Solution to Review Questions for Midterm 2

MAT1322D, Fall 2017

1. (4 marks) Suppose salted water of concentration 5 g / m^3 is added to a reservoir of volume 100 m^3 at a rate 2 m^3/ minute. Assume the water in the reservoir is well mixed and the same amount of mixed water is removed from the reservoir. The reservoir is originally filled with fresh water (i.e., concentration of salt is 0 g / m^3). Let $Q(t)$ be the quantity, in grams, of salt in the reservoir at time t.

(a) Find the differential equation that $Q(t)$ satisfies. What is the initial condition?

Solution. Rate of increasing $r_{\text{in}} = 5 \times 2 = 10$ g / min. Rate of decreasing r_{out} = concentration × quantity, where concentration is $Q / 100$ and quantity is 2. Hence, $r_{\text{out}} = (Q / 100) \times 2$ $= 0.02Q$ g / min. The differential equation is $\frac{dQ}{dt} = 10 - 0.02Q$. The initial condition is $Q(0) = 0$.

(b) Solve this initial-value problem.

Solution. $\frac{dQ}{dt} = 10 - 0.02Q$, $Q(0) = 0$. $\int \frac{1}{10-0.02Q}\,dQ = dt, -\frac{1}{0.02}\ln|10-0.02Q| = t + C$.

$|10 - 0.02Q| = K_1 e^{-002t}$, where $K_1 = e^{-0.02C} > 0$. $10 - 0.02Q = K_2 e^{-0.02t}$, where $K_2 = \pm K_1 \neq 0$. By the initial condition, $K_2 = 10$. Hence, $10 - 0.02Q = 10e^{-0.02t}$. $Q = 500(1 - e^{-0.02t})$.

2. Consider the initial-value problem $\frac{dy}{dt} = \frac{\cos t}{e^{2y}}$, $y(0) = 0$.

(a) Use Euler's method to find an approximation of $y(0.3)$ with step-size $h = 0.1$.

Solution. $t_0 = 0$, $y(0) = y_0 = 0$.

$t_1 = 0.1$, $y(0.1) \approx y_1 = y_0 + h\frac{\cos t_0}{e^{2y_0}} = 0 + 0.1 \times \frac{\cos 0}{e^0} = 0.1$.

$t_2 = 0.2$, $y(0.2) \approx y_2 = y_1 + h\frac{\cos t_1}{e^{2y_1}} = 0.1 + 0.1 \times \frac{\cos 0.1}{e^{0.2}} \approx 0.181464$.

$t_3 = 0.3$, $y(0.3) \approx y_3 = y_2 + h\frac{\cos t_2}{e^{2y_2}} = 0.181464 + 0.1 \times \frac{\cos 0.2}{e^{0.362928}} \approx 0.249641$.

(b) Solve this initial-value problem.

Solution. $e^{2y}\,dy = \cos t\,dt$. $\frac{1}{2}e^{2y} = \sin t + C$. $\frac{1}{2}e^{0} = \sin 0 + C$, $C = \frac{1}{2}$. Then $\frac{1}{2}e^{2y} = \sin t + \frac{1}{2}$, $e^{2y} = 2\sin t + 1$. $y = \frac{1}{2}\ln(2\sin t + 1)$.

3. Solve the initial-value problem $\frac{dy}{dt} = 2ty^2 + y^2 + 2t + 1$, $y(0) = 0$.

Solution. Write the equation as $\frac{dy}{dt} = (2t+1)(y^2+1)$. Then

$$\int \frac{1}{y^2+1}\,dy = \int (2t+1)\,dt,\ \arctan y = t^2 + t + C.$$

By the initial condition, $C = 0$. $\arctan y = t^2 + t$, $y = \tan(t^2 + t)$.

4. Suppose the population of a country increases exponentially. At the beginning of 2010, the population is 2 million. At the beginning of 2014, the population is 2.1 million. What is the population at the beginning of 2018?

Solution. Let t be the time, in years, after the beginning of 2010. The model is $P(t) = P(0)e^{rt}$. Since $P(0) = 2$, and $P(4) = P(0)e^{4r} = 2e^{4r} = 2.1$. $e^{4r} = 1.05$. The population at the beginning of 2018 is $P(8) = P(0)e^{8r} = 2(e^{4r})^2 = 2 \times 1.05^2 = 2.205$ million.

5. Solve the initial-value problem $\frac{dy}{dt} = y^2 - 2y - 3$, $y(0) = 0$.

Solution. $y^2 - 2y - 3 = (y+1)(y-3)$. Use partial fraction. Let

$\frac{1}{(y+1)(y-3)} = \frac{A}{y+1} + \frac{B}{y-3} = \frac{A(y-3)+B(y+1)}{(y+1)(y-3)}$. Then $A(y-3) + B(y+1) = 1$. Let $y = -1$. $-4A = 1$, $A = -\frac{1}{4}$. Let $y = 3$. $4B = 1$, $B = \frac{1}{4}$. Hence,

$$\int \frac{dy}{(y+1)(y-3)} = \int dt.\ \int \frac{dy}{(y+1)(y-3)} = \frac{1}{4}\int \left(\frac{1}{y-3} - \frac{1}{y+1}\right)dy = \frac{1}{4}\ln\left|\frac{y-3}{y+1}\right| = t + C.$$

Then $\left|\frac{y-3}{y+1}\right| = K_1 e^{4t}$, where $K_1 = e^{4C} > 0$. $\frac{y-3}{y+1} = Ke^{4t}$, where $K = \pm K_1 \neq 0$. Use the initial condition, $K = -3$. $y - 3 = -3(y + 1)e^{4t}$. $(1 + 3e^{4t})y = 3(1 - e^{4t})$. $y = \frac{3(1-e^{4t})}{1+3e^{4t}}$.

6. Let sequence $\{ a_n \}$ be defined recursively by $a_1 = 1$, $a_{n+1} = \frac{1}{1+a_n}$. Suppose it is known that the limit $\lim_{n\to\infty} a_n = L$ exists. Find L.

Solution. Since $\lim_{n\to\infty} a_{n+1} = \frac{1}{1+\lim_{n\to\infty} a_n}$, and $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L$, we have $L = \frac{1}{1+L}$, $L^2 + L - 1 = 0$. Then $L = \frac{1}{2}(-1+\sqrt{5}) \approx 0.618034$.

7. Determine whether the series $\sum_{n=1}^{\infty} a_n$, where $a_n = \ln\frac{n+1}{n}$, is convergent or divergent. If it is convergent, find its sum.

Solution. $a_n = \ln(n + 1) - \ln n$. Hence,

$S_1 = a_1 = \ln 2 - \ln 1 = \ln 2$,
$S_2 = S_1 + a_2 = \ln 2 + (\ln 3 - \ln 2) = \ln 3$.
$S_3 = S_2 + a_3 = \ln 3 + (\ln 4 - \ln 3) = \ln 4$.
… .

In general, $S_k = \sum_{n=1}^{k} a_n = \ln k$.

When k approaches infinity, $\ln k$ approaches infinity. This series is divergent.

8. Determine whether each of the following statements is true or false.

(a) If $a_n > a_{n+1}$ and $a_n < N$ for all $n = 1, 2, 3, \ldots$, then sequence $\{ a_n\}$ is convergent.

(b) If $a_n \geq b_n \geq c_n$ for all $n = 1, 2, 3, \ldots$, and sequences $\{a_n \}$ and $\{ c_n \}$ converges, then sequence $\{ b_n \}$ converges.

(c) If series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.

(d) If series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, then it is convergent.

(e) If series $\sum_{n=1}^{\infty} a_n$ is convergent, then it is absolutely convergent.

(f) If series $\sum_{n=1}^{\infty} a_n$ is convergent, then it is conditionally convergent.

(g) If series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, then it is absolutely convergent.

(h) If series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is conditionally convergent.

(i) If $\lim_{n\to\infty} a_n = 0$, then series $\sum_{n=1}^{\infty} a_n$ is convergent.

(j) If series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

(k) If $\sum_{n=1}^{\infty} (-1)^n a_n$ is an alternating series, and $\lim_{n\to\infty} a_n = 0$, then this series is convergent.

Answers. (a), (c), (d) and (j) are true. (b) is false because we must have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$.
(e) and (f) are false, because a convergent series may be absolutely convergent or conditionally convergent. (g) and (h) are false because $\sum_{n=1}^{\infty} a_n$ is absolutely convergent means that $\sum_{n=1}^{\infty} |a_n|$ is convergent, while $\sum_{n=1}^{\infty} a_n$ is conditionally convergent means that $\sum_{n=1}^{\infty} |a_n|$ is divergent. (i) is false and the harmonic series is a counterexample. (k) is false, because a_n must be decreasing.

9. Determine whether each of the following series is convergent or divergent. Justify your answer by appropriate test method and state the condition to use these tests:

(a) $\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$.

Solution. Since the general term is positive, decreasing and continuous, we can use the integral test.

$$\int_1^{\infty} x^{-2}\ln x dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}dx .$$

Use integration by parts, we have $\int \frac{\ln x}{x^2}dx = -\frac{1+\ln x}{x}+C$. Then

$$\lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}dx = -\lim_{b\to\infty}\left[\frac{1+\ln x}{x}\right]_{x=1}^{b} = \lim_{b\to\infty}\left(1-\frac{1+\ln b}{b}\right) = 1 < \infty.$$ This series is convergent.

(b) $\sum_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$.

Solution. Since the general term is alternating, we can use the alternating series test. Since the general term decreases and approaches 0, this series is convergent.

(c) $\sum_{n=2}^{\infty}\frac{(-1)^n n}{(\ln n)^2}$.

Solution. By L'Hospital's rule,

$$\lim_{n\to\infty}\frac{n}{(\ln n)^2} = \lim_{n\to\infty}\frac{1}{(2\ln n)(1/n)} = \lim_{n\to\infty}\frac{n}{2\ln n} = \lim_{n\to\infty}\frac{1}{2(1/n)} = \lim_{n\to\infty}\frac{n}{2} = \infty .$$

Since the general term of this series does not approach zero, it is divergent.

(d) $\sum_{n=1}^{\infty}\frac{2n-1}{\sqrt{n^3+n}}$.

Solution. Since this is a positive series, we can use the limit comparison test.

Compare this series with the series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$. We have

$$\lim_{n\to\infty}\frac{2n-1}{\sqrt{n^3+n}}\sqrt{n}=\lim_{n\to\infty}\frac{\sqrt{n}(2n-1)/(n\sqrt{n})}{\sqrt{n^3+n}/(n\sqrt{n})}=\lim_{n\to\infty}\frac{2-1/n}{\sqrt{1+1/n^2}}=2.$$

Since $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is a p-series with $p = 1/2$, it is divergent. Series $\sum_{n=1}^{\infty}\frac{2n-1}{\sqrt{n^3+n}}$ is also divergent.

(e) $\sum_{n=1}^{\infty}\frac{2n-\sin n}{n^2+1}$.

Solution. Since this series is positive, we can use the comparison test. Since $2n - \sin n > n$, and $n^2 + 1 < 2n^2$, $\frac{2n-\sin n}{n^2+1}>\frac{n}{2n^2}=\frac{1}{2n}$. Since $\sum_{n=0}^{\infty}\frac{1}{2n}$ diverges, this series diverges.

10. Suppose $S_5 = \sum_{n=1}^{5}\frac{1}{n^5} \approx 1.036662$. Find an upper bound and a lower bound of $S = \sum_{n=1}^{\infty}\frac{1}{n^5}$.

Solution. $S_5 + \int_6^{\infty}\frac{1}{x^5}dx < S < S_5 + \int_5^{\infty}\frac{1}{x^5}dx$. Since $\int\frac{1}{x^5}dx=-\frac{1}{4x^4}+C$,

$\int_6^{\infty}\frac{1}{x^5}dx=\lim_{b\to\infty}\left[-\frac{1}{4x^4}\right]_{x=6}^{b}=\frac{1}{4\times 6^4}\approx 0.000193$, and $\int_5^{\infty}\frac{1}{x^5}dx=\lim_{b\to\infty}\left[-\frac{1}{4x^4}\right]_{x=4}^{b}=\frac{1}{4\times 5^4}\approx 0.000400$.

Hence, $1.036662 + 0.000193 = 1.036855 < S < 1.036662 + 0.000400 = 1.037062$.

11. Suppose $S_5 = \sum_{n=1}^{5}(-1)^{n-1}\frac{1}{n^5} \approx 0.972209$. Find an upper bound and a lower bound of $S = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^5}$.

Solution. Since $a_6 = (-1)^5\frac{1}{6^5} < 0$, S_5 is an overestimate, which is an upper bund of S.

$S_6 - S < |a_6| \approx 0.000129$. Hence, $0.972209 - 9{,}999129 = 0.972080 < S < 0.972209$.