Sol. Attached.

CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	Number	Sections
Mathematics	201	All
Examination	**Date**	**Duration**
Midterm	23 October, 2016	1 h 30 min
Special Instructions:	Only approved calculators are allowed **Show all your work for full marks**	

1. [12] (a) Find the midpoint M of the line segment joining A$(-3, 1)$ and B$(3, -3)$, and then find the distance between M and the point A.

(b) Find the equation of the line that passes through the point $(8, -7)$ and is perpendicular to the line $4x = 20 + 2y$.

(c) Find the coordinates of the center and the radius of the circle whose equation is $x(4 - x) = y(y + 3)$

2. [9] Consider the quadratic function $f(x) = 3x^2 - 9x + 12$.

(a) Express $f(x)$ in the vertex form.

(b) Find the coordinates of the vertex and indicate whether it corresponds to the maximum or minimum of $f(x)$.

(c) Find the x- and y-intercepts of the graph $y = f(x)$.

3. [5] Consider the functions $f(x) = \sqrt{4x + 3}$ and $g(x) = x^2 - 1$. Find $(f \circ g)$ and $(g \circ f)$, and determine the domain of $(f \circ g)$ and the domain of $(g \circ f)$.

4. [6] Given the rational function $f(x) = \dfrac{x^4 - 16}{3(x^2 + 1)(x^2 - x - 6)}$, find:

(a) the x- and y- intercepts,

(b) all vertical asymptotes, if any,

(c) all horizontal asymptotes, if any.

5. [12] Find the solutions of the following equations:

(a) $9^x = 3^x + 12$

(b) $\log_2 x + \log_2(x - 14) = 5$

(c) $5^{\log_5(x^2)} - 4 \cdot 3^{\log_3(x+1)} = 8$

6. [6] Consider the function $f(x) = \ln(x + 3) + 6$.

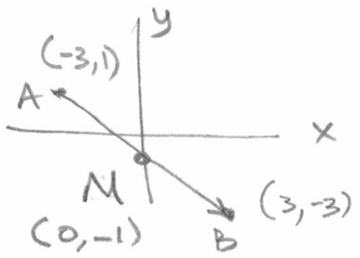
(a) Find the inverse function $f^{-1}(x)$.

(b) Find the domain and range of $f(x)$ and the domain and range of $f^{-1}(x)$.

Bonus. [3]: Let $f(x)$ be a function defined for all real x. **(a)** If we now that the range of f is $(-2, 1)$, what is the range of $F(x) = [f(x)]^2$? And **(b)** explain why this information is sufficient to claim that $F(x)$ cannot be invertible function even if $f(x)$ is invertibe.

Math 201 Mid Term Oct 2016

1a)



(i) Mid pt AB $\left\{ \begin{array}{l} x = \frac{x_1+x}{2} = \frac{-3+3}{2} = \frac{0}{2} = 0 \\ y = \frac{y_1+y}{2} = \frac{1+(-3)}{2} = -\frac{2}{2} = -1 \end{array} \right\} \Rightarrow$ Mid pt is M $(0,-1)$

(ii) dist. $MA = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

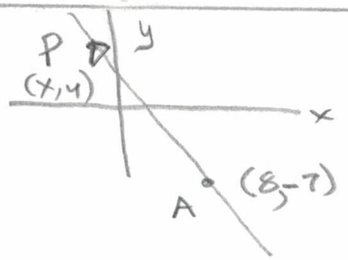
$= \sqrt{(-3-0)^2+(1-[-1])^2}$

$= \sqrt{9+4} = \sqrt{13}$

4

---

b)



Step 1 Get slope of Required line

(i) slope of given line $4x = 20+2y$

$4x-20 = 2y$

$y = 2x-10$

$\Rightarrow$ Slope $= 2$ or $\frac{2}{1}$

(ii) Slope of $\perp$ line is $-\frac{1}{2}$

Step 2 let $P(x,y)$ be Any pt. on Required line

Slope $AB = \frac{y_2-y_1}{x_2-x_1}$

$-\frac{1}{2} = \frac{y-(-7)}{x-8}$

$2(y+7) = -1(x-8)$

$2y+14 = -x+8$

$2y = -x-6 \implies y = -\frac{1}{2}x-3$

4

---

c) Step 1 $x(4-x) = y(y+3) \Rightarrow 4x-x^2 = y^2+3y \Rightarrow -x^2-y^2+4x-3y = 0$

(Get into Standard form) $\Rightarrow x^2+y^2-4x+3y = 0$

Step 2 (Complete square)

$x^2-4x+\underline{4}+y^2+3y+\underline{\frac{9}{4}} = 0+\underline{4}+\underline{\frac{9}{4}}$

$(x-2)(x-2)+(y+\frac{3}{2})(y+\frac{3}{2}) = \frac{4}{1}+\frac{9}{4}$

$(x-2)^2+(y+\frac{3}{2})^2 = \frac{25}{4}$

$\sqrt{(x-2)^2+(y+\frac{3}{2})^2} = \sqrt{\frac{25}{4}} = \frac{5}{2}$

$\Longrightarrow$ Centre is $(2,-3/2)$, Radius $= \frac{5}{2}$

4

$\overline{}$ 12

2. a)

$y = 3x^2 - 9x + 12$

$y = 3(x^2 - 3x + \frac{9}{4}) + 12 - 3(\frac{9}{4})$

(3) $y = 3(x - \frac{3}{2})(x - \frac{3}{2}) + 12 - \frac{27}{4}$

$y = 3(x - \frac{3}{2})^2 + \frac{21}{4} \implies$ b) Vertex is $(\frac{3}{2}, \frac{21}{4})$

(3) opens up (concave up)

Vertex is a Minimum.

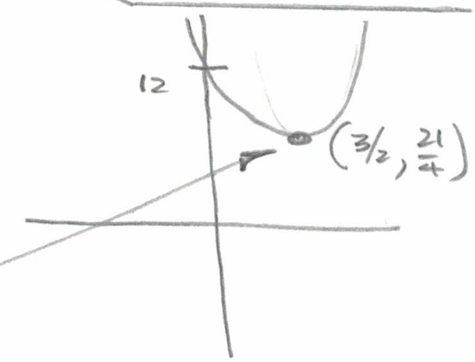
c)

x int	y int
$y = 3x^2 - 9x + 12$ $3x^2 - 9x + 12 = 0$ $x^2 - 3x + 4 = 0$ $x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(4)}}{2(1)}$	$y = 3x^2 - 9x + 12$ $y = 3(0)^2 - 9(0) + 12$ $\Rightarrow$ y int. is 12

(3)

$x = \frac{3 \pm \sqrt{-7}}{2}$

No x intercept (or see graph)



3. a) (i) $f(g(x))$: $f(x) = \sqrt{4x+3} \Rightarrow f(g(x)) = \sqrt{4(x^2-1)+3} = \underline{\underline{\sqrt{4x^2-1}}}$

(3) $g(f(x))$: $g(x) = x^2 - 1 \Rightarrow g(f(x)) = (\sqrt{4x+3})^2 - 1 = \underline{\underline{4x+2}}$

(ii) Domain $(f \circ g)(x)$ Step 1 Domain $g(x)$: $x \in R$

(2) Step 2 Domain of $f(g(x))$: $4x^2 - 1 \geq 0 \begin{cases} x \geq \frac{1}{2} \\ \text{OR} \\ x \leq -\frac{1}{2} \end{cases}$

$(g \circ f)(x)$ Step 1 Domain $f(x)$: $4x + 3 \geq \Rightarrow x \geq -\frac{3}{4}$

Step 2 Domain $g(f(x))$: $x \in R$

4 a) x int $\frac{0}{1} = \frac{x^4 - 16}{3(x^2+1)(x^2-x-6)}$

$x^4 - 16 = 0$

$x^4 = 16$

(2) $x = \underline{\underline{\pm 2}}$

but $x = -2$ makes DEN = 0

$\Rightarrow x = 2$ only

y int $y = \frac{x^4 - 16}{3(x^2+1)(x^2-x-6)}$

$y = \frac{0^4 - 16}{3(0^2+1)(0^2-0-6)}$

$y = \frac{-16}{(3)(-6)} = \frac{-16}{-18} = \underline{\underline{\frac{8}{9}}}$

b) VA: let DEN = 0

$3(x^2+1)(x^2-x-6) = 0$

$(x^2+1)(x^2-x-6) = 0$

(2)

$x^2+1=0$	$x^2-x-6=0$	
NO x	$(x+2)(x-3)=0$	
	$x+2=0$ $x=-2$	$x-3=0$ $x=3$

NUM = 0 when $x = -2$

$\Rightarrow x = 3$ is only VA.

c) HA:

(2) $\lim_{x \to \infty} f(x) = \frac{\infty}{\infty}$

$\lim_{x \to \infty} \frac{\frac{x^4 - 16}{x^4}}{3\frac{(x^2+1)}{x^2}\frac{(x^2-x-6)}{x^2}}$

$\lim_{x \to \infty} \frac{1 - \frac{16}{x^4}}{3(1 - \frac{1}{x^2})(1 - \frac{1}{x} - \frac{6}{x^2})}$

$= \frac{1}{3} \Rightarrow$ HA: $y = \frac{1}{3}$

5a) $9^x = 3^x + 12$

$3^{2x} - 3^x - 12 = 0$

$(3^x)^2 - 3^x - 12 = 0$

$(3^x - 4)(3^x + 3) = 0$

$3^x - 4 = 0 \qquad 3^x + 3 = 0$

$3^x = 4 \qquad 3^x = -3$

NO x

$\ln 3^x = \ln 4$

$x \ln 3 = \ln 4$

$x = \dfrac{\ln 4}{\ln 3}$

like $A^2 - A - 12 = 0$

$(A-4)(A+3) = 0$

Except we have

$A = 3^x$

5b) $\log_2 x + \log_2 (x-14) = 5$

$\log_2 x(x-14) = 5$

$2^5 = x(x-14)$

$x^2 - 14x - 32 = 0$

$(x+2)(x-16) = 0$

$x+2=0$	$x-16=0$
$x=-2$	$x=16$

DISCARD domain of $\log x$ is $x>0$

5c) $5^{\log_5 x^2} - 4 * 3^{\log_3 (x+1)} = 8$

$x^2 - 4(x+1) = 8$ ⊛

$x^2 - 4x - 4 - 8 = 0$

$x^2 - 4x - 12 = 0$

$(x+2)(x-6) = 0$

$x+2=0$	$x-6=0$
$x=-2$	$x=6$

DISCARD (domain of)

⊛ let $5^{\log_5 x^2} = A$

$\Rightarrow \log_5 A = \log_5 x^2$

$\Rightarrow A = x^2$

$\Rightarrow \underbrace{5}\,\underbrace{\log_5} x^2 = x^2$

(Inverses of each other)

6. a) $y = \log_e (x+3) + 6$

$x = \log_e (y+3) + 6$

$x - 6 = \log_e (y+3)$

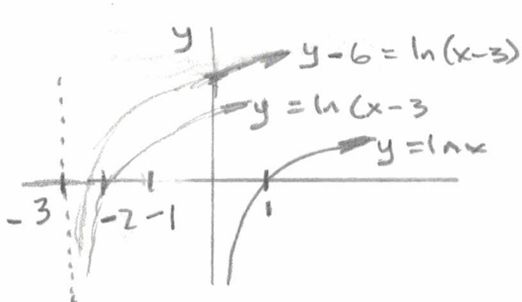
$e^{x-6} = y + 3$

$y = e^{x-6} - 3$

$\Rightarrow f^{-1}(x) = e^{x-6} - 3$

b) Domain of $f(x)$: $x+3>0 \Rightarrow x \in R \mid x > -3$

Domain of $f^{-1}(x)$: (this will be Range of $f(x) \Rightarrow x \in R$)



$\Rightarrow$ Range $f(x)$ is $x \in R$

Bonus:

a) Since Range of $f(x)$ is interval $(-2,1)$

$\Rightarrow \quad -2 < f(x) < 1$

$\Rightarrow 0 \leq [f(x)]^2 < 4$ ① $[f(x)]^2$ cannot be Neg.

② $(-2)^2 = 4$

$\Rightarrow$ Range of $[f(x)]^2$ is interval $[0,4)$

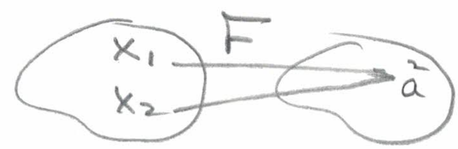
b) Since Range $f(x)$ is $(-2,1)$

$\Rightarrow \exists\ x_1, x_2\ (x_1 \neq x_2)$ and a

Such that $f(x_1) = a \Rightarrow [f(x_1)]^2 = a^2$

$f(x_2) = -a \Rightarrow [f(x_2)]^2 = a^2$

$\Rightarrow F(x) = [f(x)]^2$ is Not one to one



$\Rightarrow F(x)$ is not Invertible | to have an Inverse a func must be 1 to 1.