(A)

MAT 2384 X
DIFFERENTIAL EQUATIONS AND NUMERICAL METHODS
TEST #1
May 30, 2018

Instructor: Dr. Steve Desjardins

Duration: 80 minutes

Name: Solutions

Student Number:__________________________

Instructions:

- Print your name and student number on this page.
- Verify that your copy of the exam has all 4 pages.
- You must answer all questions.
- Write your answers in the spaces below the questions. You may use the backs of the pages if necessary.
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Question 1 (7 marks) Solve the initial value problems:

A

(a) $y' = 2x(1+y^2)$, $y(0) = 0$

DE is separable: $\frac{dy}{dx} = 2x(1+y^2)$

or $\frac{dy}{1+y^2} = 2x\,dx$

integrate $\int \frac{dy}{1+y^2} = \int 2x\,dx + C$

we get $\arctan y = x^2 + C$

$y = \tan(x^2 + C)$ (general solution)

$y(0) = 0 \Rightarrow 0 = \tan(C) \Rightarrow C = 0$

$\therefore$ $\boxed{y = \tan(x^2)}$ (unique solution)

(b) $(2x + y)\,dx - x\,dy = 0$, $y(1) = 2$ (use the substitution $u = y/x$)

let $u = y/x$ or $y = ux$ and $dy = u\,dx + x\,du$

DE becomes $(2x + ux)\,dx - x(u\,dx + x\,du) = 0$

$2x\,dx + \cancel{ux\,dx} - \cancel{xu\,dx} - x^2\,du = 0$

or $2x\,dx - x^2\,du = 0$

so $du = \frac{2}{x}\,dx$

integrate $\int du = \int \frac{2}{x}\,dx + C$

$u = 2\ln|x| + C$

but $u = y/x$, so $y = x(2\ln|x| + C)$ (general solution)

$y(1) = 2 \Rightarrow 2 = (1)(2\ln(1) + C) \Rightarrow C = 2$

$\therefore$ $\boxed{y = 2x(\ln|x| + 1)}$ (unique solution)

Question 2 (6 marks) Solve the initial value problem:

A

$$(21y^2 + 16x\cos y)\,dx + (14xy - 4x^2\sin y)\,dy = 0, \quad y(1) = 0$$

$M(x,y) = 21y^2 + 16x\cos y \Rightarrow M_y = 42y - 16x\sin y$
$N(x,y) = 14xy - 4x^2\sin y \Rightarrow N_x = 14y - 8x\sin y$
$\}\ M_y \neq N_x$ DE not exact

$\frac{M_y - N_x}{N} = \frac{28y - 8x\sin y}{14xy - 4x^2\sin y} = \frac{2}{x}$ (function of x only)

$\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$ and DE becomes

$(21x^2y^2 + 16x^3\cos y)dx + (14x^3y - 4x^4\sin y)dy = 0$

$M^*(x,y) = 21x^2y^2 + 16x^3\cos y \Rightarrow M_y^* = 42x^2y - 16x^3\sin y$
$N^*(x,y) = 14x^3y - 4x^4\sin y \Rightarrow N_x^* = 42x^2y - 16x^3\sin y$
$\}\ M_y^* = N_x^*$ DE exact

$F(x,y) = \int M^*(x,y)dx + g(y)$ (or $\int N^*(x,y)dy + g(x)$)

$= \int (21x^2y^2 + 16x^3\cos y)dx + g(y)$

$= 7x^3y^2 + 4x^4\cos y + g(y)$

then $\frac{dF}{dy} = 14x^3y - 4x^4\sin y + g'(y) = N^*(x,y) = 14x^3y - 4x^4\sin y$

so $g'(y) = 0 \Rightarrow g(y) = \text{constant} \Rightarrow$ take $g(y) = 0$

so $F(x,y) = 7x^3y^2 + 4x^4\cos y$

and the general solution is $7x^3y^2 + 4x^4\cos y = C$

$y(1) = 0 \Rightarrow 7(1)^3(0)^2 + 4(1)^4\cos(0) = C \Rightarrow C = 4$

$\therefore$ unique solution is $\boxed{7x^3y^2 + 4x^4\cos y = 4}$

Question 3 (7 marks) Solve the initial value problems:

A

(a) $y' - \frac{2}{x} y = x^4$, $y(1) = 2$

DE is linear $f(x) = -\frac{2}{x}$, $r(x) = x^4$

$\mu(x) = e^{\int -\frac{2}{x} dx} = x^{-2}$

So general solution is $y = \frac{1}{x^{-2}} \left[\int (x^{-2})(x^4)\, dx + C \right]$

$= x^2 \left(\int x^2 dx + C \right)$

$= x^2 \left(\frac{1}{3} x^3 + C \right)$

$= Cx^2 + \frac{1}{3} x^5$

$y(1) = 2 \Rightarrow 2 = C(1)^2 + \frac{1}{3}(1)^5 \Rightarrow C = 5/3$

$\therefore$ unique solution is $\boxed{y = \frac{5}{3} x^2 + \frac{1}{3} x^5}$

(b) $y' + y = y^2$, $y(0) = 1/2$

DE is Bernoulli, $p(x) = g(x) = 1$, $a = 2$

So let $u = y^{1-2} = y^{-1}$ and DE becomes

$u' - u = -1$ linear with $f(x) = r(x) = -1$

So $\mu(x) = e^{\int -dx} = e^{-x}$

and then $u = \frac{1}{e^{-x}} \left[\int (e^{-x})(-1)\, dx + C \right]$

$= e^x \left(\int -e^{-x} dx + C \right)$

$= e^x \left(e^{-x} + C \right)$

$= Ce^x + 1$

and so general solution is $y = \frac{1}{Ce^x + 1}$

$y(0) = 1/2 \Rightarrow 1/2 = \frac{1}{Ce^0 + 1} = \frac{1}{C+1} \Rightarrow C = 1$

$\therefore$ unique solution is $\boxed{y = \frac{1}{e^x + 1}}$

(B)

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B

Question 1 (7 marks) Solve the initial value problems:

(a) $y' = 4x^3(1+y^2)$, $y(0) = 0$

$$\int \frac{dy}{1+y^2} = \int 4x^3\,dx + C$$

$$\arctan y = x^4 + C$$

$$y = \tan(x^4 + C)$$

$$y(0) = 0 \Rightarrow C = 0$$

$$\boxed{y = \tan(x^4)}$$

(b) $(3x + y)\,dx - x\,dy = 0$, $y(1) = 1$ (use the substitution $u = y/x$)

$$(3x + ux)dx - x(u\,dx + x\,du) = 0$$

$$3x\,dx + \cancel{ux\,dx} - \cancel{ux\,dx} - x^2\,du = 0$$

$$du = \frac{3}{x}dx$$

$$\int du = \int \frac{3}{x}dx + C$$

$$u = 3\ln|x| + C$$

$$y = x(3\ln|x| + C)$$

$$y(1) = 1 \Rightarrow C = 1$$

$$\boxed{y = x(3\ln|x| + 1)}$$

B

Question 2 (6 marks) Solve the initial value problem:

$$(24xy^2 + 9\sin y)\,dx + (12x^2y + 3x\cos y)\,dy = 0, \quad y(0) = 0$$

$$\left.\begin{aligned} M(x,y) &= 24xy^2 + 9\sin y \Rightarrow M_y = 48xy + 9\cos y \\ N(x,y) &= 12x^2y + 3x\cos y \Rightarrow N_x = 24xy + 3\cos y \end{aligned}\right\} M_y \neq N_x$$

$$\frac{M_y - N_x}{N} = \frac{24xy + 6\cos y}{12x^2y + 3x\cos y} = \frac{2}{x} \Rightarrow \mu = x^2$$

DE is $(24x^3y^2 + 9x^2\sin y)\,dx + (12x^4y + 3x^3\cos y)\,dy = 0$

$$\left.\begin{aligned} M^*(x,y) &= 24x^3y^2 + 9x^2\sin y \Rightarrow M_y^* = 48x^3y + 9x^2\cos y \\ N^*(x,y) &= 12x^4y + 3x^3\cos y \Rightarrow N_x^* = 48x^3y + 9x^2\cos y \end{aligned}\right\} M_y^* = N_x^*$$

$$F(x,y) = \int (24x^3y^2 + 9x^2\sin y)\,dx + g(y)$$
$$= 6x^4y^2 + 3x^3\sin y + g(y)$$

$$\frac{\partial F}{\partial y} = 12x^4y + 3x^3\cos y + g'(y) = N^*(x,y) = 12x^4y + 3x^3\cos y$$
$$\Rightarrow g'(y) = 0 \Rightarrow \text{take } g(y) = 0$$

$$F(x,y) = 6x^4y^2 + 3x^3\sin y$$

general solution $6x^4y^2 + 3x^3\sin y = C$

$y(0) = 0 \Rightarrow 6(0)^4(0)^2 + 3(0)^3\sin(0) = C \Rightarrow C = 0$

$\therefore$ unique solution $\boxed{6x^4y^2 + 3x^3\sin y = 0}$

B

Question 3 (7 marks) Solve the initial value problems:

(a) $y' - \frac{3}{x} y = x^5, \quad y(1) = 3$ $\mu(x) = x^{-3}$

$$y = x^3 \left[\int (x^{-3})(x^5)\,dx + C \right]$$
$$= x^3 \left(\int x^2\,dx + C \right)$$
$$= x^3 \left(\frac{1}{3}x^3 + C \right)$$
$$= Cx^3 + \frac{1}{3}x^6$$

$y(1) = 3 \Rightarrow 3 = C + 1/3 \Rightarrow C = 8/3$

$$\therefore \boxed{y = \frac{8}{3}x^3 + \frac{1}{3}x^6}$$

(b) $y' + y = y^2, \quad y(0) = 1/4$

general solution $y = \dfrac{1}{Ce^x + 1}$

$y(0) = 1/4 \Rightarrow C = 3$

$$\therefore \boxed{y = \frac{1}{3e^x + 1}}$$

C

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C

Question 1 (7 marks) Solve the initial value problems:

(a) $y' = 3x^2(1+y^2)$, $y(0) = 0$

$$\int \frac{dy}{1+y^2} = \int 3x^2 dx + C$$

$$\arctan y = x^3 + C$$

$$y = \tan(x^3 + C)$$

$$y(0) = 0 \Rightarrow C = 0$$

$$\therefore \boxed{y = \tan(x^3)}$$

(b) $(2x + y)\,dx - x\,dy = 0$, $y(1) = 3$ (use the substitution $u = y/x$)

$$y = x(2\ln|x| + C)$$

$$y(1) = 3 \Rightarrow C = 3$$

$$\therefore \boxed{y = x(2\ln|x| + 3)}$$

C

Question 2 (6 marks) Solve the initial value problem:

$$(15y^3 + 6\cos y)\,dx + (15xy^2 - 2x\sin y)\,dy = 0, \quad y(1) = 0$$

$M(x,y) = 15y^3 + 6\cos y \Rightarrow M_y = 45y^2 - 6\sin y$

$N(x,y) = 15xy^2 - 2x\sin y \Rightarrow N_x = 15y^2 - 2\sin y$

$\Big\}\ M_y \neq N_x$

$$\frac{M_y - N_x}{N} = \frac{30y^2 - 4\sin y}{15xy^2 - 2x\sin y} = \frac{2}{x} \Rightarrow \mu(x) = x^2$$

DE is $(15x^2y^3 + 6x^2\cos y)dx + (15x^3y^2 - 2x^3\sin y)dy = 0$

$M^*(x,y) = 15x^2y^3 + 6x^2\cos y \Rightarrow M^*_y = 45x^2y^2 - 6x^2\sin y$

$N^*(x,y) = 15x^3y^2 - 2x^3\sin y \Rightarrow N^*_x = 45x^2y^2 - 6x^2\sin y$

$\Big\}\ M^*_y = N^*_x$

$F(x,y) = \int (15x^2y^3 + 6x^2\cos y)dx + g(y)$

$= 5x^3y^3 + 2x^3\cos y + g(y)$

$\frac{\partial F}{\partial y} = 15x^3y^2 - 2x^3\sin y + g'(y) = N^*(x,y) = 15x^3y^2 - 2x^3\sin y$

so $g'(y) = 0 \Rightarrow$ take $g(y) = 0$

$F(x,y) = 5x^3y^3 + 2x^3\cos y$

general solution $5x^3y^3 + 2x^3\cos y = C$

$y(1) = 0 \Rightarrow C = 2$

$$\therefore \boxed{5x^3y^3 + 2x^3\cos y = 2}$$

C

Question 3 (7 marks) Solve the initial value problems:

(a) $y' - \frac{2}{x}y = x^5$, $y(1) = 4$

$\mu(x) = x^{-2}$

$y = x^2 \left(\int (x^{-2})(x^5)\,dx + c \right)$

$= x^2 \left(\int x^3\,dx + c \right)$

$= x^2 \left(\frac{1}{4}x^4 + c \right)$

$= cx^2 + \frac{1}{4}x^6$

$y(1) = 4 \Rightarrow c + 1/4 = 4 \Rightarrow c = 15/4$

$\therefore \boxed{y = \frac{15}{4}x^2 + \frac{1}{4}x^6}$

(b) $y' + y = y^2$, $y(0) = 1/5$

general solution $y = \dfrac{1}{Ce^x + 1}$

$y(0) = 1/5 \Rightarrow c = 4$

$\therefore$ unique $\boxed{y = \dfrac{1}{4e^x + 1}}$