Mathematics 342. Problem Set 1. Due Thursday, January 18, 2018.

(1) How many errors can be detected with the following q-ary codes? How many errors can be corrected? Explain your answers.
  (a) $C_1 = \{(0,0,0,0,1),(0,1,1,1,1),(1,1,1,0,0)\}$. Here $q = 2$.
  (b) $C_2 = \{(0,1,2,0,1,2),(2,1,0,2,1,0),(2,2,2,2,2,2)\}$. Here $q = 3$.
  (c) $C_3 = \{(0,1,2,3,4,5,6),(1,2,3,4,5,6,0),(2,3,4,5,6,0,1),(3,4,5,6,0,1,2),$ $(4,5,6,0,1,2,3),(5,6,0,1,2,3,4),(6,0,1,2,3,4,5)\}$. Here $q = 7$.

(2) Assume the code C_1 from Problem 1 was used in transmission, and the following words were received. Decode each of these words using the nearest neighbour decoding algorithm. (The incomplete decoding version: if there is more than one nearest neighbour, declare an error.)
  (a) $(0,0,1,1,1)$, (b) $(1,1,0,0,0)$, (c) $(1,1,1,1,1)$, (d) $(1,0,1,0,1)$.

Recall that the triangle inequality for the Hamming distance says that

$$d(\mathbf{a},\mathbf{b}) + d(\mathbf{b},\mathbf{c}) \leq d(\mathbf{a},\mathbf{c}).$$

Here **a**, **b** and **c** are q-ary words of length n. We will say that **b** lies *between* **a** and **c** if equality holds in the above formula, i.e.,

$$d(\mathbf{a},\mathbf{b}) + d(\mathbf{b},\mathbf{c}) = d(\mathbf{a},\mathbf{c}).$$

The purpose of the next four exercises is to discover and prove a formula for the number of words that lie between **a** and **c**.

(3) How many words lie between **a** and **a**?

(4) Assume $q = 2$ and $n = 3$. How many words lie
  (i) between $(0,0,0)$ and $(1,1,1)$?
  (ii) between $(0,0,0)$ and $(1,1,0)$?
  (iii) between $(0,0,0)$ and $(1,0,0)$?

(5) Suppose $q = 2$, **a** and **c** are binary words of length n and $d(\mathbf{a},\mathbf{c}) = d$. Based on your answers in Problems 3 and 4, guess a formula for the number of binary words of length n lying between **a** and **c**. Prove this formula.

(6) We now allow q to be arbitrary and ask the same question as in Problem 5. Given q-ary words **a** and **b** of length n and at Hamming distance d, how many q-ary words of length n lie between **a** and **c**? Prove your answer.

(7) How many binary words of length 6 are at Hamming distance
  (a) at Hamming distance 6 from $(1,0,1,0,1,0)$?
  (b) at Hamming distance 5 from $(1, 0, 1, 0, 1, 0)$?

(8) Is the code C_3 in Problem 1 equivalent to the 7-ary repetition code

$$C_4 = \{(0,...,0),(1,...,1),...,(6,...,6)\}$$

of length 7? Prove your answer.