SO,

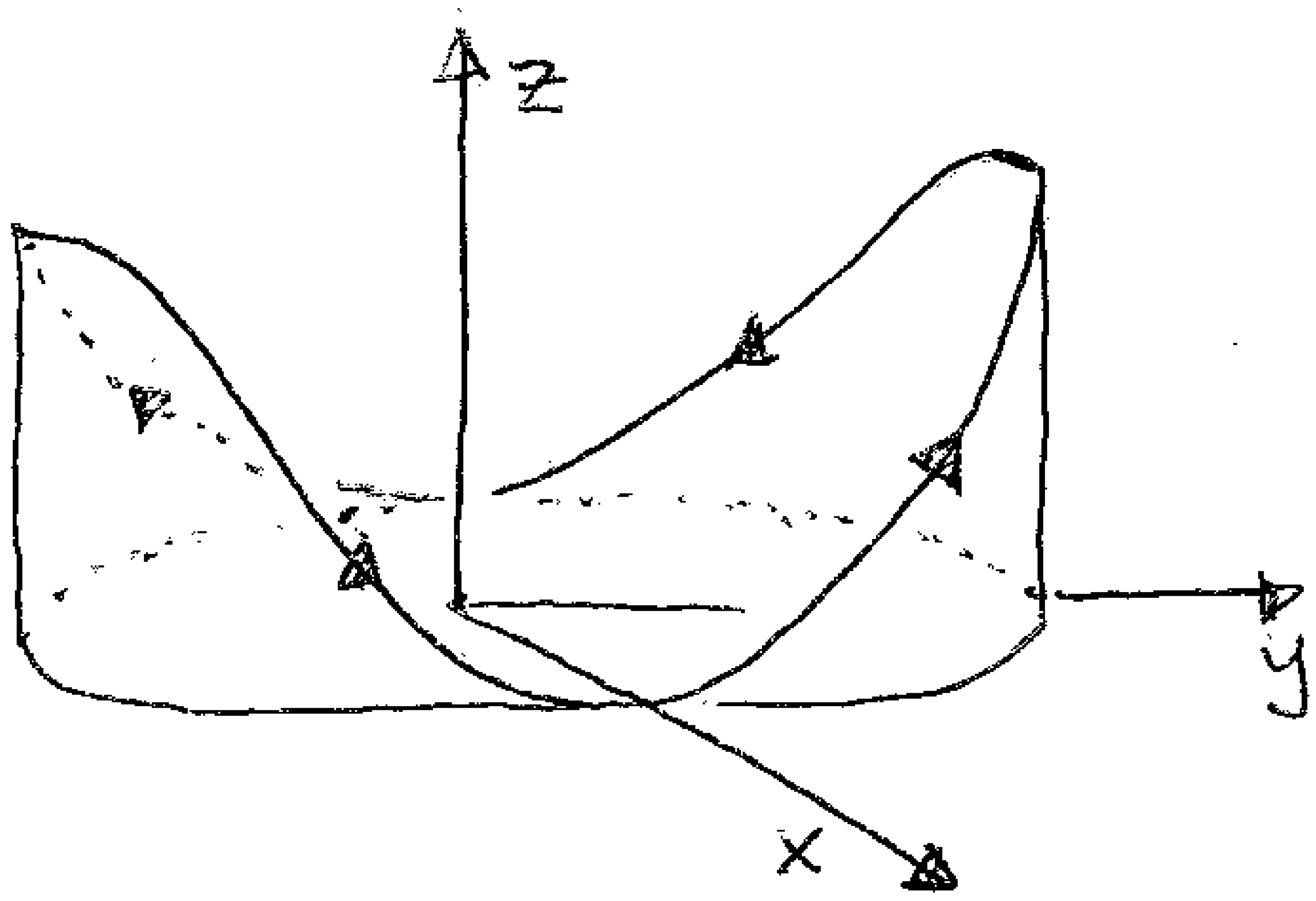
$$\iint_{\Sigma_{SPHERE}} [\vec{\nabla}\times\vec{F}]\cdot\hat{n}_{SPHERE}\,d\sigma = \iint_{\Sigma_{DISK}} 3x^2 - 2y\,dx\,dy$$

CONVERT TO POLAR COORDINATES: $x = r\cos\theta$ & $y = r\sin\theta$ $dx\,dy = r\,d\theta\,dr$

$$= \int_0^{2\pi}\int_0^2 \left[3\cos^2\theta\, r^2 + \sin\theta\, r\right] r\,dr\,d\theta$$

$$= 3\left[\int_0^{2\pi}\cos^2\theta\,d\theta\right]\left[\int_0^2 r^3\,dr\right] = 12\pi.$$

EX. EVALUATE $\oint_{\partial\Sigma} \vec{F}\cdot d\vec{s}$ WHERE $\vec{F} = (2yz)\hat{i} + (xz)\hat{j} + (xy)\hat{k}$ AND $\partial\Sigma$ IS THE INTERSECTION OF THE CYLINDER $x^2+y^2=1$ AND THE PARABOLIC SHEET $z=y^2$, ORIENTED COUNTER-CLOCKWISE WHEN VIEWED IN THE DIRECTION OF $(0,0,-1)$.



A. BY STOKES'S THEOREM,

$$\oint_{\partial\Sigma} \vec{F}\cdot d\vec{s} = \iint_{\Sigma} [\vec{\nabla}\times\vec{F}]\cdot\hat{n}\,d\sigma$$

WHERE Σ IS THE SURFACE $z=y^2$ AND $x^2+y^2\le 1$

PARAMETERIZE Σ AS:

$$\vec{S}(r,\theta) = (r\cos\theta, r\sin\theta, r^2\sin^2\theta)$$
$0\le r\le 1$ & $0\le\theta\le 2\pi$. $z=y^2$

THE TANGENT VECTORS ARE:

$$\vec{T}_r = (\cos\theta, \sin\theta, 2r\sin^2\theta)$$

$$\vec{T}_\theta = (-r\sin\theta, r\cos\theta, 2r^2\cos\theta\sin\theta)$$

AND $\vec{N} = \vec{T}_r\times\vec{T}_\theta = (0, -2r^2\sin\theta, r)$

CORRECT ORIENTATION!

~~FINALLY~~, ALSO,

$$\vec{\nabla}\times\vec{F} = (0, y, -z)$$
$$= (0, r\sin\theta, -r^2\sin^2\theta)$$

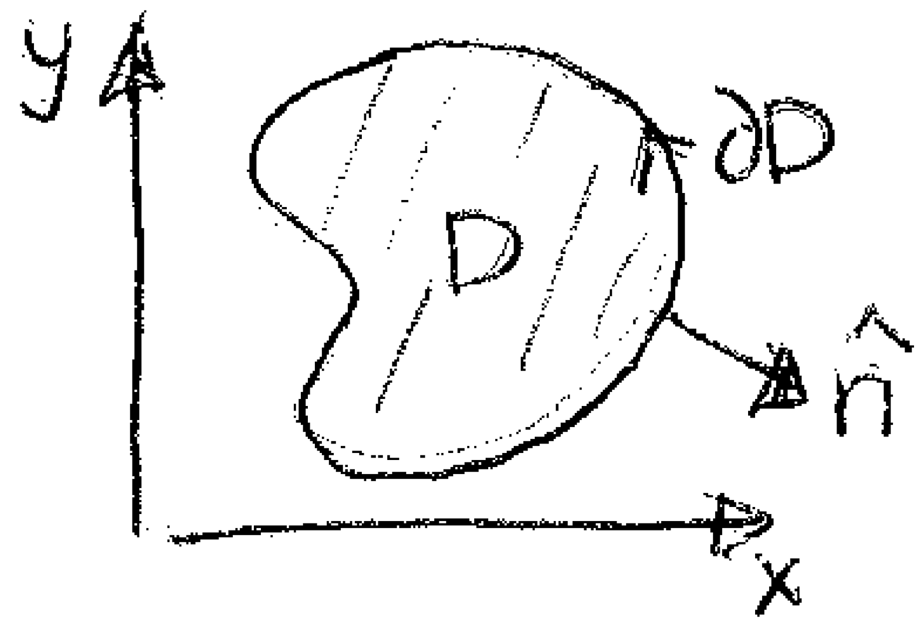
ALTOGETHER,

$$\oint_{\partial\Sigma} \vec{F}\cdot d\vec{s} = \iint_{\Sigma} [\vec{\nabla}\times\vec{F}]\cdot\hat{n}\, d\sigma = \int_0^{2\pi}\int_0^1 \underbrace{(0, r\sin\theta, \overline{r}^2\sin^2\theta)}_{\vec{\nabla}\times\vec{F}} \cdot \underbrace{(0, -2r^2\sin\theta, r)}_{\vec{N}}\, dr\, d\theta$$

$$= -3\int_0^{2\pi}\int_0^1 r^3\sin^2\theta\, dr\, d\theta = -3\left[\int_0^{2\pi}\sin^2\theta\, d\theta\right]\left[\int_0^1 r^3 dr\right] = \frac{-3\pi}{4}.$$

DIVERGENCE THEOREM

IN $\mathbb{R}^2$, WE HAD THE DIVERGENCE THEOREM



$$\oint_{\partial D} \vec{F}\cdot\hat{n}\,ds = \iint_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) dx\,dy$$

OR, WITH $\vec{\nabla}\cdot = (\partial/\partial x)\hat{i} + (\partial/\partial y)\hat{j}$,

$$\oint_{\partial D} \vec{F}\cdot\hat{n}\,ds = \iint_D \vec{\nabla}\cdot\vec{F}\,dx\,dy$$

CLOSED LINE INTEGRAL "FLUX" — AREA INTEGRAL OF "DIVERGENCE".

THIS THEOREM GENERALIZES STRAIGHTFORWARDLY TO $\mathbb{R}^3$

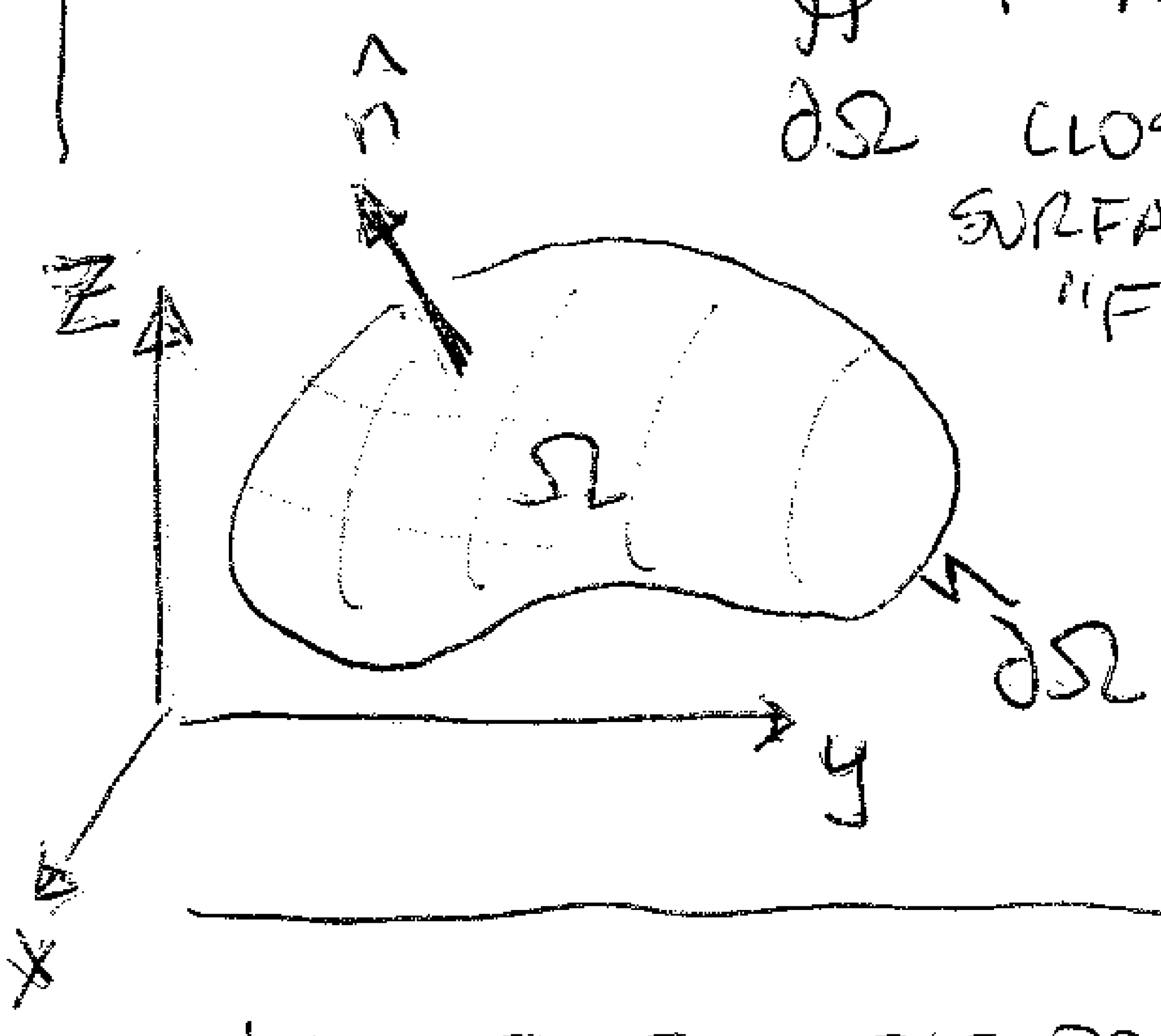
GAUSS'S THEOREM (OR "DIVERGENCE THEOREM" OR "OSTROGRADSKI THEOREM")

LET Ω BE A BOUNDED SUBSET OF $\mathbb{R}^3$ WITH BOUNDARY $\partial\Omega$ WHICH IS A SINGLE C^1 ORIENTABLE <u>CLOSED</u> SURFACE.

IF $\vec{F}$ IS C^1 ON AN OPEN SET CONTAINING $\partial\Omega \cup \Omega$, THEN

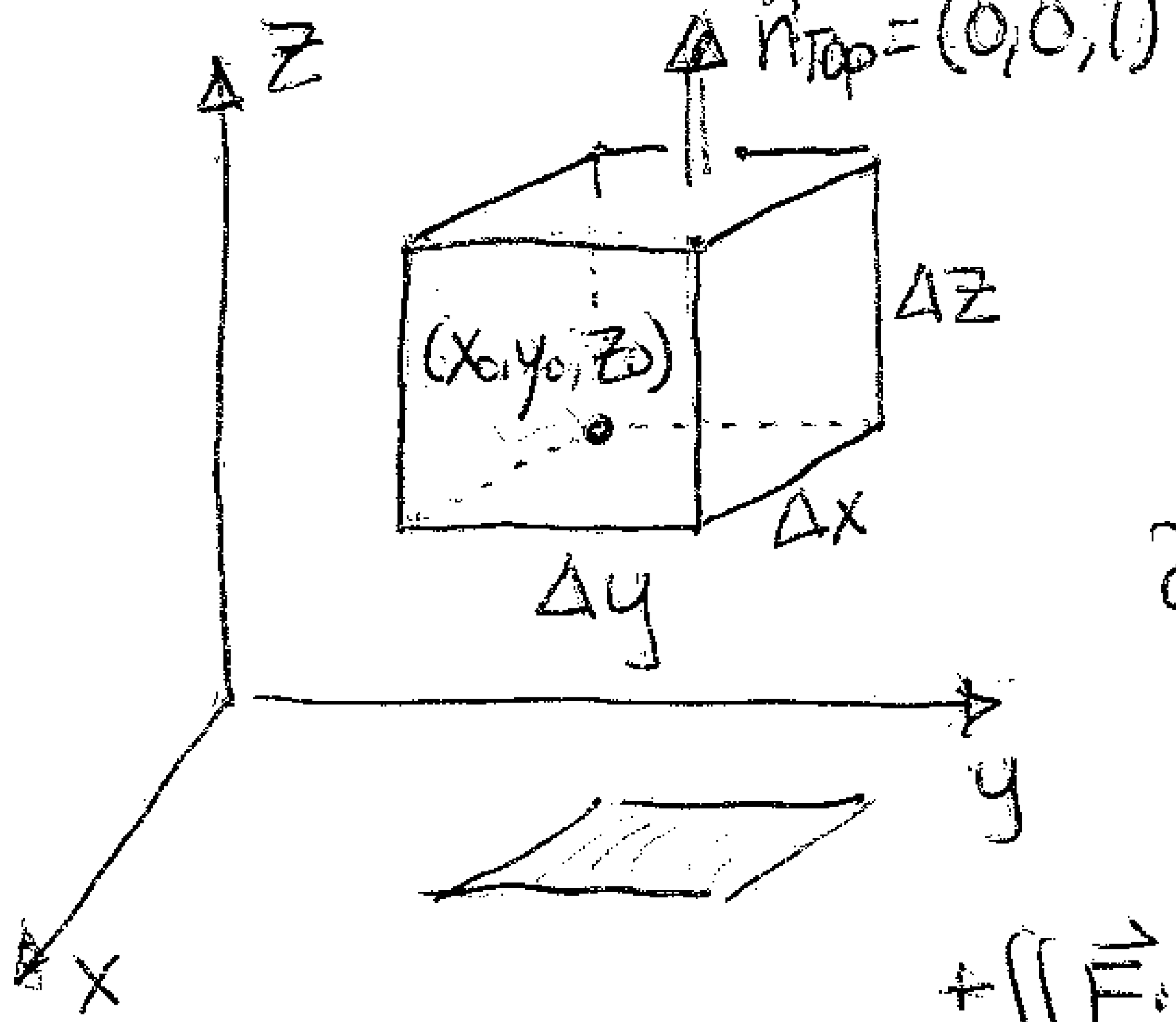
$$\oiint_{\partial\Omega} \vec{F}\cdot\hat{n}\,d\sigma = \iiint_\Omega \vec{\nabla}\cdot\vec{F}\,dV$$

CLOSED SURFACE INTEGRAL "FLUX" — VOLUME INTEGRAL OF "DIVERGENCE"



WHERE $\hat{n}$ IS THE OUTWARD UNIT NORMAL TO $\partial\Omega$.

WE'LL SKETCH THE PROOF BY CONSIDERING A CUBE,



THE FLUX OUT OF THE CUBE IS THE SUM OF FLUX ACROSS ALL 6 FACES:

$$\oiint_{\partial\Sigma} \vec{F}\cdot\hat{n}\,d\sigma = \underset{\substack{\Sigma_{BOTTOM}\\ z=z_0}}{\iint} \vec{F}\cdot\hat{n}\,d\sigma \quad \underset{\substack{\Sigma_{TOP}\\ z=z_0+\Delta z}}{\iint} \vec{F}\cdot\hat{n}\,d\sigma$$

$$+\underset{\substack{\Sigma_{LEFT}\\ y=y_0}}{\iint} \vec{F}\cdot\hat{n}\,d\sigma + \underset{\substack{\Sigma_{RIGHT}\\ y=y_0+\Delta y}}{\iint} \vec{F}\cdot\hat{n}\,d\sigma + \underset{\substack{\Sigma_{BACK}\\ x=x_0}}{\iint} \vec{F}\cdot\hat{n}\,d\sigma + \underset{\substack{\Sigma_{FRONT}\\ x=x_0+\Delta x}}{\iint} \vec{F}\cdot\hat{n}\,d\sigma$$

LOOK, FIRST, AT THE TOP & BOTTOM. THE NORMAL VECTOR IS ALIGNED WITH THE Z-AXIS: $\hat{n}_{TOP} = (0,0,1)$ & $\hat{n}_{BOTTOM} = (0,0,-1)$ FOR $\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$,

$$\underbrace{\iint_{\Sigma_{BOTTOM}} \vec{F}\cdot\underbrace{(0,0,-1)}_{\hat{n}_{BOTTOM}}\,d\sigma}_{z=z_0} + \underbrace{\iint_{\Sigma_{TOP}} \vec{F}\cdot\underbrace{(0,0,1)}_{\hat{n}_{TOP}}\,d\sigma}_{z=z_0+\Delta z} = \iint_{D_{xy}} F_3(x,y,z_0+\Delta z) - F_3(x,y,z_0)\,dx\,dy$$

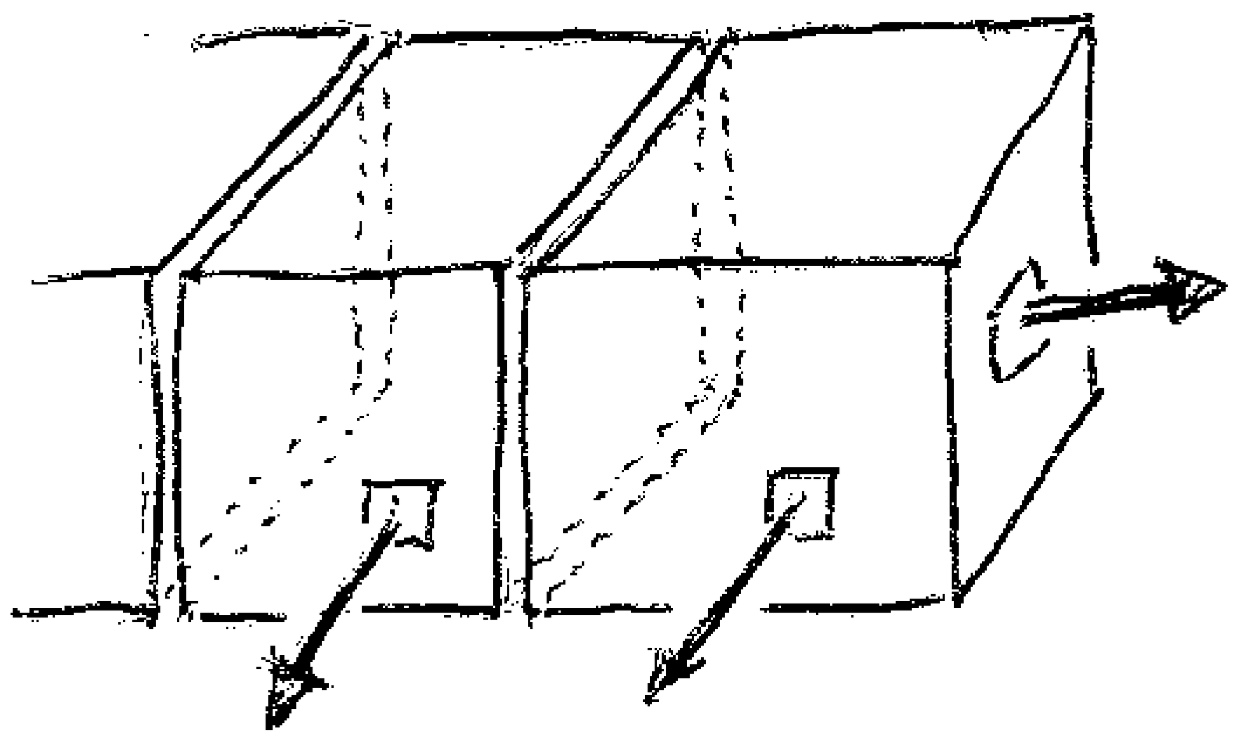
$$= \int_{x_0}^{x_0+\Delta x}\int_{y_0}^{y_0+\Delta y}\left[\int_{z_0}^{z_0+\Delta z} \frac{\partial F_3}{\partial z}\,dz\right]dy\,dx = \iiint_{\Omega} \frac{\partial F_3}{\partial z}\,\underbrace{dx\,dy\,dz}_{dV}$$

VOLUME INTEGRAL

REPEAT WITH LEFT/RIGHT & FRONT/BACK; AND SUM,

$$\oiint_{\partial\Sigma} \vec{F}\cdot\hat{n}\,d\sigma = \iiint_{\Omega} \vec{\nabla}\cdot\vec{F}\,dV$$

TO COMPLETE THE PROOF, DIVIDE A CLOSED VOLUME INTO SMALL CUBES



AS WITH STOKES'S THEOREM, FOR INTERNAL FACES, FLUX BETWEEN NEIGHBOURS CANCEL ONE ANOTHER OUT, LEAVING ONLY BOUNDARY FACES Σ

THE DIVERGENCE THEOREM IS VERY USEFUL IF WE HAVE AN EXPRESSION FOR THE FLUX ACROSS THE <u>SURFACE</u> OF A ~~SOLID~~ CLOSED REGION & WE WANT TO INFER BEHAVIOUR OF THE FIELD <u>INSIDE</u> THE REGION.

WE WILL SEE SHORTLY HOW THIS IDEA IS USED TO DERIVE THE BASIC EQUATIONS OF CLASSICAL PHYSICS.

<u>SOME EXAMPLES</u> -

1. LET $\vec{F} = bxy^2\hat{i} + bx^2y\hat{j} + (x^2+y^2)z^2\hat{k}$ AND Σ IS THE CLOSED SURFACE BOUNDING THE SOLID CYLINDER Ω DEFINED BY $x^2+y^2 \leq a^2$ AND $0 \leq z \leq b$. FIND $\oiint_\Sigma \vec{F}\cdot\hat{n}\,d\sigma$

<u>A</u>. BY THE DIVERGENCE THEOREM,

IN CYLINDRICAL POLAR COORDINATES, $dV = r\,d\theta\,dr\,dz$

$$\oiint_\Sigma \vec{F}\cdot\hat{n}\,d\sigma = \iiint_\Omega \vec{\nabla}\cdot\vec{F}\,dV = \iiint_\Omega (x^2+y^2)(b+2z)\,dV$$

$$= \left[\int_0^b (b+2z)\,dz\right]\left[\int_0^{2\pi} d\theta\right]\left[\int_0^a r^2\,r\,dr\right] = \ldots = \underline{\pi a^4 b^2}$$

2. EVALUATE $\oiint_\Sigma (x^2+y^2)\,d\sigma$ [SCALAR SURFACE INTEGRAL] WHERE Σ IS THE SPHERE $x^2+y^2+z^2=a^2$.

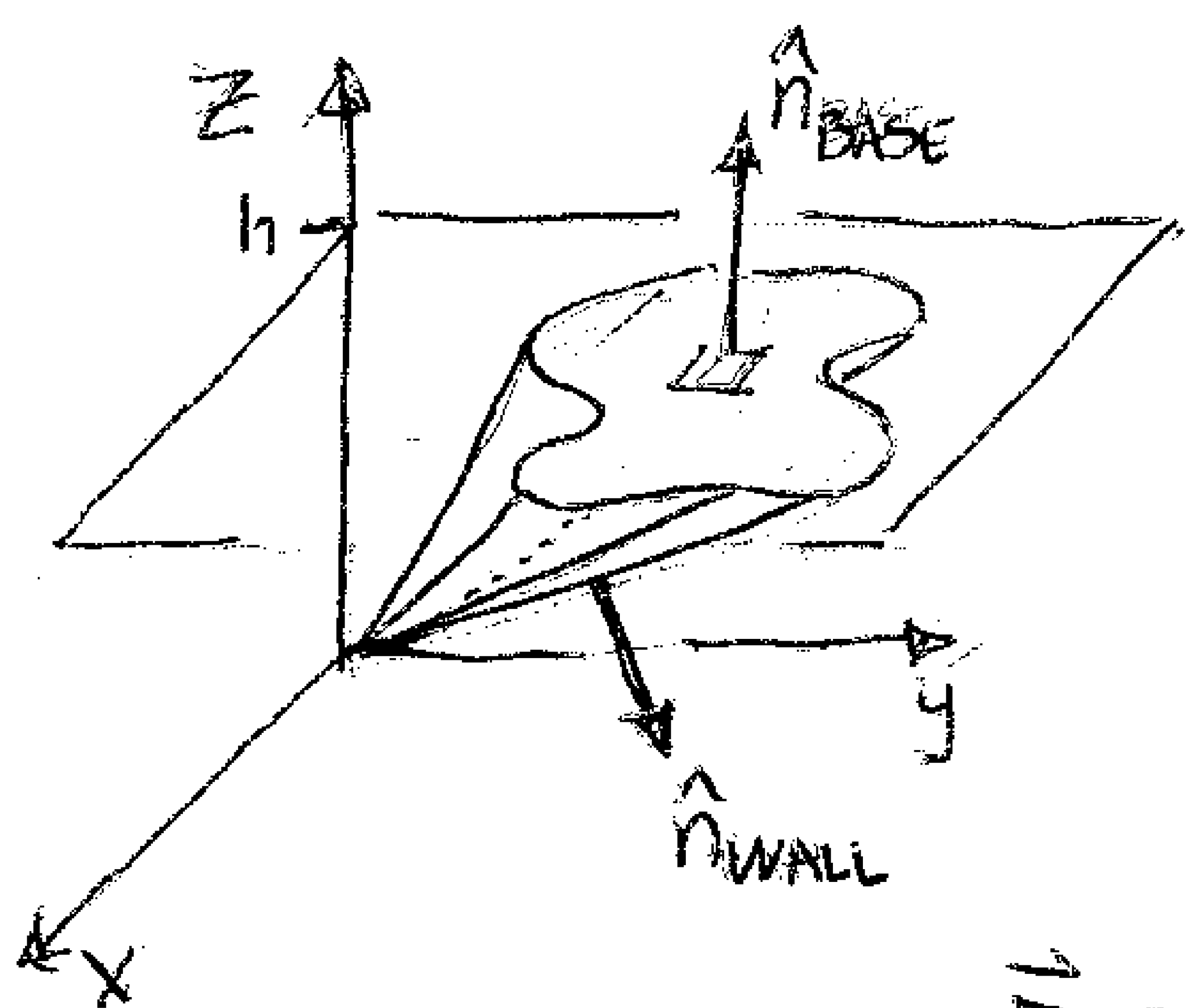
<u>A</u>. ON THE SPHERE, THE OUTWARD NORMAL IS $\hat{n} = \dfrac{x\hat{i}+y\hat{j}+z\hat{k}}{a}$

TO USE THE DIVERGENCE THEOREM, FIND AN $\vec{F}$ SO THAT $\vec{F}\cdot\hat{n} = x^2+y^2$. ONE CHOICE IS $\vec{F} = ax\hat{\imath} + ay\hat{\jmath} + 0\hat{k}$.

THEN
$$\oiint_{\Sigma} (x^2+y^2)\, d\sigma = \oiint_{\Sigma} \vec{F}\cdot\hat{n}\, d\sigma = \iiint_{\Omega} \vec{\nabla}\cdot\vec{F}\, dV = \iiint_{\Omega} (2a)\, dV$$

$$= (2a)\underbrace{\left[\tfrac{4}{3}\pi a^3\right]}_{\text{VOLUME OF SPHERE}} = 8/3\,\pi a^4$$

3. USE THE DIVERGENCE THEOREM WITH $\vec{F} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ TO COMPUTE THE VOLUME OF A CONE WITH BASE AREA $\mathbf{A}$ AND HEIGHT h.



A. LET THE VERTEX OF THE CONE BE AT THE ORIGIN & THE BASE IN THE PLANE $z=h$.

THE CONE IS ENCLOSED BY THE WALL Σ_{WALL} AND THE BASE Σ_{BASE} WITH AREA $\mathbf{A}$.

WITH THE COORDINATE SYSTEM AND THE CONE ALIGNED IN THIS WAY, $\vec{F}$ POINTS AWAY FROM THE ORIGIN SO THAT
$$\vec{F}\cdot\hat{n}_{WALL} = 0$$

ON THE BASE, $\hat{n}_{BASE} = \hat{k}$ AND $z=h$ SO $\vec{F}\cdot\hat{n}_{BASE} = z = h$

FINALLY, $\vec{\nabla}\cdot\vec{F} = 1+1+1 = 3$ PUTTING THIS ALTOGETHER,

$$\text{VOLUME} = \iiint_{\Omega} dV = \iiint_{\Omega}\left[\frac{\vec{\nabla}\cdot\vec{F}}{3}\right] dV = \frac{1}{3}\iiint_{\Omega} \vec{\nabla}\cdot\vec{F}\, dV$$

$$= \frac{1}{3}\left[\iint_{\Sigma_{WALL}} \vec{F}\cdot\hat{n}_{WALL}\, d\sigma + \iint_{\Sigma_{BASE}} \vec{F}\cdot\hat{n}_{BASE}\, d\sigma\right]$$

$$= \frac{1}{3}\left[0 + h\iint_{\Sigma_{BASE}} d\sigma\right] = \frac{1}{3} h\mathbf{A}$$

4. IF Ω SATISFIES THE CONDITIONS OF THE DIVERGENCE THEOREM, AND HAS SURFACE Σ, AND IF $\vec{F}$ IS A C^1 VECTOR FIELD AND Φ IS A C^1 SCALAR FIELD, THEN SHOW THAT

$$\iiint_{\Omega} \vec{\nabla}\times\vec{F}\, dV = -\oiint_{\Sigma} \vec{F}\times\hat{n}\, d\sigma. \qquad (1)$$

AND

$$\iiint_{\Omega} \vec{\nabla}\Phi\, dV = \oiint_{\Sigma} \Phi\hat{n}\, d\sigma. \qquad (2)$$

A. NOTICE BOTH EQUATIONS ARE VECTOR EQUATIONS. THEY CAN BE DERIVED BY APPLYING THE DIVERGENCE THEOREM TO (1) $\vec{F}\times\vec{c}$ AND (2) $\Phi\vec{c}$, WHERE $\vec{c}$ IS AN ARBITRARY CONSTANT VECTOR.

FOR (1): FROM THE DIFFERENTIAL VECTOR IDENTITIES, WE HAVE:

$$\vec{\nabla}\cdot(\vec{F}\times\vec{c}) = (\vec{\nabla}\times\vec{F})\cdot\vec{c} - \vec{F}\cdot(\vec{\nabla}\times\underset{\text{CONSTANT}}{\vec{c}}) = (\vec{\nabla}\times\vec{F})\cdot\vec{c}$$

AND,

$$(\vec{F}\times\vec{c})\cdot\hat{n} = (\hat{n}\times\vec{F})\cdot\vec{c} = -(\vec{F}\times\hat{n})\cdot\vec{c}$$

FROM THE DIVERGENCE THEOREM:

$$0 = \iiint_{\Omega} \vec{\nabla}\cdot(\vec{F}\times\vec{c})\, dV - \oiint_{\Sigma}(\vec{F}\times\vec{c})\cdot\hat{n}\, d\sigma$$

$$= \iiint_{\Omega}(\vec{\nabla}\times\vec{F})\cdot\vec{c}\, dV - \oiint_{\Sigma}(\vec{F}\times\vec{c})\cdot\hat{n}\, d\sigma$$

$$= \left[\iiint_{\Omega}(\vec{\nabla}\times\vec{F})\, dV + \oiint_{\Sigma}(\vec{F}\times\hat{n})\, d\sigma\right]\cdot\vec{c}$$

BUT $\vec{c}$ IS ARBITRARY, SO WHAT IS IN THE SQUARE BRACKETS MUST BE THE ZERO VECTOR $\vec{0}$.

TRY (2) AS AN EXERCISE.