MATH 2107 TEST 1

Only nonprogrammable calculators are allowed.

Duration: 50 minutes. **Total marks:30**

NAME (in ink): **STUDENT NO (in ink):**

PART I: [3] True/False questions. Circle the correct answer in ink.

1. If $\{v_1, v_2, v_3\}$ is independent, so is $\{v_1 + v_2 + v_3\}$. [True | False]
2. Basis is a spanning set of the vector space that is as large as possible. [True | False]
3. Three vectors in P_3 can be linearly independent. [True | False]

PART II: [6] Multiple choice questions. Circle the correct answer in ink.

1. Let $H = span\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\end{bmatrix}\right\}$. The dimension of H equals to

   a) 4 b) 3 c) 2 d) 0

2. Which of the following is a basis of P_2 ?

   $B_1 = \{x, 1+x, x-x^2\}$, $B_2 = \{1-x, 1-x^2, x+x^2\}$

   $B_3 = \{1, 1+x+x^2\}$, $B_4 = \{1, 2-x, 3-x^2, x+2x^2\}$

   a) B_1 and B_3 b) B_2 only c) B_1 and B_2 d) B_1, B_2 and B_4

3. Let $S = \left\{\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}x\\1\\0\end{bmatrix}, \begin{bmatrix}0\\2\\3\end{bmatrix}\right\}$. Which values of x makes S linearly dependent?

   a) $x = 0$ b) $x = 1$ c) $x = -1$ d) none of the above

PART III: [21] Long answer questions. Show all your work.

[3] 1. Determine whether the set $B = \{x, 2x - x^2, 2 + 3x + 2x^2\}$ is a basis of P_2. Explain.

ANS:

$\dim P_2 = 3$. The set B has three vectors. In that case if the vectors in B are linearly independent, then they form a basis.

Let $ax + b(2x - x^2) + c(2 + 3x + 2x^2) = O$

$\Rightarrow 2c + (a + 2b + 3c)x + (-2b + 2c)x^2 = O$

$$\begin{aligned} &2c = 0 && c = 0 \\ \Rightarrow\ &a + 2b + 3c = 0 \Rightarrow && b = 0 \\ &-2b + 2c = 0 && a = 0 \end{aligned}$$

$\Rightarrow$ The vectors are linearly independent.

So the set B is a basis of P_2.

[7] 2. Let $A=\begin{bmatrix}1 & -2 & 1 & 1 & 2\\ -1 & 3 & 0 & 2 & -2\\ 0 & 1 & 1 & 3 & 4\\ 1 & 2 & 5 & 13 & 5\end{bmatrix}\sim\begin{bmatrix}1 & -2 & 1 & 1 & 2\\ 0 & 1 & 1 & 3 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$.

(a) [2] Find a basis for the column space of A .

From the row echelon form of A, the columns #1, #2,#5 have pivots. Therefore a basis for the column space of A is

$$\left\{\begin{bmatrix}1\\ -1\\ 0\\ 1\end{bmatrix},\begin{bmatrix}-2\\ 3\\ 1\\ 2\end{bmatrix},\begin{bmatrix}2\\ -2\\ 4\\ 5\end{bmatrix}\right\}$$

(b) [3] Find a basis for the null space of A.

From the row echelon form of A, x_3, x_4 are free variables.

Let $x_3 = s, x_4 = t$

Also $x_5 = 0$.

From the second row, we have $x_2 + x_3 + 3x_4 = 0 \Rightarrow x_2 = -s - 3t$

From the first row, we have

$x_1 - 2x_2 + x_3 + x_4 + 2x_5 = 0 \Rightarrow x_1 = 2(-s-3t) - s - t = -3s - 7t$

Therefore the solution of $AX = 0$ is given by

$$X=\begin{bmatrix}-3s-7t\\ -s-3t\\ s\\ t\\ 0\end{bmatrix}=-s\begin{bmatrix}3\\ 1\\ -1\\ 0\\ 0\end{bmatrix}-t\begin{bmatrix}7\\ 3\\ 0\\ -1\\ 0\end{bmatrix}.$$

So a basis of the null space of A is

$$\left\{\begin{bmatrix}3\\ 1\\ -1\\ 0\\ 0\end{bmatrix},\begin{bmatrix}7\\ 3\\ 0\\ -1\\ 0\end{bmatrix}\right\}$$

(c) [2] Find the values of $rank\ A$ and $Nullity\ A$.

$rank\ A = 3$

$Nullity\ A = 2$

[4] 3. Let $\{v_1, v_2, v_3\}$ be a basis for a vector space V . Consider the set of vectors $S = \{v_1, v_1 + v_2, v_1 + v_2 + v_3\}$.

a) Show that S is linearly independent.

ANS:

Let $sv_1 + t(v_1 + v_2) + u(v_1 + v_2 + v_3)$ be a linear combination of $v_1, v_1 + v_2, v_1 + v_2 + v_3$

$sv_1 + t(v_1 + v_2) + u(v_1 + v_2 + v_3) = (s + t + u)v_1 + (t + v)v_2 + uv_{31}$

Since $\{v_1, v_2, v_3\}$ is a basis of V , they are linearly independent. Hence

$$\left.\begin{aligned} s+t+u&=0\\ t+u&=0\\ u&=0\end{aligned}\right\}\Rightarrow s=t=u=0$$

Therefore $S = \{v_1, v_1 + v_2, v_1 + v_2 + v_3\}$ is linearly independent.

b) Is S a basis for V ? Explain.

ANS:

yes, since dim(V)=3 and S has 3 linearly independent vectors.
Therefore they span V too and form a basis.

[3] 4. Determine whether $W = \left\{ \begin{bmatrix} a & b \\ b & 2a \end{bmatrix} \right\}$ is a subspace of M_{22}. Explain.

ANS #1:

Any vector in W can be expressed as $\begin{bmatrix} a & b \\ b & 2a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

i.e. $W = span\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$, two vectors of M_{22}.

Therefore, W is a subspace of M_{22}.

ANS #2:

Show zero vector is in W.
Show that W is closed under vector addition.
Show that W is closed under scalar multiplication.
Conclude that it is a subspace.

[4] 5. Find a basis of M_{22} containing the linearly independent set
$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

The vectors $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ are linearly independent.

$\dim M_{22} = 4$

A vector from the standard basis $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ of M_{22} can be added to extend the given set to form a basis of M_{22}.

Any linear combination of the three given set is given by

$a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a-c & b \\ b & a+c \end{bmatrix}$.

Any one of the two vectors $\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ can be added to get the basis

(The last step can be done in many ways)

They need to show that the final set is linearly independent.

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