i & 4-2i \\ 0 & 1 & i & -1 & -1+2i \\ 0 & 0 & 1 & i & -i \end{array}\right] \begin{matrix} R_1-(3-i)R_3 \\ R_2-iR_3 \\ \\ \end{matrix} \sim$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1-2i & 5+i \\ 0 & 1 & 0 & 0 & -2+2i \\ 0 & 0 & 1 & i & -i \end{array}\right]$$

Hence, the system is consistent with one parameter. Let $z_4 = t \in \mathbb{C}$. Then, the general solution is $\vec{z} = \begin{bmatrix} 5+i \\ -2+2i \\ -i \\ 0 \end{bmatrix} + t \begin{bmatrix} -1+2i \\ 0 \\ -i \\ 1 \end{bmatrix}$, $t \in \mathbb{C}$.

Homework Problems

B1 (a) The general solution is $\vec{z} = \begin{bmatrix} 7-2i \\ -1+3i \\ 2i \end{bmatrix}$.

(b) The general solution is $\vec{z} = \begin{bmatrix} -1 \\ 0 \\ 1-2i \\ 0 \end{bmatrix} + s\begin{bmatrix} -2-i \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ -2i \\ 1 \end{bmatrix}, \quad s,t \in \mathbb{C}$.

(c) The system is inconsistent.

(d) The general solution is $\vec{x} = t\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -i \\ 0 \\ i \\ 1 \end{bmatrix}, \quad s,t \in \mathbb{C}$.

9.3 Vector Spaces over C

Practice Problems

A1 (a) $\begin{bmatrix} -2+i \\ 1 \end{bmatrix} - \begin{bmatrix} 3+4i \\ 1-i \end{bmatrix} = \begin{bmatrix} -5-3i \\ i \end{bmatrix}$

(b) $\begin{bmatrix} 2-i \\ 3+i \\ 2-5i \end{bmatrix} + \begin{bmatrix} 3-2i \\ 4+7i \\ -3-4i \end{bmatrix} = \begin{bmatrix} 5-3i \\ 7+8i \\ -1-9i \end{bmatrix}$

(c) $2i\begin{bmatrix} 2+5i \\ 3-2i \end{bmatrix} = \begin{bmatrix} -10+4i \\ 4+6i \end{bmatrix}$

(d) $(-1-2i)\begin{bmatrix} 2-i \\ 3+i \\ 2-5i \end{bmatrix} = \begin{bmatrix} -4-3i \\ -1-7i \\ -12+i \end{bmatrix}$

A2 (a) We have

$$[L] = \begin{bmatrix} L(1,0) & L(0,1) \end{bmatrix} = \begin{bmatrix} 1+2i & 3+i \\ 1 & 1-i \end{bmatrix}$$

(b) We have

$$L(2+3i, 1-4i) = [L]\begin{bmatrix} 2+3i \\ 1-4i \end{bmatrix} = \begin{bmatrix} 1+2i & 3+i \\ 1 & 1-i \end{bmatrix}\begin{bmatrix} 2+3i \\ 1-4i \end{bmatrix} = \begin{bmatrix} 3-4i \\ -1-2i \end{bmatrix}$$

(c) Every vector in the range of L can be written as a linear combination of the columns of $[L]$. Hence, every vector $\vec{z}$ in Range(L) has the form

$$\begin{aligned} \vec{z} &= t_1\begin{bmatrix} 1+2i \\ 1 \end{bmatrix} + t_2\begin{bmatrix} 3+i \\ 1-i \end{bmatrix} \\ &= t_1\begin{bmatrix} 1+2i \\ 1 \end{bmatrix} + (1-i)t_2\begin{bmatrix} 1+2i \\ 1 \end{bmatrix} \\ &= (t_1 + (1-i)t_2)\begin{bmatrix} 1+2i \\ 1 \end{bmatrix} \end{aligned}$$

Thus, $\left\{\begin{bmatrix}1+2i\\1\end{bmatrix}\right\}$ spans the range of L and is clearly linearly independent, so it forms a basis for Range(L).

Let $\vec{z}$ be any vector in Null(L). Then we have

$$\begin{bmatrix}0\\0\end{bmatrix} = [L]\vec{z} = z_1\begin{bmatrix}1+2i\\1\end{bmatrix} + z_2\begin{bmatrix}3+i\\1-i\end{bmatrix} = \begin{bmatrix}(1+2i)z_1+(3+i)z_2\\z_1+(1-i)z_2\end{bmatrix}$$

Row reducing the coefficient matrix corresponding to the homogeneous system gives

$$\begin{bmatrix}1+2i & 3+i\\1 & 1-i\end{bmatrix} \sim \begin{bmatrix}1 & 1-i\\0 & 0\end{bmatrix}$$

Hence, a basis for Null(L) is $\left\{\begin{bmatrix}-1+i\\1\end{bmatrix}\right\}$.

A3 (a) Row reducing the matrix A to its reduced row echelon form R gives

$$\begin{bmatrix}1+i & 1 & i\\-2i & 2i & 2+2i\end{bmatrix} \sim \begin{bmatrix}1 & 0 & i\\0 & 1 & 1\end{bmatrix}$$

The non-zero rows of R form a basis for the rowspace of A. Hence, a basis for Row(A) is $\left\{\begin{bmatrix}1\\0\\i\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$. The columns from A which correspond to columns in R which have leadings ones form a basis for the columnspace of A. So, a basis for Col(A) is $\left\{\begin{bmatrix}1+i\\-2i\end{bmatrix}, \begin{bmatrix}1\\2i\end{bmatrix}\right\}$. Solve the homogeneous system $A\vec{z} = \vec{0}$, we find that a basis for Null(A) is $\left\{\begin{bmatrix}-i\\-1\\1\end{bmatrix}\right\}$.

(b) Row reducing the matrix B to its reduced row echelon form R gives

$$\begin{bmatrix}1 & i\\1+i & -1+i\\-1 & i\end{bmatrix} \sim \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}$$

The non-zero rows of R form a basis for the rowspace of B. Hence, a basis for Row(B) is $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$. The columns from B which correspond to columns in R which have leadings ones form a basis for the columnspace of B. So, a basis for Col(B) is $\left\{\begin{bmatrix}1\\1+i\\-1\end{bmatrix}, \begin{bmatrix}i\\-1+i\\i\end{bmatrix}\right\}$. Solve the homogeneous system $B\vec{z} = \vec{0}$, we find that a basis for Null(B) is the empty set.

(c) Row reducing the matrix C to its reduced row echelon form R gives

$$\begin{bmatrix}1 & i & -1+i & -1\\2 & 1+2i & -2+3i & -2\\1+i & i & -2+i & -1-i\end{bmatrix} \sim \begin{bmatrix}1 & 0 & i & -1\\0 & 1 & i & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

The non-zero rows of R form a basis for the rowspace of C. Hence, a basis for Row(C) is $\left\{\begin{bmatrix}1\\0\\i\\-1\end{bmatrix},\begin{bmatrix}0\\1\\i\\0\end{bmatrix}\right\}$.
The columns from C which correspond to columns in R which have leadings ones form a basis for the columnspace of C. So, a basis for Col(C) is $\left\{\begin{bmatrix}1\\2\\1+i\end{bmatrix},\begin{bmatrix}i\\1+2i\\i\end{bmatrix}\right\}$. Solve the homogeneous system $A\vec{z}=\vec{0}$, we find that a basis for Null(C) is $\left\{\begin{bmatrix}-i\\-i\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\1\end{bmatrix}\right\}$.

Homework Problems

B1 (a) $\begin{bmatrix}6+i\\-1\end{bmatrix}$ (b) $\begin{bmatrix}1+i\\-5-2i\\4i\end{bmatrix}$ (c) $\begin{bmatrix}9-3i\\-9-15i\end{bmatrix}$ (d) $\begin{bmatrix}1-i\\1-5i\\-5+9i\end{bmatrix}$

B2 (a) $[L]=\begin{bmatrix}-i & 1+i\\-1+i & -2i\end{bmatrix}$

(b) $L(2-i,-4+i)=\begin{bmatrix}-6-5i\\1+11i\end{bmatrix}$

(c) A basis for Range(L) is $\left\{\begin{bmatrix}-i\\-1+i\end{bmatrix}\right\}$. A basis for Null($L$) is $\left\{\begin{bmatrix}1-i\\1\end{bmatrix}\right\}$

B3 (a) Since the dimension of $\mathbb{C}^3$ as a complex vector space is three and the set contains three linearly independent vectors, it is a basis.

(b) The set is a linearly independent spanning set and hence a basis for $\mathbb{C}^3$.

B4 (a) A basis for Row(A) is $\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$. A basis for Col($A$) is $\left\{\begin{bmatrix}1\\i\\1+i\end{bmatrix},\begin{bmatrix}i\\1\\-1-i\end{bmatrix}\right\}$, and a basis for Null($A$) is $\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix}\right\}$.

(b) A basis for Row(B) is $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$. A basis for Col($B$) is $\left\{\begin{bmatrix}1+i\\1\\-1+i\end{bmatrix},\begin{bmatrix}2-i\\-i\\2i\end{bmatrix}\right\}$, and a basis for Null($B$) is the empty set.

(c) A basis for Row(C) is $\left\{\begin{bmatrix}1\\0\\-\frac{5}{2}i\\\frac{1}{4}\end{bmatrix},\begin{bmatrix}0\\1\\\frac{1}{2}\\-\frac{3}{4}i\end{bmatrix}\right\}$. A basis for Col($C$) is $\left\{\begin{bmatrix}i\\2i\end{bmatrix},\begin{bmatrix}-1\\6\end{bmatrix}\right\}$, and a basis for Null($C$) is $\left\{\begin{bmatrix}\frac{5}{2}i\\-\frac{1}{2}\\1\\0\end{bmatrix},\begin{bmatrix}-\frac{1}{4}\\\frac{3}{4}i\\0\\1\end{bmatrix}\right\}$.

(d) A basis for Row(D) is $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$. A basis for Col($D$) is $\left\{\begin{bmatrix}0\\1+4i\\1\end{bmatrix},\begin{bmatrix}i\\i\\0\end{bmatrix},\begin{bmatrix}2-i\\3\\i\end{bmatrix}\right\}$, and a basis for Null($D$) is the empty set.

Conceptual Problems

D1 (a) We have $\alpha z = (a+ib) = ax - by + i(bx + ay)$. Thus, define $M_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$M_\alpha(x, y) = (ax - by, bx + ay)$$

The standard matrix of this mapping is

$$[M_\alpha] = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

as claimed.

(b) We may write

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \sqrt{a^2+b^2}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where $\cos\theta = a/\sqrt{a^2+b^2}$ and $\sin\theta = b/\sqrt{a^2+b^2}$. Hence, this matrix describes a dilation or contraction by factor $\sqrt{a^2+b^2}$ following a rotation through angle θ.

(c) If $\alpha = 3 - 4i$, then

$$[M_\alpha] = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix} = 5\begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix}$$

so $\cos\theta = 3/5$, $\sin\theta = -4/5$, and hence $\theta \approx -0.927$ radians. Thus multiplication of a complex number by α increases the modulus by a factor of 5 and rotates by angle of θ. (Compare this to the polar form of α.)

D2 (a) Verify that $\mathbb{C}(2,2)$ satisfies all 10 vector spaces axioms.

(b) A basis for $\mathbb{C}(2,2)$ is $\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$ and hence the dimension is 4.

D3 No changes are required in the previous definitions and theorems, except that the scalars are now allowed to be complex numbers.

D4 If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$, then it is linearly independent. Since all of the entries of these vectors are real, this set will still be linearly independent in $\mathbb{C}^3$. Hence, it is a basis for $\mathbb{C}^3$ since $\mathbb{C}^3$ is 3 dimensional as a complex vector space.

9.4 Eigenvectors in Complex Vector Spaces

Practice Problems

A1 (a) We have

$$C(\lambda) = \begin{vmatrix} -\lambda & 4 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 4 = (\lambda - 2i)(\lambda + 2i)$$

Thus, the eigenvalues are $\lambda_1 = 2i$ and $\lambda_2 = -2i$. Since A has distinct eigenvalues, it is diagonalizable. Moreover, A is diagonalized to $D = \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix}$.

To find a real canonical form, we pick $\lambda_1 = 0 + 2i$. Thus, we have $a = 0$ and $b = 2$. So, a real canonical form of A is $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$.

To find the corresponding change of coordinates matrix P, we need to find an eigenvector of λ_1. We have

$$A - 2iI = \begin{bmatrix} -2i & 4 \\ -1 & -2i \end{bmatrix} \sim \begin{bmatrix} 1 & 2i \\ 0 & 0 \end{bmatrix}$$

Hence, an eigenvector corresponding to λ_1 is $\vec{z} = \begin{bmatrix} -2i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} -2 \\ 0 \end{bmatrix}$. The columns of P are the real and imaginary parts of $\vec{z}$. Hence, $P = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}$.

(b) We have

$$C(\lambda) = \begin{vmatrix} -1 - \lambda & 2 \\ -1 & -3 - \lambda \end{vmatrix} = \lambda^2 + 4\lambda + 5$$

Hence, $\lambda = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$. Thus, the eigenvalues are $\lambda_1 = -2 + i$ and $\lambda_2 = -2 - i$. Since A has distinct eigenvalues, it is diagonalizable. Moreover, A is diagonalized to $D = \begin{bmatrix} -2 + i & 0 \\ 0 & -2 - i \end{bmatrix}$.

To find a real canonical form, we pick $\lambda_1 = -2 + i$. Thus, we have $a = -2$ and $b = 1$. So, a real canonical form of A is $\begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix}$.

To find the corresponding change of coordinates matrix P, we need to find an eigenvector of λ_1. We have

$$A - (-2 + i)I = \begin{bmatrix} 1 - i & 2 \\ -1 & -1 - i \end{bmatrix} \sim \begin{bmatrix} 1 & 1 + i \\ 0 & 0 \end{bmatrix}$$

Hence, an eigenvector corresponding to λ_1 is $\vec{z} = \begin{bmatrix} -1 - i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. The columns of P are the real and imaginary parts of $\vec{z}$. Hence, $P = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$.

(c) We have

$$C(\lambda) = \begin{vmatrix} 2 - \lambda & 2 & -1 \\ -4 & 1 - \lambda & 2 \\ 2 & 2 & -1 - \lambda \end{vmatrix} = -\lambda(\lambda^2 - 2\lambda + 5)$$

Hence, the eigenvalues are 0 and $\frac{2\pm\sqrt{4-20}}{2} = 1 \pm 2i$. Since A has distinct eigenvalues, it is diagonalizable. Moreover, A is diagonalized to $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1+2i & 0 \\ 0 & 0 & 1-2i \end{bmatrix}$.

To find a real canonical form, we have $\mu = 0$ and we pick $\lambda = 1 + 2i$. Thus, we have $a = 1$ and $b = 2$. So, a real canonical form of A is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -2 & 1 \end{bmatrix}$.

To find the corresponding change of coordinates matrix P, we need to find an eigenvector of μ and an eigenvector of λ.

We have

$$A - 0I = \begin{bmatrix} 2 & 2 & -1 \\ -4 & 1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, an eigenvector corresponding to μ is $\vec{z} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

We have

$$A - (1+2i)I = \begin{bmatrix} 1-2i & 2 & -1 \\ -4 & -2i & 2 \\ 2 & 2 & -2-2i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -i \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, an eigenvector corresponding to λ is $\vec{z} = \begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

The first columns of P is an eigenvector of μ, and the second and third columns of P are the real and imaginary parts of $\vec{z}$. Hence, $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$.

(d) We have

$$C(\lambda) = \begin{vmatrix} 2-\lambda & 1 & -1 \\ 2 & 1-\lambda & 0 \\ 3 & -1 & 2-\lambda \end{vmatrix} = -(\lambda - 1)(\lambda^2 - 4\lambda + 5)$$

Hence, the eigenvalues are 1 and $\frac{4\pm\sqrt{16-20}}{2} = 2 \pm i$. Since A has distinct eigenvalues, it is diagonalizable. Moreover, A is diagonalized to $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2+i & 0 \\ 0 & 0 & 2-i \end{bmatrix}$.

To find a real canonical form, we have $\mu = 1$ and we pick $\lambda = 2 + i$. Thus, we have $a = 2$ and $b = 1$. So, a real canonical form of A is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}$.

To find the corresponding change of coordinates matrix P, we need to find an eigenvector of μ and an eigenvector of λ.

We have

$$A - I = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 0 \\ 3 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, an eigenvector corresponding to μ is $\vec{z} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

We have

$$A - (2+i)I = \begin{bmatrix} -i & 1 & -1 \\ 2 & -1-i & 0 \\ 3 & -1 & -i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{5} - \frac{2}{5}i \\ 0 & 1 & -\frac{3}{5} - \frac{1}{5}i \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, an eigenvector corresponding to λ is $\vec{z} = \begin{bmatrix} 1+2i \\ 3+i \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + i \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

The first columns of P is an eigenvector of μ, and the second and third columns of P are the real and imaginary parts of $\vec{z}$. Hence, $P = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & 1 \\ 1 & 5 & 0 \end{bmatrix}$.

Homework Problems

B1 (a) $D = \begin{bmatrix} -1+i & 0 \\ 0 & -1-i \end{bmatrix}$, $P = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$, $P^{-1}AP = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$

(b) $D = \begin{bmatrix} 2+2i & 0 \\ 0 & 2-2i \end{bmatrix}$, $P = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}$, $P^{-1}AP = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$

(c) $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 1-i \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 2 & 2 & 0 \end{bmatrix}$, $P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

(d) $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix}$, $P = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ -2 & -2 & 0 \end{bmatrix}$, $P^{-1}AP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$

(e) $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & -2i \end{bmatrix}$, $P = \begin{bmatrix} 1 & 3 & 1 \\ 1 & -1 & -2 \\ 1 & 5 & 0 \end{bmatrix}$, $P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix}$

(f) $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 1-i \end{bmatrix}$, $P = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$, $P^{-1}AP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

Conceptual Problems

D1 If $\vec{z}$ is an eigenvector of A, then there exists an eigenvalue λ, such that $A\vec{z} = \lambda\vec{z}$. Hence,

$$\overline{A}\overline{\vec{z}} = \overline{A\vec{z}} = \overline{\lambda\vec{z}} = \overline{\lambda}\overline{\vec{z}}$$

Hence, $\overline{\vec{z}}$ is an eigenvector of $\overline{A}$ with eigenvalue $\overline{\lambda}$.

D2 We have

$$A(\vec{x} + i\vec{y}) = (a + bi)(\vec{x} + i\vec{y}) = a\vec{x} - b\vec{y} + i(b\vec{x} + a\vec{y}) \tag{1}$$

(a) If $\vec{x} = \vec{0}$, then we have $A(i\vec{y}) = -b\vec{y} + ia\vec{y}$. But, the real part of the left side of the equation is zero, hence the real part of the right side must also be zero. Hence $\vec{y} = \vec{0}$, since $b \neq 0$. But, then $\vec{z} = \vec{0}$ which contradicts the definition of an eigenvector so $\vec{x} \neq \vec{0}$.

If $\vec{y} = \vec{0}$, then we have $A\vec{x} = a\vec{x} + ib\vec{x}$. But the left side has zero imaginary part, hence we must have $\vec{x} = \vec{0}$ since $b \neq 0$. But, then $\vec{z} = \vec{0}$ which contradicts the definition of an eigenvector so $\vec{y} \neq \vec{0}$.

(b) Suppose that $\vec{x} = k\vec{y}$, $k \neq 0$. Then, the real part and imaginary parts of (1) are

$$A(k\vec{y}) = a(k\vec{y}) - b\vec{y} = (ak - b)\vec{y} \Rightarrow A\vec{y} = \frac{1}{k}(ak - b)\vec{y}$$
$$A\vec{y} = b(k\vec{y}) + a\vec{y} = (bk + a)\vec{y}$$

Since $\vec{y} \neq \vec{0}$, setting these equal to each other we get

$$\frac{1}{k}(ak - b)\vec{y} = (bk + a)\vec{y} \Rightarrow b(k^2 + 1) = 0$$

which is impossible since $b \neq 0$.

(c) Suppose that $\vec{v} = p\vec{x} + q\vec{y}$ is a real eigenvector of A, hence it must correspond to a real eigenvalue μ of A. Observe from our work in question (a) that we can not have $p = 0$ or $q = 0$ since $\vec{x}$ and $\vec{y}$ are both not the zero vector and hence not eigenvectors of A. Then, we have

$$A\vec{v} = \mu\vec{v} = \mu p\vec{x} + \mu q\vec{y},$$

and

$$A\vec{v} = A(p\vec{x} + q\vec{y}) = p(a\vec{x} - b\vec{y}) + q(b\vec{x} + a\vec{y}) = (pa + qb)\vec{x} + (qa - pb)\vec{y}.$$

Since $\{\vec{x}, \vec{y}\}$ is linearly independent, the coefficients of $\vec{x}$ and $\vec{y}$ must be equal, so we get

$$pa + qb = \mu p$$
$$qa - pb = \mu q$$

Multiplying the first by q and the second by p gives

$$qpa + q^2b = \mu pq = pqa - p^2b \Rightarrow q^2b = -p^2b,$$

which is impossible since $q, p, b \neq 0$. Thus, there can be no real eigenvector of A in $\mathrm{Span}\{\vec{x}, \vec{y}\}$.

9.5 Inner Products in Complex Vector Spaces

Practice Problems

A1 (a) We have

$$\langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \overline{\vec{v}} = \begin{bmatrix} 2+3i \\ -1-2i \end{bmatrix} \cdot \begin{bmatrix} 2i \\ 2+5i \end{bmatrix} = (2+3i)(2i) + (-1-2i)(2+5i) = 2-5i$$
$$\langle \vec{v}, \vec{u} \rangle = \overline{\langle \vec{u}, \vec{v} \rangle} = 2+5i$$
$$\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle} = \sqrt{(2+3i)(2-3i) + (-1-2i)(-1+2i)} = \sqrt{18}$$
$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{(-2i)(2i) + (2-5i)(2+5i)} = \sqrt{33}$$

(b) We have

$$\langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \overline{\vec{v}} = \begin{bmatrix} -1+4i \\ 2-i \end{bmatrix} \cdot \begin{bmatrix} 3-i \\ 1-3i \end{bmatrix} = (-1+4i)(3-i) + (2-i)(1-3i) = 6i$$
$$\langle \vec{v}, \vec{u} \rangle = \overline{\langle \vec{u}, \vec{v} \rangle} = -6i$$
$$\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle} = \sqrt{(-1+4i)(-1-4i) + (2-i)(2+i)} = \sqrt{22}$$
$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{(3+i)(3-i) + (1+3i)(1-3i)} = \sqrt{20}$$

(c) We have

$$\langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \overline{\vec{v}} = \begin{bmatrix} 1-i \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1-i \\ 1+i \end{bmatrix} = (1-i)(1-i) + 3(1+i) = 3+i$$
$$\langle \vec{v}, \vec{u} \rangle = \overline{\langle \vec{u}, \vec{v} \rangle} = 3-i$$
$$\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle} = \sqrt{(1-i)(1+i) + 3(3)} = \sqrt{11}$$
$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{(1+i)(1-i) + (1-i)(1+i)} = 2$$

(d) We have

$$\langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \overline{\vec{v}} = \begin{bmatrix} 1+2i \\ -1-3i \end{bmatrix} \cdot \begin{bmatrix} i \\ 2i \end{bmatrix} = (1+2i)(i) + (-1-3i)(2i) = 4-i$$
$$\langle \vec{v}, \vec{u} \rangle = \overline{\langle \vec{u}, \vec{v} \rangle} = 4+i$$
$$\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle} = \sqrt{(1+2i)(1-2i) + (-1-3i)(-1+3i)} = \sqrt{15}$$
$$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{(-i)(i) + (-2i)(2i)} = \sqrt{5}$$

A2 (a) We have

$$\left\langle \begin{bmatrix} 1 \\ 1+i \end{bmatrix}, \begin{bmatrix} 1+i \\ 1 \end{bmatrix} \right\rangle = 1(1-i) + (1+i)(1) = 2$$

Hence, the columns of A are not orthogonal, so A is not unitary.

(b) The columns of B clearly form an orthonormal basis for $\mathbb{C}^2$, so B is unitary.

(c) We have

$$C^*C = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ -i & -i \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ -1 & i \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I$$

Thus, C is unitary.

(d) We have

$$DD^* = \begin{bmatrix} (-1+i)/\sqrt{3} & (1-i)/\sqrt{6} \\ 1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} (-1-i)/\sqrt{3} & 1/\sqrt{3} \\ (1+i)/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} = I$$

Therefore, D is unitary.

A3 (a) We have

$$\langle \vec{u}, \vec{v} \rangle = 1(-i) + 0(1) + i(1) = 0$$

(b) Let $\mathbb{S} = \operatorname{Span}\{\vec{u}, \vec{v}\}$. Then we have

$$\begin{aligned} \operatorname{proj}_{\mathbb{S}} \vec{w} &= \frac{\langle \vec{w}, \vec{u} \rangle}{\|\vec{u}\|^2}\vec{u} + \frac{\langle \vec{w}, \vec{v} \rangle}{\|\vec{v}\|^2}\vec{v} \\ &= \frac{2-2i}{2}\begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix} + \frac{6+i}{3}\begin{bmatrix} i \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{3} + i \\ 2 + \frac{1}{3}i \\ 3 + \frac{4}{3}i \end{bmatrix} \end{aligned}$$

A4 (a) We have

$$1 = \det I = \det(U^*U) = \det(U^*)\det U = \overline{\det U}\det U = |\det U|^2$$

Therefore, $|\det U| = 1$.

(b) The matrix $U = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}$ is unitary and $\det U = i$.

Homework Problems

B1 (a) $\langle \vec{u}, \vec{v} \rangle = 4 - 12i$, $\langle \vec{v}, \vec{u} \rangle = 4 + 12i$, $\|\vec{u}\| = \sqrt{18}$, $\|\vec{v}\| = \sqrt{34}$

(b) $\langle \vec{u}, \vec{v} \rangle = 6 - i$, $\langle \vec{v}, \vec{u} \rangle = 6 + i$, $\|\vec{u}\| = \sqrt{15}$, $\|\vec{v}\| = \sqrt{7}$

(c) $\langle \vec{u}, \vec{v} \rangle = 14 - 5i$, $\langle \vec{v}, \vec{u} \rangle = 14 + 5i$, $\|\vec{u}\| = \sqrt{30}$, $\|\vec{v}\| = \sqrt{13}$

(d) $\langle \vec{u}, \vec{v} \rangle = 0$, $\langle \vec{v}, \vec{u} \rangle = 0$, $\|\vec{u}\| = 2$, $\|\vec{v}\| = 0$

B2 (a) A is unitary.

(b) B is not unitary.

(c) C is unitary.

(d) D is not unitary.

B3 (a) $\langle \vec{u}, \vec{v} \rangle = 0$ (b) $\begin{bmatrix} \frac{32}{21} + \frac{1}{7}i \\ \frac{4}{3} - \frac{25}{21}i \\ 2 - \frac{12}{7}i \end{bmatrix}$

B4 (a) $\left\{ \frac{1}{3}\begin{bmatrix} 1+i \\ 2-i \\ -1+i \end{bmatrix}, \frac{1}{\sqrt{7}}\begin{bmatrix} 0 \\ -1-i \\ -2-i \end{bmatrix} \right\}$

(b) $\frac{1}{63}\begin{bmatrix} 28 \\ 5-15i \\ 73i \end{bmatrix}$

Conceptual Problems

D1 Observe that the required statement is equivalent to

$$\|\mathbf{z}+\mathbf{w}\|^2 \leq (\|\mathbf{z}\|+\|\mathbf{w}\|)^2$$

Consider

$$\begin{aligned}\|\mathbf{z}+\mathbf{w}\|^2 - (\|\mathbf{z}\|+\|\mathbf{w}\|)^2 &= \langle \mathbf{z}+\mathbf{w}, \mathbf{z}+\mathbf{w}\rangle - (\|\mathbf{z}\|^2 + 2\|\mathbf{z}\|\|\mathbf{w}\| + \|\mathbf{w}\|^2) \\ &= \langle \mathbf{z},\mathbf{z}\rangle + \langle \mathbf{z},\mathbf{w}\rangle + \langle \mathbf{w},\mathbf{z}\rangle + \langle \mathbf{w},\mathbf{w}\rangle - (\langle \mathbf{z},\mathbf{z}\rangle + 2\|\mathbf{z}\|\|\mathbf{w}\| + \langle \mathbf{w},\mathbf{w}\rangle) \\ &= \langle \mathbf{z},\mathbf{w}\rangle + \overline{\langle \mathbf{z},\mathbf{w}\rangle} - 2\|\mathbf{w}\|\|\mathbf{w}\| \\ &= 2\,\mathrm{Re}(\langle \mathbf{z},\mathbf{w}\rangle) - 2\|\mathbf{z}\|\|\mathbf{w}\|\end{aligned}$$

But for any $\alpha \in \mathbb{C}$ we have $\mathrm{Re}(\alpha) \leq |\alpha|$, so

$$\mathrm{Re}(\langle \mathbf{z},\mathbf{w}\rangle) \leq |\langle \mathbf{z},\mathbf{w}\rangle| \leq \|\mathbf{z}\|\|\mathbf{w}\|$$

by Theorem 1 (4).

D2 For property (1) we have

$$\langle A\vec{z}, \vec{w}\rangle = (A\vec{z})^T\overline{\vec{w}} = \vec{z}^T A^T \overline{\vec{w}} = \vec{z}^T \overline{A^*}\overline{\vec{w}} = \vec{z}^T \overline{A^*\vec{w}} = \langle \vec{z}, A^*\vec{w}\rangle$$

The other properties follow from the definition of the conjugate transpose and properties of conjugates and transposes.

D3 We prove this by induction. If A is a 1×1 matrix, then the result is obvious. Assume the result holds for $n-1\times n-1$ matrices and consider an $n\times n$ matrix A. If we expand $\det \overline{A}$ along the first row, we get by definition of the determinant

$$\det \overline{A} = \sum_{i=1}^{n} \overline{a_{1i}} C_{1i}(\overline{A})$$

where the $C_{1i}(\overline{A})$ represent the cofactors of $\overline{A}$. But, each of these cofactors is the determinant of an $n-1\times n-1$ matrix, so we have by our inductive hypothesis that $C_{1i}(\overline{A}) = \overline{C_{1i}(A)}$. Hence,

$$\det \overline{A} = \sum_{i=1}^{n} \overline{a_{1i}} C_{1i}(\overline{A}) = \sum_{i=1}^{n} \overline{a_{1i}C_{1i}(A)} = \overline{\det A}$$

D4 Since $AA^* = I$ and $BB^* = I$ we get

$$(AB)(AB)^* = ABB^*A^* = I$$

Hence, AB is unitary.

D5 (a) We have

$$\|U\vec{z}\|^2 = (U\vec{z})^T(\overline{U\vec{z}}) = \vec{z}^T U^T \overline{U}\,\overline{\vec{z}} = \vec{z}^T \overline{U^*}\,\overline{U}\,\overline{\vec{z}} = \vec{z}^T \overline{I}\vec{z} = \vec{z}^T \overline{\vec{z}} = \|\vec{z}\|^2$$

(b) If λ is an eigenvalue of U with corresponding unit eigenvector $\vec{z}$, then

$$|\lambda| = |\lambda|\|\vec{z}\| = \|\lambda\vec{z}\| = \|U\vec{z}\| = \|\vec{z}\| = 1$$

(c) The matrix $U = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ is unitary and has eigenvalues i and $-i$.

D6 Let λ be any eigenvalue of a symmetric matrix A with corresponding unit eigenvector $\vec{z}$. Then

$$\begin{aligned} \lambda &= \lambda\langle\vec{z},\vec{z}\rangle = \langle\lambda\vec{z},\vec{z}\rangle = \langle A\vec{z},\vec{z}\rangle \\ &= \langle\vec{z},A\vec{z}\rangle = \langle\vec{z},\lambda\vec{z}\rangle = \overline{\lambda}\langle\vec{z},\vec{z}\rangle = \overline{\lambda} \end{aligned}$$

Thus, $\lambda = \overline{\lambda}$. Hence, λ is real.

9.6 Hermitian Matrices and Unitary Diagonalization

Practice Problems

A1 (a) Observe that $A^* = \begin{bmatrix} 4 & \sqrt{2}+i \\ \sqrt{2}-i & 2 \end{bmatrix} = A$, so A is Hermitian.

We have

$$C(\lambda) = \begin{vmatrix} 4-\lambda & \sqrt{2}+i \\ \sqrt{2}-i & 2-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 5 = (\lambda-1)(\lambda-5)$$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 5$.

For $\lambda_1 = 1$ we get

$$A - I = \begin{bmatrix} 3 & \sqrt{2}+i \\ \sqrt{2}-i & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & (\sqrt{2}+i)/3 \\ 0 & 0 \end{bmatrix}$$

A basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix} \sqrt{2}+i \\ -3 \end{bmatrix}\right\}$.

For $\lambda_2 = 5$ we get

$$A - 5I = \begin{bmatrix} -1 & \sqrt{2}+i \\ \sqrt{2}-i & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -\sqrt{2}-i \\ 0 & 0 \end{bmatrix}$$

A basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix} \sqrt{2}+i \\ 1 \end{bmatrix}\right\}$.

We normalize the basis vectors for the eigenspaces to get the orthonormal basis for $\mathbb{C}^2$ of eigenvectors of A $\left\{\begin{bmatrix} (\sqrt{2}+i)/\sqrt{12} \\ -3/\sqrt{12} \end{bmatrix}, \begin{bmatrix} (\sqrt{2}+i)/2 \\ 1/2 \end{bmatrix}\right\}$. Hence, $U = \begin{bmatrix} (\sqrt{2}+i)/\sqrt{12} & (\sqrt{2}+i)/2 \\ -3/\sqrt{12} & 1/2 \end{bmatrix}$ unitarily diagonalizes A to $U^*AU = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$.

(b) Observe that

$$B^* = \begin{bmatrix} 5 & \sqrt{2}-i \\ \sqrt{2}+i & \sqrt{3}-i \end{bmatrix} \neq B$$

So, B is not Hermitian.

(c) Observe that $C^* = \begin{bmatrix} 6 & \sqrt{3}-i \\ \sqrt{3}+i & 3 \end{bmatrix} = C$, so C is Hermitian.

We have

$$\begin{vmatrix} 6-\lambda & \sqrt{3}-i \\ \sqrt{3}+i & 3-\lambda \end{vmatrix} = \lambda^2 - 9\lambda + 14 = (\lambda-7)(\lambda-2)$$

The eigenvalues are $\lambda_1 = 7$ and $\lambda_2 = 2$.

For $\lambda_1 = 7$ we get

$$C - 7I = \begin{bmatrix} -1 & \sqrt{3}-i \\ \sqrt{3}+i & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -\sqrt{3}+i \\ 0 & 0 \end{bmatrix}$$

A basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix} \sqrt{3}-i \\ 1 \end{bmatrix}\right\}$.

For $\lambda_2 = 2$ we get

$$C - 2I = \begin{bmatrix} 4 & \sqrt{3}-i \\ \sqrt{3}+i & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & (\sqrt{3}-i)/4 \\ 0 & 0 \end{bmatrix}$$

A basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix} \sqrt{3}-i \\ -4 \end{bmatrix}\right\}$.

We normalize the basis vectors for the eigenspaces to get the orthonormal basis for $\mathbb{C}^2$ of eigenvectors of A $\left\{\begin{bmatrix} (\sqrt{3}-i)/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}, \begin{bmatrix} (\sqrt{3}-i)/\sqrt{20} \\ -4/\sqrt{20} \end{bmatrix}\right\}$. Hence, $U = \begin{bmatrix} (\sqrt{3}-i)/\sqrt{5} & (\sqrt{3}-i)/\sqrt{20} \\ 1/\sqrt{5} & -4/\sqrt{20} \end{bmatrix}$ unitarily diagonalizes C to $U^*CU = \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}$.

(d) Observe that $F^* = \begin{bmatrix} 1 & 1+i & 0 \\ 1-i & 0 & 1-i \\ 0 & 1+i & -1 \end{bmatrix} = F$, so F is Hermitian.

We have

$$C(\lambda) = \begin{vmatrix} 1-\lambda & 1+i & 0 \\ 1-i & -\lambda & 1-i \\ 0 & 1+i & -1-\lambda \end{vmatrix} = -\lambda(\lambda^2 - 5) = -\lambda(\lambda - \sqrt{5})(\lambda + \sqrt{5})$$

The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = \sqrt{5}$, and $\lambda_3 = -\sqrt{5}$.

For $\lambda_1 = 0$ we get

$$F - 0I = \begin{bmatrix} 1 & 1+i & 0 \\ 1-i & 0 & 1-i \\ 0 & 1+i & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & (-1+i)/2 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix} -2 \\ 1-i \\ 2 \end{bmatrix}\right\}$.

For $\lambda_2 = \sqrt{5}$ we get

$$F - \sqrt{5}I = \begin{bmatrix} 1-\sqrt{5} & 1+i & 0 \\ 1-i & -\sqrt{5} & 1-i \\ 0 & 1+i & -1-\sqrt{5} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -(3+\sqrt{5})/2 \\ 0 & 1 & -(1-i)(1+\sqrt{5})/2 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix} 3+\sqrt{5} \\ (1-i)(1+\sqrt{5}) \\ 2 \end{bmatrix}\right\}$.

For $\lambda_3 = -\sqrt{5}$ we get

$$F - (-\sqrt{5})I = \begin{bmatrix} 1+\sqrt{5} & 1+i & 0 \\ 1-i & \sqrt{5} & 1-i \\ 0 & 1+i & -1+\sqrt{5} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -(3-\sqrt{5})/2 \\ 0 & 1 & -(1-i)(1-\sqrt{5})/2 \\ 0 & 0 & 0 \end{bmatrix}$$

A basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix} 3-\sqrt{5} \\ (1-i)(1-\sqrt{5}) \\ 2 \end{bmatrix}\right\}$.

We normalize the basis vectors for the eigenspaces to get the orthonormal basis for $\mathbb{C}^3$ of eigenvectors of A. Let

$$a = \left\|\begin{bmatrix} 3+\sqrt{5} \\ (1-i)(1+\sqrt{5}) \\ 2 \end{bmatrix}\right\| = 4 + (3+\sqrt{5})^2 + 2(1+\sqrt{5})^2 \approx 52.361$$

$$b = \left\|\begin{bmatrix} 3-\sqrt{5} \\ (1-i)(1-\sqrt{5}) \\ 2 \end{bmatrix}\right\| = 4 + (3-\sqrt{5})^2 + 2(1-\sqrt{5})^2 \approx 7.639$$

Hence,

$$U = \begin{bmatrix} -2/\sqrt{10} & (3+\sqrt{5})/a & (3-\sqrt{5})/b \\ (1-i)/\sqrt{10} & (1-i)(1+\sqrt{5})/a & (1-i)(1-\sqrt{5})/b \\ 2/\sqrt{10} & 2/a & 2/b \end{bmatrix}$$

unitarily diagonalizes F to $U^*FU = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}$.

Homework Problems

B1 (a) A is not Hermitian.

(b) B is Hermitian. It is unitarily diagonalized by $U = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{2}-i \\ -\sqrt{2}-i & 1 \end{bmatrix}$ to $D = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$.

(c) C is Hermitian. It is unitarily diagonalized by $U = \begin{bmatrix} (-\sqrt{3}-i)/\sqrt{26} & (\sqrt{3}+i)/\sqrt{11} \\ 4/\sqrt{26} & 1/\sqrt{11} \end{bmatrix}$ to $D = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$.

(d) F is Hermitian. It is unitarily diagonalized by $U = \begin{bmatrix} (1+i)/\sqrt{8} & (-1-3i)/4 & (-1+i)/4 \\ (1+i)/\sqrt{8} & (-1+i)/4 & (-1-3i)/4 \\ 1/\sqrt{2} & 1/2 & 1/2 \end{bmatrix}$ to $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

Conceptual Problems

D1 (a) The matrices $A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1+i \\ 1-i & 0 \end{bmatrix}$ are both Hermitian, but $AB = \begin{bmatrix} 1+i & 1+i \\ 1-i & 1-i \end{bmatrix}$ is not Hermitian.

(b) $(A^2)^* = (AA)^* = A^*A^* = AA = A^2$. Thus, A^2 is Hermitian.

(c) $(A^{-1})^* = (A^*)^{-1} = A^{-1}$. Thus, A^{-1} is Hermitian.

D2 By Problem 9.5.D4, we have

$$\det A = \det A^* = \det \overline{A}^T = \det \overline{A} = \overline{\det A}$$

Hence, $\det A$ is real.

D3 (a) If A is Hermitian and unitary, then

$$\begin{aligned} I = A^*A &= \begin{bmatrix} a & b+ci \\ b-ci & d \end{bmatrix}\begin{bmatrix} a & b+ci \\ b-ci & d \end{bmatrix} \\ &= \begin{bmatrix} a^2+b^2+c^2 & b(a+d)+c(a+d)i \\ b(a+d)-c(a+d)i & b^2+c^2+d^2 \end{bmatrix} \end{aligned}$$

Either $a + d = 0$ or $b = c = 0$.

If $b = c = 0$, then $A = \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$.

If $a + d = 0$, we have $a = -d$, so $a^2 = d^2$ and the diagonal entries both require that $a^2 + b^2 + c^2 = 1$. Thus, $b^2 + c^2 = 1 - a^2$. Clearly $|a| \leq 1$. Moreover, since b and c satisfy the equation of a circle, there exists some angle θ such that $b = \sqrt{1-a^2}\cos\theta$ and $c = \sqrt{1-a^2}\sin\theta$. Note that in this case $b + ci = \sqrt{1-a^2}e^{i\theta}$ and $b - ci = \sqrt{1-a^2}e^{-i\theta}$. Hence, the general form of A is

$$A = \begin{bmatrix} a & \sqrt{1-a^2}e^{i\theta} \\ \sqrt{1-a^2}e^{-i\theta} & -a \end{bmatrix}$$

where $|a| \leq 1$ and θ is any angle.

(b) If we add the additional requirement that A is diagonal, then the only possibilities are $A = \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$.

(c) For a 3×3 matrix to be Hermitian and diagonal requires that $A = \operatorname{diag}(a, b, c)$ where $a, b, c \in \mathbb{R}$.

If we require that the matrix is also unitary, then the columns must have length 1, so $A = \operatorname{diag}(\pm 1, \pm 1, \pm 1)$.

(d) Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ be an orthonormal basis of $\mathbb{V}$. Assume that L is Hermitian. By definition of $[L]_\mathcal{B}$ and our formula for finding coordinates with respect to a basis, we get

$$([L]_\mathcal{B})_{jk} = \langle L(\vec{v}_j), \vec{v}_k \rangle = \langle \vec{v}_j, L(\vec{v}_k) \rangle = \overline{\langle L(\vec{v}_k), \vec{v}_j \rangle} = ([L]_\mathcal{B})_{kj}$$

Hence, $[L]_\mathcal{B}$ is Hermitian.

On the other hand, assume that $[L]_\mathcal{B}$ is Hermitian. Then, we have

$$\langle \vec{v}_j, L(\vec{v}_k) \rangle = \overline{\langle L(\vec{v}_k), \vec{v}_j \rangle} = \langle L(\vec{v}_j), \vec{v}_k \rangle$$

Let $\vec{x}, \vec{y} \in \mathbb{V}$. Then we can write $\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n$ and $\vec{y} = d_1\vec{v}_1 + \cdots + d_n\vec{v}_n$. Observe that

$$\begin{aligned}
\langle \vec{x}, L(\vec{y})\rangle &= \langle c_1\vec{v}_1 + \cdots + c_n\vec{v}_n, L(d_1\vec{v}_1 + \cdots + d_n\vec{v}_n)\rangle \\
&= \langle c_1\vec{v}_1 + \cdots + c_n\vec{v}_n, d_1L(\vec{v}_1) + \cdots + d_nL(\vec{v}_n)\rangle \\
&= \sum_{j,k} c_j\overline{d_k}\langle \vec{v}_j, L(\vec{v}_k)\rangle \\
&= \sum_{j,k} c_j\overline{d_k}\langle L(\vec{v}_j), \vec{v}_k\rangle \\
&= \langle c_1L(\vec{v}_1) + \cdots + c_nL(\vec{v}_n), d_1\vec{v}_1 + \cdots + d_n\vec{v}_n\rangle \\
&= \langle L(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n), d_1\vec{v}_1 + \cdots + d_n\vec{v}_n\rangle \\
&= \langle L(\vec{x}), \vec{y}\rangle
\end{aligned}$$

Hence, L is Hermitian.

(e) The proof is by induction on n. If $n = 1$, then A is a diagonal matrix and hence is unitarily diagonalizable with $U = [1]$.

Suppose the result is true for all $(n-1)\times(n-1)$ Hermitian matrices, and consider an $n \times n$ Hermitian matrix A. Pick an eigenvalue λ_1 of A and find a corresponding unit eigenvector $\vec{v}_1$. Extend the set $\{\vec{v}_1\}$ to an orthonormal basis $\{\vec{v}_1, \vec{w}_2, \dots, \vec{w}_n\}$ for $\mathbb{C}^n$. The the matrix $U_0 = \begin{bmatrix} \vec{v}_1 & \vec{w}_2 & \cdots & \vec{w}_n \end{bmatrix}$ is unitary. We find that

$$U_0^*AU_0 = \begin{bmatrix} \lambda_1 & O_{1,n-1} \\ O_{n-1,1} & A_1 \end{bmatrix}$$

where A_1 is an $(n-1)\times(n-1)$ Hermitian matrix. Then, by our hypothesis there is an $(n-1)\times(n-1)$ unitary matrix U_1 such that

$$U_1^T A_1 U_1 = D_1$$

where D_1 is an $(n-1)\times(n-1)$ diagonal matrix. Define

$$U_2 = \begin{bmatrix} 1 & O_{1,n-1} \\ O_{n-1,1} & P_1 \end{bmatrix}$$

The columns of U_2 form an orthonormal basis for $\mathbb{C}^n$, hence U_2 is unitary. Since a product of unitary matrices is unitary, we get that $U = U_2U_0$ is an $n\times n$ orthogonal matrix. Moreover, by block multiplication,

$$U^*AU = \begin{bmatrix} \lambda_1 & O_{1,n-1} \\ O_{n-1,1} & D_1 \end{bmatrix} = D$$

is diagonal as required.

Chapter 9 Quiz

Problems

E1 We have $|z_1| = \sqrt{1^2 + (-\sqrt{3})^2} = 2$ and any argument θ of z_1 satisfies

$$1 = 2\cos\theta \quad \text{and} \quad -\sqrt{3} = 2\sin\theta$$

Hence, $\cos\theta = \frac{1}{2}$ and $\sin\theta = -\frac{\sqrt{3}}{2}$. Thus, $\theta = -\frac{\pi}{3} + 2\pi k$, $k \in \mathbb{Z}$. So, a polar form of z_1 is

$$z_1 = 2\left(\cos\frac{-\pi}{3} + i\sin\frac{-\pi}{3}\right)$$

We have $|z_2| = \sqrt{2^2 + 2^2} = \sqrt{8}$ and any argument θ of z_2 satisfies

$$2 = \sqrt{8}\cos\theta \quad \text{and} \quad 2 = \sqrt{8}\sin\theta$$

Hence, $\cos\theta = \frac{1}{\sqrt{2}}$ and $\sin\theta = \frac{1}{\sqrt{2}}$. Thus, $\theta = \frac{\pi}{4} + 2\pi k$, $k \in \mathbb{Z}$. So, a polar form of z_2 is

$$z_2 = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Thus,

$$\begin{aligned}
z_1 z_2 &= 2(2\sqrt{2})\left(\cos\left(\frac{-\pi}{3} + \frac{\pi}{4}\right) + i\sin\left(\frac{-\pi}{3} + \frac{\pi}{4}\right)\right) \\
&= 4\sqrt{2}\left(\cos\frac{-\pi}{12} + i\sin\frac{-\pi}{12}\right) \\
\frac{z_1}{z_2} &= \frac{2}{2\sqrt{2}}\left(\cos\left(\frac{-\pi}{3} - \frac{\pi}{4}\right) + i\sin\left(\frac{-\pi}{3} - \frac{\pi}{4}\right)\right) \\
&= \frac{1}{\sqrt{2}}\left(\cos\frac{-7\pi}{12} + i\sin\frac{-7\pi}{12}\right)
\end{aligned}$$

E2 We have $i = e^{i\frac{\pi}{2} + 2\pi k}$, so the square roots are

$$(i)^{1/2} = e^{i(\frac{\pi}{2} + 2\pi k)/2}, \quad k = 0, 1$$

In particular,

$$\begin{aligned}
w_0 &= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \\
w_1 &= \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i
\end{aligned}$$

E3 (a) $2\vec{u} + (1+i)\vec{v} = \begin{bmatrix} 7 - i \\ 3 + 5i \\ 9 + 3i \end{bmatrix}$.

(b) $\overline{\vec{u}} = \begin{bmatrix} 3 + i \\ -i \\ 2 \end{bmatrix}$.

(c) $\langle \vec{u}, \vec{v} \rangle = (3 - i)(1) + i(3) + 2(4 + i) = 11 + 4i$.

(d) $\langle \vec{v}, \vec{u} \rangle = \overline{\langle \vec{u}, \vec{v} \rangle} = 11 - 4i$.

(e) $\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle} = \sqrt{^2 + 3^2 + (4-i)(4+i)} = \sqrt{1+9+17} = \sqrt{27}$.

(f) $\operatorname{proj}_{\vec{u}} \vec{v} = \dfrac{\langle \vec{v}, \vec{u} \rangle}{\|\vec{u}\|^2}\vec{u} = \dfrac{11-4i}{15}\begin{bmatrix} 3-i \\ i \\ 2 \end{bmatrix} = \dfrac{1}{15}\begin{bmatrix} 29-23i \\ 4+11i \\ 22-8i \end{bmatrix}$.

E4 (a) We have

$$C(\lambda) = \begin{vmatrix} -\lambda & 13 \\ -1 & 4-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 13$$

Thus, the eigenvalues of A are $\lambda = \frac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i$.
For $\lambda_1 = 2 + 3i$ we get

$$\begin{bmatrix} -2-3i & 13 \\ -1 & 2-3i \end{bmatrix} \sim \begin{bmatrix} 1 & -2+3i \\ 0 & 0 \end{bmatrix}$$

Hence, an eigenvector corresponding to λ_1 is $\vec{z}_1 = \begin{bmatrix} 2-3i \\ 1 \end{bmatrix}$.

For $\lambda_2 = 2 - 3i$ we get

$$\begin{bmatrix} -2+3i & 13 \\ -1 & 2+3i \end{bmatrix} \sim \begin{bmatrix} 1 & -2-3i \\ 0 & 0 \end{bmatrix}$$

Hence, an eigenvector corresponding to λ_2 is $\vec{z}_2 = \begin{bmatrix} 2+3i \\ 1 \end{bmatrix}$.

Hence, $P = \begin{bmatrix} 2-3i & 2+3i \\ 1 & 1 \end{bmatrix}$ diagonalizes A to $P^{-1}AP = D = \begin{bmatrix} 2+3i & 0 \\ 0 & 2-3i \end{bmatrix}$.

(b) Using our work in (a), pick $\lambda = 2 + 3i$, then $a = 2$ and $b = 3$. Also, we have a corresponding eigenvector is

$$\vec{z} = \begin{bmatrix} 2-3i \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + i\begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

Thus, the change of coordinates matrix $P = \begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix}$ brings A into the real canonical form $\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$.

E5 We have

$$UU^* = \frac{1}{\sqrt{3}}\begin{bmatrix} 1-i & -i \\ 1 & -1+i \end{bmatrix} \frac{1}{\sqrt{3}}\begin{bmatrix} 1+i & 1 \\ i & -1-i \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so U is unitary.

E6 (a) A is Hermitian if $A^* = A$. Thus, we must have $3 + ki = 3 - i$ and $3 - ki = 3 + i$, which is only true when $k = -1$. Thus, A is Hermitian if and only if $k = -1$.

(b) If $k = -1$, then $A = \begin{bmatrix} 0 & 3-i \\ 3+i & 3 \end{bmatrix}$, and

$$\det(A - \lambda I) = \begin{vmatrix} 0-\lambda & 3-i \\ 3+i & 3-\lambda \end{vmatrix} = (\lambda+2)(\lambda-5)$$

Thus the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 5$. For $\lambda_1 = -2$,

$$A - \lambda_1 I = \begin{bmatrix} 2 & 3-i \\ 3+i & 5 \end{bmatrix} \sim \begin{bmatrix} 2 & 3-i \\ 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace of λ_1 is $\left\{\begin{bmatrix} 3-i \\ -2 \end{bmatrix}\right\}$. For $\lambda_2 = 5$,

$$A - \lambda_2 I = \begin{bmatrix} -5 & 3-i \\ 3+i & -2 \end{bmatrix} \sim \begin{bmatrix} -5 & 3-i \\ 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace of λ_2 is $\left\{\begin{bmatrix} 3-i \\ 5 \end{bmatrix}\right\}$.

We can easily verify that $\left\langle \begin{bmatrix} 3-i \\ -2 \end{bmatrix}, \begin{bmatrix} 3-i \\ 5 \end{bmatrix} \right\rangle = 0$.

Chapter 9 Further Problems

Problems

F1 Suppose that U is a unitary matrix such that $U^*AU = D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Define $C = \operatorname{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$. Then observe that taking $B = UCU^*$ gives

$$B^2 = (UCU^*)(UCU^*) = UCU^*UCU^* = UC^2U^* = UDU^* = A$$

and

$$B^* = (UCU^*) = UC^*U^* = UCU^* = B$$

as required.

F2 (a) We have $(A^*A)^* = A^*(A^*)^* = A^*A$, so A^*A is Hermitian.

Let λ be any eigenvalue of A^*A with corresponding unit eigenvector $\vec{z}$. Since, A^*A is Hermitian, we have that λ is real. Also,

$$\langle A\vec{z}, A\vec{z}\rangle = (A\vec{z})^T \overline{A\vec{z}} = \vec{z}^T A^T \overline{A}\,\overline{\vec{z}} = \vec{z}^T \overline{A^*}\,\overline{A}\,\overline{\vec{z}} = \vec{z}^T \overline{A^*A\vec{z}} = \vec{z}^T (\overline{\lambda\vec{z}}) = \lambda \vec{z}^T \overline{\vec{z}} = \lambda$$

Thus, $\lambda \geq 0$.

(b) If A is invertible, we can repeat our work in (a) to get $\lambda = \langle A\vec{z}, A\vec{z}\rangle$. But, since $\vec{z}$ is an eigenvector, we have that $\vec{z} \neq \vec{0}$, so $A\vec{z} \neq \vec{0}$ since A is invertible. Thus, $\lambda > 0$.

F3 We prove this by induction. If $n = 1$, then A is upper triangular. Assume the result holds for some $n = k \geq 1$. Let λ_1 be an eigenvalue of A with corresponding unit eigenvector $\vec{z}_1$. Extend $\{\vec{z}_1\}$ to an orthonormal basis $\{\vec{z}_1, \dots, \vec{z}_n\}$

of $\mathbb{C}^n$ and let $U_1 = \begin{bmatrix} \vec{z}_1 & \cdots & \vec{z}_n \end{bmatrix}$. Then, U_1 is unitary and we have

$$\begin{aligned}
U_1^* A U_1 &= \begin{bmatrix} \overline{\vec{z}_1}^T \\ \vdots \\ \overline{\vec{z}_n}^T \end{bmatrix} A \begin{bmatrix} \vec{z}_1 & \cdots & \vec{z}_n \end{bmatrix} \\
&= \begin{bmatrix} \overline{\vec{z}_1}^T \\ \vdots \\ \overline{\vec{z}_n}^T \end{bmatrix} \begin{bmatrix} A\vec{z}_1 & \cdots & A\vec{z}_n \end{bmatrix} \\
&= \begin{bmatrix} \overline{\vec{z}_1}^T \lambda_1 \vec{z} & \overline{\vec{z}_1}^T A\vec{z}_2 & \cdots & \overline{\vec{z}_1}^T A\vec{z}_n \\ \overline{\vec{z}_2}^T \lambda_1 \vec{z} & \overline{\vec{z}_2}^T A\vec{z}_2 & \cdots & \overline{\vec{z}_2}^T A\vec{z}_n \\ \vdots & \vdots & & \vdots \\ \overline{\vec{z}_n}^T \lambda_1 \vec{z} & \overline{\vec{z}_n}^T A\vec{z}_2 & \cdots & \overline{\vec{z}_n}^T A\vec{z}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & B \\ 0 & A_1 \end{bmatrix}
\end{aligned}$$

where A_1 is an $(n-1)\times(n-1)$ matrix. Thus, by the inductive hypothesis we get that there exists a unitary matrix Q such that $Q^* A_1 Q = T_1$ is upper triangular. Let $U_2 = \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix}$, then U_2 is clearly unitary and hence $U = U_1 U_2$ is unitary. Thus

$$\begin{aligned}
U^* A U &= (U_1 U_2)^* A (U_1 U_2) = U_2^* U_1^* A U_1 U_2 \\
&= \begin{bmatrix} 1 & 0 \\ 0 & Q^* \end{bmatrix} \begin{bmatrix} \lambda_1 & B \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} \lambda_1 & BQ \\ 0 & T_1 \end{bmatrix} = T
\end{aligned}$$

which is upper triangular.

F4 (a) If A is unitarily diagonalizable, then there exists a unitary matrix U and diagonal matrix D such that $U^* A U = D$ or $A = UDU^*$. Since D is diagonal, observe that $DD^* = D^* D$. Hence,

$$\begin{aligned}
AA^* &= (UDU^*)(UDU^*)^* = UDU^* UD^* U^* = UDD^* U^* \\
&= UD^* DU^* = UD^* U^* UDU^* = (UDU^*)^* (UDU^*) = AA^*
\end{aligned}$$

Thus, A is normal.

(b) By Schur's Theorem, there exist an upper triangular matrix T and unitary matrix U such that $U^* A U = T$. Observe that T is normal since

$$\begin{aligned}
TT^* &= (U^* A U)(U^* A^* U) = U^* A A^* U \\
&= U^* A^* A U = (U^* A^* U)(U^* A U) = T^* T
\end{aligned}$$

(c) Comparing the diagonal entries of TT^* and T^*T we get

$$\begin{aligned}
|t_{11}|^2 + |t_{12}|^2 + \cdots + |t_{1n}|^2 &= |t_{11}|^2 \\
|t_{22}|^2 + \cdots + |t_{2n}|^2 &= |t_{12}|^2 + |t_{22}|^2 \\
&\vdots \\
|t_{nn}|^2 &= |t_{1n}|^2 + |t_{2n}|^2 + \cdots + |t_{nn}|^2
\end{aligned}$$

Hence, we must have $t_{ij} = 0$ for all $i \neq j$ and so T is diagonal. Thus, from our work in (b), we have that A is unitarily similar to the diagonal matrix T as required.