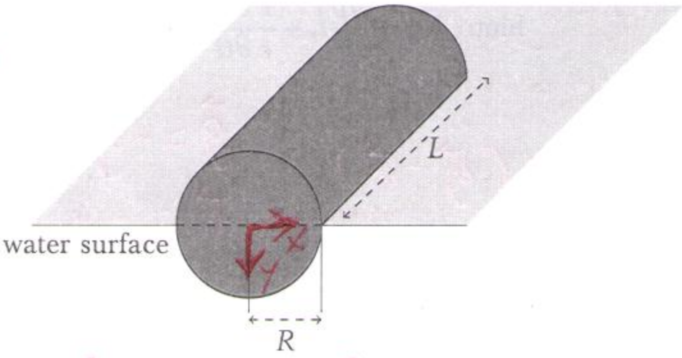
Name:

1) A log of length, L, and radius, R, floats on a lake such that 50% of the log is above water.

a) What is the average density of the log? $\boxed{\rho_{log} = \frac{1}{2}\rho_{H_2O}}$

b) Verify Archimedes' principle by direct integration of the pressure force acting on the log.



$x = R\cos\theta \quad y = R\sin\theta \quad z = z$

$\vec{r} = R\cos\theta\,\hat{i} + R\sin\theta\,\hat{j} + z\hat{k}$

$\frac{\partial r}{\partial \theta} = -R\sin\theta\,\hat{i} + R\cos\theta\,\hat{j} + 0\hat{k}$

$\frac{\partial r}{\partial z} = 0\hat{i} + 0\hat{j} + 1\hat{k}$

$P = \rho_{H_2O}\, g R \sin\theta$

$\hat{n}\,dS = \begin{vmatrix} i & j & k \\ -R\sin\theta & R\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} d\theta\, dz = (R\cos\theta\,\hat{i} + R\sin\theta\,\hat{j})\, d\theta\, dz$

wrong way $\times -1$

$\hat{n}\,dS = -(R\cos\theta\,\hat{i} + R\sin\theta\,\hat{j})\, d\theta\, dz$

Archimedes Principle: $\boxed{\vec{F}_p = -\iiint_V \rho \vec{g}\, dV = -\rho g \frac{V_{log}}{2}\hat{j}}$ $(g = 9.81\, m/s^2)$

$\vec{F}_p = \iint_S -p\hat{n}\, dS = \int_0^L \int_0^\pi -(\rho g R\sin\theta)(-R\cos\theta\,\hat{i} - R\sin\theta\,\hat{j})\, d\theta\, dz$

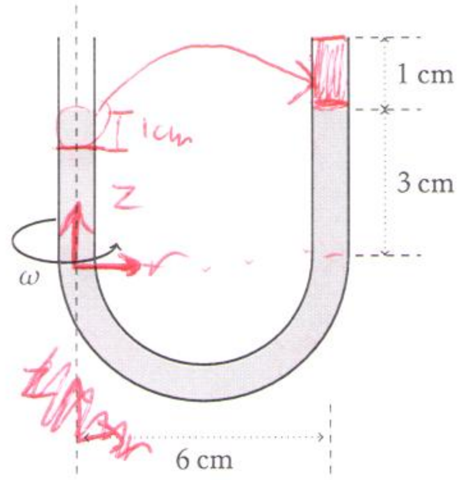
x-component is 0 by symmetry

$F_{py} = \int_0^L \int_0^\pi -\rho g R^2 \sin^2\theta\, d\theta\, dz = -L \int_0^\pi \rho g R^2 \left[\frac{1-\cos 2\theta}{2}\right] d\theta$

$= -L\rho g R^2 \left[\frac{\theta}{2} - \cancel{\frac{\sin 2\theta}{4}}\right]_0^\pi = -\rho g \frac{\pi R^2 L}{2} \Rightarrow \boxed{\vec{F}_p = -\rho g \frac{V}{2}\hat{j}}$ ✓

2) A U-tube is initially filled with water as shown in the image. Very gradually, it is then spun around one arm at ever increasing angular velocity. At what angular velocity, ω, will the water begin to spill from the other arm?

hint: $\vec{\nabla}p = \frac{\partial p}{\partial r}\hat{i}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\hat{i}_\theta + \frac{\partial p}{\partial z}\hat{i}_z$



$\vec{\nabla}p = \rho(\vec{g} - \vec{a})$ $\quad \vec{g} = -g\hat{i}_z$

$\vec{a} = -\omega^2 r \hat{i}_r$

$\frac{1}{r}\frac{\partial p}{\partial \theta} = 0$

$\frac{\partial p}{\partial r} = \rho\omega^2 r \Rightarrow p = \frac{\rho\omega^2 r^2}{2} + c_1(z)$

$\frac{\partial p}{\partial z} = -\rho g \Rightarrow p = -\rho g z + c_2(r)$

$p = \frac{\rho\omega^2 r^2}{2} - \rho g z + C$

@ $r = 0 \quad z = 2\,(cm)$

$P = P_{atm}, \ P_{rel} = 0$

$P_{rel} = 0 = -\rho g(0.02m) + C$

$C = \rho g(0.02m)$

@ $r = 6cm \quad z = 4cm, \ P_{rel} = 0$

$0 = \cancel{\rho}\frac{\omega^2(0.06m)^2}{2} - \cancel{\rho}(9.81m/s^2)(0.04m) + \cancel{\rho}(9.81m/s^2)(0.02m)$

$\omega^2 = 109 \frac{rad^2}{s^2}$ $\quad \boxed{\omega = 10.4\ rad/s}$

3) At a depth of 50 m, a bubble of air is released from a scuba diver. The bubble has an initial diameter of 1 cm. The density of the water is constant and equal to $1000\,\text{kg/m}^3$ and the temperature is constant everywhere. What will be the diameter of the bubble just as it reaches the surface of the water? Assume the bubble remains spherical and that the atmospheric pressure at the surface is 101.325 kPa.

$$P_{rel} \text{ @ } 50\,m = \rho g h = 1000\,kg/m^3 (9.81\,m/s^2)(50\,m)$$

$$= 490\,500\,Pa$$

$$P_{abs} \text{ @ } 50\,m = \rho g h + 101\,325\,Pa = 591\,825\,Pa$$

Boyle's law $\quad (P_1 V_1)\big|_{50m} = (P_2 V_2)\big|_{surface}$

$$V \propto D^3$$

$$\frac{P_1}{P_2} D_1^3 = D_2^3$$

$$\left(\frac{591\,825\,Pa}{101\,325\,Pa}\right)(1\,cm)^3 = 5.84\,cm^3$$

$$\boxed{D_2 = 1.8\,cm}$$