

Student number: _____, Total marks: _____ out of 30

Problem	1	2	3	4	5	6
Marks						

Question 1. [6 points] (a) Compute the derivative of the function $g(x) = \frac{1+x^2}{\sqrt{x+x^{-1}}}$.

[Do not simplify your result.]

$$g(x) = \frac{1+x^2}{\sqrt{x+x^{-1}}} = \frac{1+x^2}{\sqrt{x} + \frac{1}{\sqrt{x}}}$$

$$g'(x) = \frac{(1+x^2)' \cdot (\sqrt{x} + \frac{1}{\sqrt{x}}) - (1+x^2) (\sqrt{x} + \frac{1}{\sqrt{x}})'}{(\sqrt{x} + \frac{1}{\sqrt{x}})^2}$$

$$= \frac{2x(\sqrt{x} + \frac{1}{\sqrt{x}}) - (1+x^2)(\frac{1}{2\sqrt{x}} - \frac{1}{x^2})}{(\sqrt{x} + \frac{1}{\sqrt{x}})^2}$$

$$g'(x) = \frac{2x(\sqrt{x} + \frac{1}{\sqrt{x}}) - (1+x^2)(\frac{1}{2\sqrt{x}} - \frac{1}{x^2})}{(\sqrt{x} + \frac{1}{\sqrt{x}})^2}$$

(b) Let $f(x) = \frac{x^2 - 4}{|x - 2|}$, $|x - 2| = \begin{cases} x - 2, & \text{if } x - 2 \geq 0 \text{ or } x \geq 2 \\ -(x - 2), & \text{if } x - 2 < 0 \text{ or } x < 2 \end{cases}$

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} \frac{\boxed{x \geq 2} (x-2)(x+2)}{(x-2)} = \lim_{x \rightarrow 2^+} (x+2) = 4$$

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} \frac{\boxed{x < 2} (x-2)(x+2)}{-(x-2)} = \lim_{x \rightarrow 2^-} -(x+2) = -4$$

Is f continuous at $x = 2$? Answer

No

Question 2. [4 points] Use the definition of the derivative to calculate the derivative of the function

$$f(x) = \frac{1}{2x-3}$$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \\
 &= \lim_{h \rightarrow 0} \frac{\frac{1}{2(x+h)-3} - \frac{1}{2x-3}}{h} = \\
 &= \lim_{h \rightarrow 0} \frac{2x-3 - [2(x+h)-3]}{[2(x+h)-3][2x-3] \cdot h} = \\
 &= \lim_{h \rightarrow 0} \frac{2x-3 - 2x - 2h + 3}{[2(x+h)-3][2x-3] \cdot h} = \\
 &= \lim_{h \rightarrow 0} \frac{-2\cancel{h}}{[2(x+h)-3][2x-3] \cdot \cancel{h}} = \\
 &= \lim_{h \rightarrow 0} \frac{-2}{[2(x+h)-3][2x-3]} = \frac{-2}{(2x-3)^2}
 \end{aligned}$$

Question 3. [6 points] The angle of the sun above the horizon at 12 noon in Ottawa has its highest value in June with 68.1 degrees and its lowest value in December with 21.3 degrees. Assume that the height above the horizon can be written in standard cosine form.

(a) Find the values of the parameters A, B, Φ, T in the standard cosine description, i.e.,

$$f(t) = A + B \cos(2\pi(t - \Phi)/T),$$

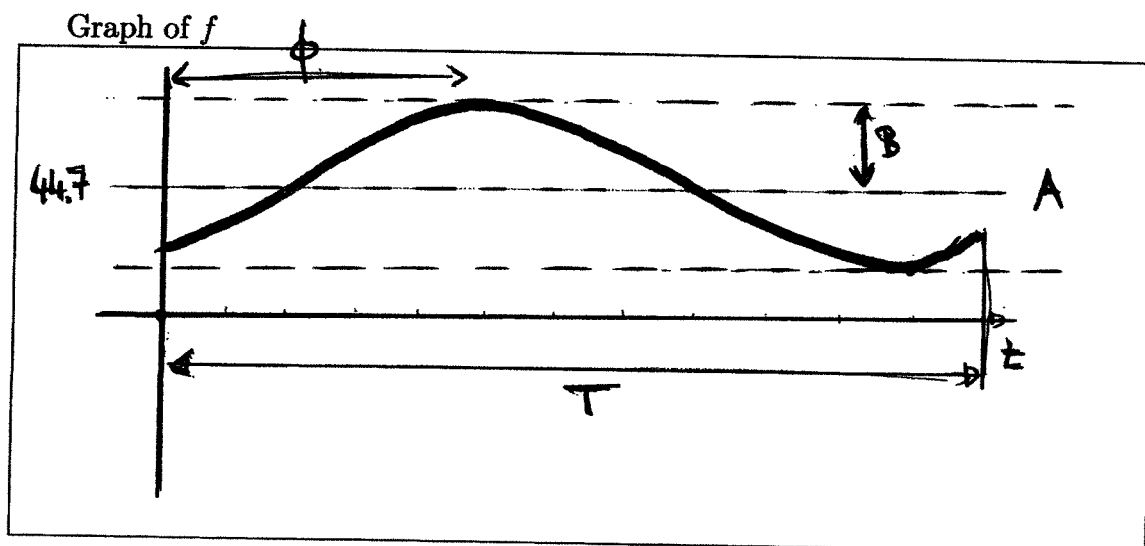
where t is in months, and $t = 0$ corresponds to the month of January.

$$f(t) = 44.7 + 23.4 \cos\left(\frac{2\pi}{12}(t - 5)\right)$$

(b) Give the names of the four parameters A, B, Φ, T .

A : average
 B : amplitude
 ϕ : phase
 T : period

(c) Draw the graph of the function and identify the four parameters A, B, Φ, T in the graph.



Question 4. [2 points] Is the following function continuous at $x = 1$? Justify your answer in a short sentence.

$$f(x) = \cos(2x) + \frac{3x^2 - 5x}{x^2 - 2}$$

Answer:

The function is a sum of a continuous function ($\cos(2x)$) and a rational function. The rational function is continuous where its denominator is not zero ($x \neq \sqrt{2}$ and $x \neq -\sqrt{2}$). $(x^2 - 2)|_{x=1} \neq 0$. Hence the function is continuous at $x=1$.

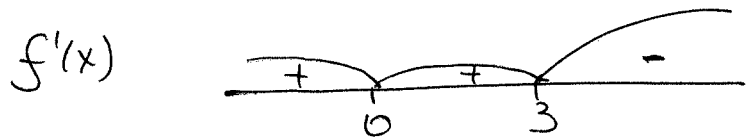
Question 5. [4 points]

(a) Find the critical point(s) of the function

$$f(x) = x^3 e^{-x}$$

$$\begin{aligned} f'(x) &= (x^3)' e^{-x} + x^3 \cdot (e^{-x})' = 3x^2 e^{-x} + x^3 \cdot (e^{-x})(-1) \\ &= 3x^2 e^{-x} - x^3 e^{-x} = x^2 e^{-x} (3 - x) = 0 \\ x_1^* &= 0, \quad x_2^* = 3. \end{aligned}$$

$$\begin{aligned} f'(4) &< 0 \\ f'(-1) &> 0 \\ f'(1) &> 0 \end{aligned}$$



Answer: $x_1^* = 0, \quad x_2^* = 3$

(b) Find the intervals where the function is increasing and where it is decreasing.

Increasing: $x < 3$

Decreasing: $x > 3$

Question 6. [8 points] Consider the discrete-time dynamical system (DTDS)

$$M_{t+1} = -0.8M_t + 6$$

(a) [1 point] Find the updating function of the DTDS.

$$-0.8M + 6 = f(M)$$

(b) [1 point] Find the equilibrium point of the DTDS.

$$M^* = \frac{b}{1-a} = \frac{6}{1-(-0.8)} = \frac{10}{3}$$

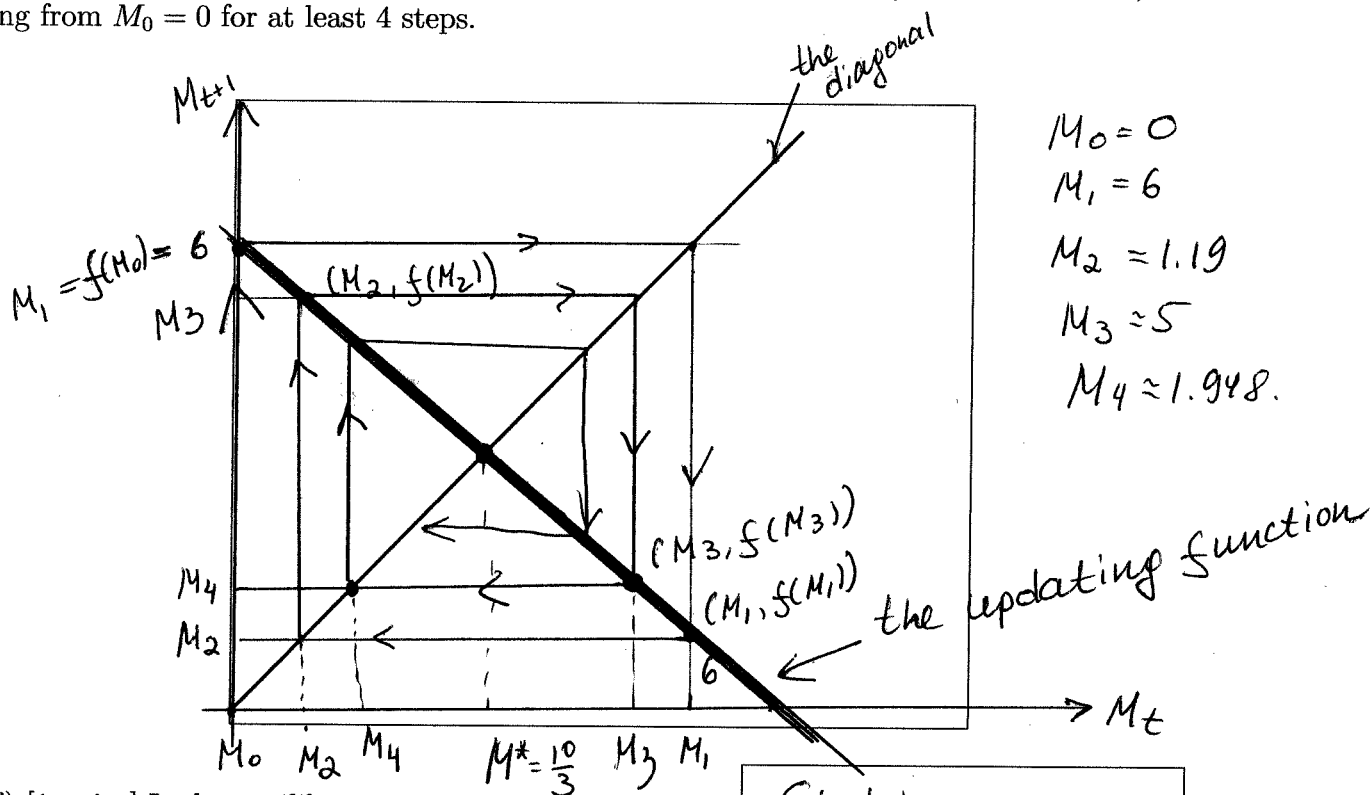
(c) [2 points] Give the general solution formula for the DTDS:

$$M_t = a^t M_0 + b \left(\frac{1-a^t}{1-a} \right) = (-0.8)^t M_0 + 6 \left(\frac{1-(-0.8)^t}{1-(-0.8)} \right)$$

(d) [1 point] Calculate M_{10} if $M_0 = 0$.

$$M_{10} = (-0.8)^{10} \cdot 0 + \frac{10}{3} (1 - (-0.8)^{10}) = 2.975$$

(e) [2 points] Graph the updating function and draw the cobweb diagram of the DTDS, starting from $M_0 = 0$ for at least 4 steps.



(f) [1 point] Is the equilibrium point stable or unstable?

Stable