MATHEMATICS 1LS3 TEST 2

Day Class
Duration of Examination: 60 minutes
McMaster University, 30 October 2017

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First name (PLEASE PRINT): SOLUTIONS

Family name (PLEASE PRINT): ______________________

Student No.: ______________________

THIS TEST HAS 8 PAGES AND 6 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. USE PEN TO WRITE YOUR TEST. IF YOU USE A PENCIL YOUR TEST WILL NOT BE ACCEPTED FOR REMARKING (IF NEEDED).

Total number of points is 40. Marks are indicated next to the problem number. Calculator allowed: McMaster standard calculator Casio fx991MS or Casio fx991MS PLUS or lower Casio which has two lines of discplay and no graphing capabilities.

EXCEPT ON QUESTIONS 1 AND 2, you must show work to receive full credit.

Problem	Points	Mark
1	10	
2	6	
3	6	
4	5	
5	6	
6	7	
TOTAL	40	

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1. **Multiple choice questions: circle ONE answer. No justification is needed.**

(a)[2] If $f(x) = e^{\tan(\pi x)}$, then $f'(0) =$

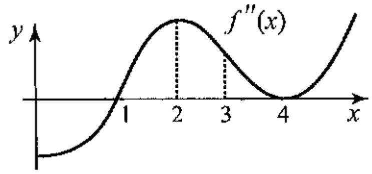
(A) 0 (B) 1 (C) -1 (D) π

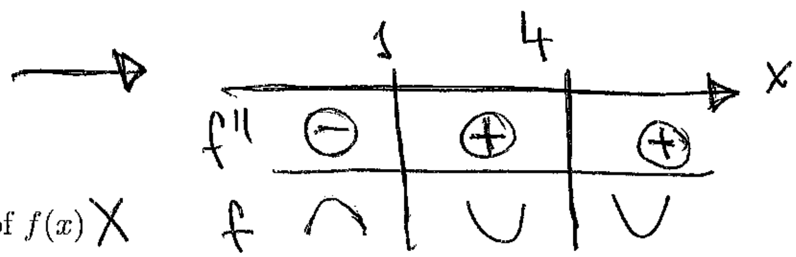
(E) $\pi/4$ (F) $-\pi$ (G) $-\pi/4$ (H) 2

$$f'(x) = e^{\tan(\pi x)} \cdot \sec^2(\pi x) \cdot \pi$$

$$f'(0) = e^0 \cdot 1 \cdot \pi = \pi$$

(b)[2] The graph of the second derivative $f''(x)$ of a function $f(x)$ is given. Which statements is/are true?





(I) $x = 4$ is a point of inflection of $f(x)$ X

(II) The graph of $f(x)$ is concave up on $(1, 4)$ ✓

(III) $x = 1$ is a point of inflection of $f(x)$ ✓

(A) none (B) I only (C) II only (D) III only

(E) I and II (F) I and III (G) II and III (H) all three (c)[2] It is known that the function $f(x)$ is defined for all real numbers, and its derivative is given by $f'(x) = \dfrac{(x-3)\,e^{-2x}}{(4-x)^{1/3}}$. Find all its critical points.

(A) no critical points (B) 0 only (C) 3 only (D) 4 only
(E) 0 and 3 (F) 0 and 4 (G) 3 and 4 (H) 0, 3 and 4

$f' = 0 \rightarrow x = 3$
f' dne $\rightarrow x = 4$

(d)[2] $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = \lim\limits_{x\to 0} \dfrac{e^x}{1} = 1$

$\frac{0}{0}$

(A) 0 (B) ∞ (C) 1 (D) 1/2
(E) 1/3 (F) 1/4 (G) 1/6 (H) $-\infty$

(e)[2] $\lim\limits_{x\to 0^+} x \ln x = \lim\limits_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x}} = \lim\limits_{x\to 0^+} \dfrac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim\limits_{x\to 0^+}(-x) = 0$

$0\cdot(-\infty)$

(A) 0 (B) ∞ (C) 1 (D) -1
(E) 1/2 (F) $-1/2$ (G) 1/4 (H) $-\infty$

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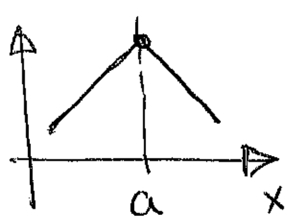
2. **True/false questions: circle ONE answer. No justification is needed.**

(a)[2] If $f(x)$ has a relative maximum at $x = a$, then $f(x)$ must satisfy $f'(a) = 0$.

TRUE FALSE

f could be



(b)[2] If $f(x) = g(x)h(x)$, then by the product rule, $f''(x) = g''(x)h(x) + g(x)h''(x)$.

TRUE FALSE

$f' = g'h + gh'$

$\to f'' = g''h + g'h' + g'h' + gh''$

(c)[2] The formula $\lim_{h\to 0} \frac{e^{2+h} - e^2}{h} = e^2$ is correct.

YES NO

looks like $\boxed{\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x)}$

$f(x) = e^x$

$x = 2$

$\lim_{h\to 0} \frac{e^{2+h} - e^2}{h} = f'(2) = e^x$ when $x = 2$

$= e^2$

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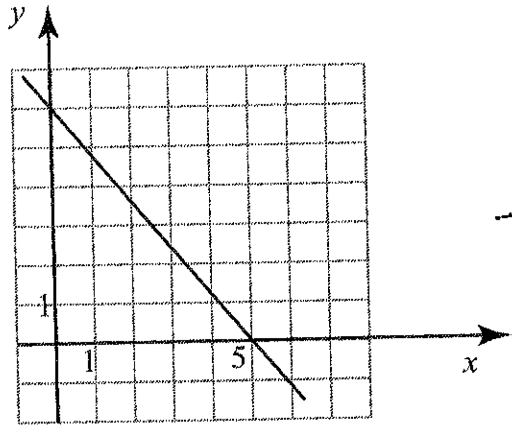
Questions 3-6: You must show correct work to receive full credit.

3. (a)[3] Let $h(x) = x + \sin(f(x))$. The graph of $f(x)$ is a line shown below. Find $h'(5)$.



$$h'(x) = 1 + \cos(f(x)) \cdot f'(x)$$

$$\rightarrow h'(5) = 1 + \cos(f(5)) \cdot f'(5)$$

$$= 1 + \underbrace{\cos(0)}_{1} \cdot \left(-\frac{6}{5}\right)$$

$$= -\frac{1}{5}$$

(b)[3] Find $y'(0)$, if $\arcsin(xy) = x^3 + y^2 - 1$, and $y(0) = 1$.

$$\frac{1}{\sqrt{1-(xy)^2}} \cdot (y + xy') = 3x^2 + 2yy'$$

$$\begin{matrix} x=0 \\ y=1 \end{matrix} \rightarrow \quad 1(1+0) = 0 + 2y'$$

$$2y' = 1$$

$$y' = \frac{1}{2}$$

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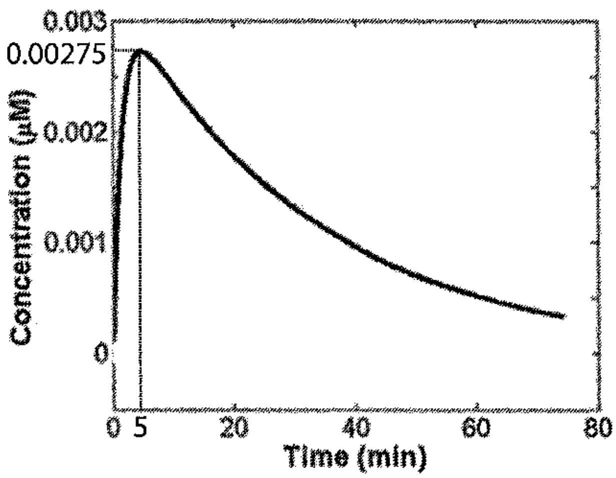
4. (a)[3] Find the relative maximum of $f(t) = Ate^{-\beta t}$ and the value of t where it occurs. (Your answer will contain A and β.)

$$f'(t) = A\left(e^{-\beta t} + te^{-\beta t}(-\beta)\right)$$
$$= \underbrace{Ae^{-\beta t}}_{\neq 0}(1-\beta t) = 0$$
$$1-\beta t = 0 \rightarrow t = \frac{1}{\beta}$$
$$f\left(\frac{1}{\beta}\right) = A\left(\frac{1}{\beta}\right)e^{-\beta \cdot \frac{1}{\beta}} = \frac{A}{\beta}e^{-1} = \frac{A}{\beta e}$$

(b)[2] The following graph is taken from *Modeling of pharmacokinetics of cocaine in human reveals the feasibility for development of enzyme therapies for drugs of abuse.* C.G. Zhan and F. Zheng. PLoS Computational Biology. 8.7 (July 2012).



You need to find a formula for the concentration as a function of time.

Going through your textbook, you notice that the graph of $f(t) = Ate^{-\beta t}$ (with $A > 0$ and $\beta > 0$) matches the shape of the given graph. Using your answer to (a), identify the values of β and A, and write the formula for the concentration.

$$t = \frac{1}{\beta} = 5 \rightarrow \beta = 0.2$$

$$\frac{A}{\beta e} = 0.00275 \rightarrow A = \beta e(0.00275) \simeq 0.001468 \approx 0.0015$$

$$\text{so } c(t) = 0.0015 \cdot t \cdot e^{-0.2t}$$

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5. (a)[3] In the article *Migration behaviour of grizzly bears in Northern British Columbia: contribution to a modelling approach.* G. Brown et al, Bear Science 4 (June 2012), we find the formula

$$P(t) = \arctan(1.67t) + 4.71$$

where t represents time.

Next, we read "initially, $P(t) \approx 1.67t + 4.71$, which gives a linear relationship." Explain how linear approximation can be used to verify this statement.

$P(t) \approx P(0) + P'(0)t \leftarrow$ linear approximation

$\underline{P(0)} = \arctan 0 + 4{,}71 = 4{,}71$

$P'(t) = \dfrac{1}{1+(1{,}67t)^2} \cdot 1{,}67 \rightarrow \underline{P'(0)} = 1{,}67$

thus $P(t) \approx 4{,}71 + 1{,}67t$

(b)[3] The linear model for the ratio S of cancer cells surviving radiation treatment states that

$$S(x) = e^{-ax+b}$$

where a and b are constants and x is a radiation dose. This formula is sometimes simplified using a quadratic approximation near $x = 0$. Find that approximation.

$\hookrightarrow T_2(x) = S(0) + S'(0)x + \dfrac{S''(0)}{2!}x^2$

S, S', S'' ..	at $x=0$
$S = e^{-ax+b}$	e^b
$S' = -ae^{-ax+b}$	$-ae^b$
$S'' = a^2e^{-ax+b}$	a^2e^b

so

$T_2(x) = e^b - ae^b x + \dfrac{a^2e^b}{2}x^2$

or: $Q(x)$

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6. The function $c(t) = t^2e^{-6t}$ has been used to model the absorption of a drug (such as morphine); $c(t)$ is the concentration (in milligrams per millilitre, mg/mL) of the drug in the bloodstream, and $t \geq 0$ is time (in hours).

(a)[3] The function $c(t)$ has two critical points such that $t \geq 0$. Find them.

$$c'(t) = 2te^{-6t} + t^2e^{-6t}(-6)$$
$$= 2te^{-6t}(1-3t) = 0 \longrightarrow t=0,\ t=\frac{1}{3}$$

(b)[2] Give a precise statement of the Extreme Value Theorem.

IF $f(x)$ continuous, defined on a closed interval $[a,b]$

THEN $f(x)$ has an abs. max. and an abs. min. in $[a,b]$

(c)[2] Find the absolute maximum and the absolute minimum values that the concentration $c(t)$ reaches during the first hour after the drug is administered, i.e., over the interval $[0, 1]$.

t	$c(t)$
$1/3$	$\left(\frac{1}{3}\right)^2 e^{-6(\frac{1}{3})} = \frac{1}{9e^2} \approx 0.015$
0	0
1	$e^{-6} = \frac{1}{e^6} \approx 0.002$

abs. max. at $t=\frac{1}{3}$, value $c(\frac{1}{3}) \approx 0.015$ mg/mL

abs. min. at $t=0$, value $c(0)=0$ mg/mL

THE END