

Solution to Midterm Test 2 (DGD 1, 2)

MAT 1322D, Fall 2017

Total = 20 marks

Part I. Multiple-choice questions ($3 \times 4 = 12$ marks)

Version A: DECA

Version B: ABDD

1. Suppose Euler's method with step size $h = 0.025$ is used to estimate $y(0.05)$, where $y(t)$ is the solution to the initial-value problem $y' = \frac{1+ty}{y^2}$, $y(0) = 1$. Which one of the following values is closest to the result that you obtained?

Version A

(A) 1.0250; (B) 1.0733; (C) 1.0967; **(D) 1.0494;** (E) 1.0613; (F) 1.0323.

Version B

(A) 1.0494; (B) 1.0733; (C) 1.0967; (D) 1.0250; (E) 1.0323 (F) 1.0613.

Solution. Use the iteration formula $y_{i+1} = y_i + 0.025 \frac{1+t_i y_i}{y_i^2}$. We have

i	t_i	y_i
0	0	1
1	0.025	$1 + 0.025 \times \frac{1+0 \times 1}{1^2} = 1.0250$
2	0.05	$1.025 + 0.025 \times \frac{1+0.025 \times 1.025}{1.025^2} \approx 1.0494$

2. Suppose fresher air with 0.1% (i.e., 0.001) of carbon dioxide is blowing at a rate 20 m^3 per minute into a hall of capacity 10000 m^3 . Suppose the air is well mixed and the mixed air is removed from the room by the ventilation system at the same rate (i.e., 20 m^3 per minute). Let $Q(t)$ be the amount of carbon dioxide, in cubic meters, in the hall at time t . The differential equation satisfied by function $Q(t)$ is

Version A:

(A) $Q' = 0.02Q - 0.002Q^2$; (B) $Q' = 2 - 0.02Q$; (C) $Q' = 0.002 - 0.02Q$;
 (D) $Q' = 2Q - 0.02Q^2$; **(E) $Q' = 0.02 - 0.002Q$;** (F) $Q' = 0.02 - 0.002Q^2$.

Version B:

(A) $Q' = 2Q - 0.02Q^2$; **(B) $Q' = 0.02 - 0.002Q$;** (C) $Q' = 0.02 - 0.002Q^2$;

(D) $Q' = 0.02Q - 0.002Q^2$; (E) $Q' = 2 - 0.02Q$; (F) $Q' = 0.002 - 0.02Q$;

Solution. $r_{\text{in}} = 20 \times 0.001 = 0.02$. $r_{\text{out}} = 20 \times \frac{Q}{10000} = 0.002Q$. The equation is $Q' = 0.02 - 0.002Q$.

3. Let $y = f(t)$ be the solution to the initial-value problem $y' = y^2 \cos t$, $y(0) = 1$. Then $y\left(\frac{\pi}{6}\right) =$

Version A:

(A) $\frac{1}{2}$; (B) $\frac{\pi}{2}$; **(C) 2**; (D) $\frac{3}{2}$; (E) 3; (F) π .

Version B:

(A) $\frac{1}{2}$; (B) $\frac{\pi}{2}$; (C) π ; **(D) 2**; (E) $\frac{3}{2}$; (F) 3.

Solution. Separating variables, $\int \frac{1}{y^2} dy = \int \cos t dt$. Then $-\frac{1}{y} = \sin t + C$. By the initial condition, $C = -1$. $y = \frac{1}{1 - \sin t}$. When $t = \frac{\pi}{6}$, $\sin t = \frac{1}{2}$. $y\left(\frac{\pi}{6}\right) = 2$.

4. Which one(s) of the following statements is(are) always true?

(i) If series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \rightarrow \infty} a_n = 0$.

(ii) If $\lim_{n \rightarrow \infty} a_n = 0$, then series $\sum_{n=1}^{\infty} a_n$ is convergent.

(iii) If $\sum_{n=1}^{\infty} |a_n|$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(iv) If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Version A:

(A) (i) and (iii) only; (B) (i) and (iv) only; (C) (ii) and (iii) only;
 (D) (ii) and (iv) only; (E) (i), (ii) and (iii) only; (F) (i), (ii) and (iv) only.

Version B:

- (A) (ii) and (iv) only; (B) (i), (ii) and (iii) only; (C) (i), (ii) and (iv) only;
(D) (i) and (iii) only; (E) (i) and (iv) only; (F) (ii) and (iii) only.

Part II. Detailed Answer questions (8 marks)

5. (4 marks) Find $y = f(t)$, where y is the solution to the initial-value problem $y' = y(2 - y)$, $y(0) = 3$.

Solution. $\int \frac{1}{y(2-y)} dy = \int dt.$

Since $\frac{1}{y(2-y)} = \frac{1}{2} \left(\frac{1}{y} + \frac{1}{2-y} \right),$

$$\int \frac{1}{y(2-y)} dy = \frac{1}{2} (\ln |y| - \ln |2-y|) = \frac{1}{2} \ln \left| \frac{y}{2-y} \right| = t + C.$$

Then $\left| \frac{y}{2-y} \right| = K_1 e^{2t}$, where $K_1 = e^{2C} > 0$. $\frac{y}{2-y} = K e^{2t}$, where $K = \pm K_1 \neq 0$.

By the initial values condition, $K = -3$. Hence, $\frac{y}{2-y} = -3e^{2t}$, $y = -6e^{5t} + 3ye^{5t}$.

$$y = -\frac{6e^{2t}}{1-3e^{5t}} = \frac{6}{3-e^{5t}}.$$

6. ($2 \times 2 = 4$ marks) Use appropriate test method to determine whether each of the following series is convergent or divergent. State the condition(s) to use this test.

a. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2};$ b. $\sum_{n=2}^{\infty} (-1)^n \frac{n}{3n^2 + 1}.$

Solution. a. Since function $y = \frac{1}{x(\ln x)^2}$ is positive, decreasing and continuous when $x \geq 2$, we can use the integral test.

$$\int_2^{\infty} \frac{1}{x(\ln x)^2} dx = \lim_{b \rightarrow \infty} \int_2^b \frac{1}{x(\ln x)^2} dx = \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln b} \frac{1}{u^2} du = \lim_{b \rightarrow \infty} \left(-\frac{1}{\ln b} - \left(-\frac{1}{\ln 2} \right) \right) = \frac{1}{\ln 2}.$$

This series converges.

b. This is an alternating series. Since $\frac{n}{3n^2+1}$ is decreasing and $\lim_{n \rightarrow \infty} \frac{n}{3n^2+1} = 0$, this series converges.