MAT1341 - Midterm exam

June 18, 2018. Duration: 80 minutes.

Instructor: Antoine Poirier.

θ	$\sin\theta$	$\cos\theta$
0	0	1
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1	0

1	
2	
3	
4	
5	
6	
Total	

Family Name:______________________

First Name:______________________

Student Number:____________________

DGD room::____________________

PLEASE READ THESE INSTRUCTIONS CAREFULLY.

- You have 80 minutes to complete this quiz.
- This is a closed book quiz, and no notes of any kind are permitted. The use of calculators, cell phones, or similar devices is not permitted.
- Read each question carefully. You will save yourself time and unnecessary grief later on.
- Questions 1-3 are multiple choice and are worth 1 points each.
- Questions 4-6 require complete solution, and are worth 6 points each.
- **The correct answer in questions 4-6 requires justification written legibly and logically: you must convince the marker that you know why your solution is correct.**
- Good luck!

1. Which of the following are subspaces of $\mathbb{R}^3$?

$U = \{(x-y,\ x+y,\ x-y) \mid x,\ y \in \mathbf{R}\}$
$V = \{(x,\ y,\ -y) \mid x,\ y \in \mathbf{R}\}$
$W = \{(x^2,\ y,\ x+y) \mid x,\ y \in \mathbf{R}\}$
$X = \{(x,y,z) \mid x-y=0\}$

$U = \text{span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\-1\end{pmatrix}\right\}$, so U is a subspace of $\mathbb{R}^3$

$V = \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}\right\}$, so V is also a subspace of $\mathbb{R}^3$.

A. U and V only.
B. U and W only.
C. W and X only.
D. U, W and X only.
[✓ E.] U, V and X only.
F. V and W only.

$\begin{pmatrix}1\\2\\3\end{pmatrix} \in W$ $(x=1,\ y=2)$ but $2\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}2\\4\\6\end{pmatrix} \notin W$

$(y=4\ \ x=2\ \ x^2 \neq 2)$

so W is not a subspace of $\mathbb{R}^3$

$X = \text{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ so it is a subspace of $\mathbb{R}^3$

2. Suppose we have a vector space V spanned by $\{\mathbf{v}_1, \ldots, \mathbf{v}_8\}$. Suppose that $\{\mathbf{u}_1, \ldots, \mathbf{u}_6\}$ is a linearly independent subset of V. Which of the following statements are true?

(I) Any set of 7 vectors in V is linearly dependent.
(II) $6 \leq \dim(V)$.
(III) $\dim(V) \leq 8$.
(IV) There exists a subset of $\{\mathbf{v}_1, \ldots, \mathbf{v}_8\}$ (possibly the whole set itself) which is a basis of V.

A. (I) and (II) only.
B. (II) and (III) only.
C. (I) and (IV) only.
D. (I), (II) and (III) only.
[✓ E.] (II), (III) and (IV) only.
F. (I), (III) and (IV) only.

(I) is false since $\dim(V)$ could be equal to 8, should $\{v_1, \ldots v_8\}$ be a basis.

(II) is true by the following inequality:

size of LI set $\leq \dim(V) \leq$ size of spanning set

(III) is true by the same inequality

(IV) is true, since we can reduce a spanning set of V until it becomes LI, and still maintain the span

3. Let

$$\mathbf{u} = \begin{bmatrix}1\\2\\1\end{bmatrix}, \mathbf{v} = \begin{bmatrix}2\\-2\\6\end{bmatrix}, \mathbf{w} = \begin{bmatrix}4\\14\\0\end{bmatrix}, \mathbf{x} = \begin{bmatrix}1\\-4\\t\end{bmatrix}$$

For what value of t is the vector $\mathbf{x}$ in span$\{\mathbf{u},\mathbf{v},\mathbf{w}\}$?

A. $t = 0$.
B. $t = 1$.
C. $t = 2$.
D. $t = 3$.
E. $t = 4$.
[✓ F. $t = 5$.]

We solve

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 1 \\ 2 & -2 & 14 & -4 \\ 1 & 6 & 0 & t \end{array}\right) \underset{R_3 \to R_3 - R_1}{\overset{R_2 \to R_2 - 2R_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 1 \\ 0 & -6 & 6 & -6 \\ 0 & 4 & -4 & t-1 \end{array}\right)$$

$$\overset{R_2 \to \frac{-1}{6}R_2}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 4 & -4 & t-1 \end{array}\right) \overset{R_3 \to R_3 - 4R_2}{\sim} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & t-5 \end{array}\right)$$

$\bar{x} \in$ span$\{u, v, w\}$ if and only if the previous system is consistent, so if and only if $t=5$. If $t \neq 5$, the rank of the augmented matrix is strictly greater than the rank of the coefficient matrix, meaning the system is inconsistent.

Let us check our linear combination

$$1+x^2 + -1 + x^3 - (2x+x^2) = -2x+x^3$$

$\Rightarrow k \in U$.

d) Since $k \in U$, $\text{span}\{f, g, h, k\} = \text{span}\{f, g, h\} = U$
so $\dim(\text{span}\{f, g, h, k\}) = \dim(U) = 3$.

4. Consider the following functions of the vector space $\mathbb{P}_3 = \{a+bx+cx^2+dx^3 \mid a,b,c,d \in \mathbb{R}\}$:

$$\begin{aligned} f(x) &= 1+x^2 \\ g(x) &= -1+x^3 \\ h(x) &= 2x+x^2 \end{aligned}$$

Let $U = \operatorname{span}\{f,g,h\}$.

a) Show that $\{f,g,h\}$ is linearly independent.

b) Find a basis for U and $\dim(U)$.

c) Let $k(x) = -2x+x^3$. Show that $k \in U$.

d) What is the dimension of $\operatorname{span}\{f,g,h,k\}$?

a) $a(1+x^2)+b(-1+x^3)+c(2x+x^2)=0$

for $x=0$: $a-b=0$
for $x=1$: $2a+3c=0$
for $x=-1$: $2a-2b-c=0$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 2 & 0 & 3 & 0 \\ 2 & -2 & -1 & 0 \end{array}\right) \underset{R_3\to R_3-2R_1}{\overset{R_2\to R_2-2R_1}{\sim}} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right)$$

$$\underset{R_3\to -R_3}{\overset{R_2\to \frac{1}{2}R_2}{\sim}} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 3/2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Thus system has a unique solution:
$a=0, b=0, c=0 \Rightarrow$ LI

b) Since $\{f,g,h\}$ is a LI spanning set for U, it is a basis, so $\dim(U)=3$.

c) We need to solve $a(1+x^2)+b(-1+x^3)+c(2x+x^2)=-2x+x^3$

for $x=0$: $a-b=0$
for $x=1$: $2a+3c=-1$
for $x=-1$: $2a-2b-c=1$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 2 & 0 & 3 & -1 \\ 2 & -2 & -1 & 1 \end{array}\right) \underset{R_3\to R_3-2R_1}{\overset{R_2\to R_2-2R_1}{\sim}} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{array}\right)$$

$$\underset{R_3\to -R_3}{\overset{R_2\to \frac{1}{2}R_2}{\sim}} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 3/2 & -1/2 \\ 0 & 0 & 1 & -1 \end{array}\right) \overset{R_2\to R_2-3/2R_3}{\sim} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right) \overset{R_1\to R_1+R_2}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$\Rightarrow a=1, b=1, c=-1$

5. Consider the following linear system with the three variables x, y and z:

$$\begin{aligned} x+y+2z &= 1\\ 2x+y+az &= 2\\ 3x+4y+az &= c \end{aligned}$$

a) For what values of a and c is this system inconsistent?

b) For what values of a and c does this system have a unique solution?

c) For what values of a and c does this system have infinitely many solutions?

d) Using your answer in c), find the general solution when this system has infinitely many solutions.

$$\left(\begin{array}{ccc|c}1&1&2&1\\2&1&a&2\\3&4&a&c\end{array}\right) \underset{R_3\to R_3-3R_1}{\overset{R_2\to R_2-2R_1}{\sim}} \left(\begin{array}{ccc|c}1&1&2&1\\0&-1&a-4&0\\0&1&a-6&c-3\end{array}\right) \overset{R_2\to -R_2}{\sim} \left(\begin{array}{ccc|c}1&1&2&1\\0&1&4-a&0\\0&1&a-6&c-3\end{array}\right)$$

$$\underset{\sim}{R_3\to R_3-R_2} \left(\begin{array}{ccc|c}1&1&2&1\\0&1&4-a&0\\0&0&2a-10&c-3\end{array}\right)$$

a) If $a=5$ and $c\neq 3$, then

$$\left(\begin{array}{ccc|c}1&1&2&1\\0&1&-1&0\\0&0&0&c-3\end{array}\right) \overset{R_3\to\frac{1}{c-3}R_3}{\sim} \left(\begin{array}{ccc|c}1&1&2&1\\0&1&-1&0\\0&0&0&1\end{array}\right)$$

This system is inconsistent

b) If $a\neq 5$ then

$$\left(\begin{array}{ccc|c}1&1&2&1\\0&1&4-a&0\\0&0&2a-10&c-3\end{array}\right) \overset{R_3\to\frac{1}{2a-10}R_3}{\sim} \left(\begin{array}{ccc|c}1&1&2&1\\0&1&4-a&0\\0&0&1&\frac{c-3}{2a-10}\end{array}\right)$$

We have a pivot in every column of the coefficient matrix and no pivots in the last column. The system has a unique solution.

c) If $a=5$ and $c=3$

$$\left(\begin{array}{ccc|c}1&1&2&1\\0&1&-1&0\\0&0&0&0\end{array}\right)$$

This system is consistent (no pivots in the last column) and has a free variable $\Longrightarrow$ infinitely many solutions.

d) If $a=5$ and $c=3$:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \underset{\sim}{R_1 \to R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x = 1 - 3z$

$y = z$

$z = z$ (z is free)

$\Rightarrow$ general solution: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-3z \\ z \\ z \end{pmatrix} = z\begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$

6. Determine if the following statements are (always) true or (sometimes) false. Justify your answer.

- If you say that a statement is false, justify your answer by providing an example using vectors or functions.
- If you say that a statement is true, you must give a clear explanation why.

I. A linear system with 10 equations and 15 variables has infinitely many solutions.

It could be inconsistent: one of its equations could be $0=1$.

ANSWER FALSE

II. If the augmented matrix associated with a linear homogeneous system of 8 equations and 7 variables has a rank of 7, then the system has a unique solution.

The augmented matrix of a homogeneous system cannot have a pivot in its last column (it is full of zeroes), so its 7 pivots must be in the 7 variable columns. Therefore, there are no free variables, and since a homogeneous system must be consistent, this system has a unique solution.

ANSWER TRUE

III. If $\{\mathbf{u}, \mathbf{v}\}$ is a linearly independent subset of $\mathbb{R}^{10}$, then $\{\mathbf{u}-\mathbf{v}, \mathbf{v}\}$ is also linearly independent.

ANSWER TRUE

We look at the equation

$$a(\vec{u}-\vec{v})+b\vec{v}=\vec{0}$$

$\Rightarrow a\vec{u}+(b-a)\vec{v}=\vec{0}$ Since $\{\vec{u},\vec{v}\}$ is LI, it implies that $a=0$ and $b-a=0 \Rightarrow b=0$

So $a=0$ and $b=0 \Rightarrow \{\vec{u}-\vec{v},\vec{v}\}$ is LI.

IV. If there is a column of zeroes in the coefficient matrix of a consistent linear system of equations, then the system has infinitely many solutions.

ANSWER TRUE

The system is consistent, so it either has a unique solution, or infinitely many.

Since there is a column of zeroes in the coefficient matrix, this column has no pivot: the corresponding variable is free. This implies an infinite number of solutions.