NAME: ____________________

STUDENT NUMBER: ____________________

- No calculators or other electronic aids allowed.
- No notes, books or other papers allowed.
- Answer all questions in the space provided. You must justify your answers and explain your reasoning.
- There are 4 pages. In all there are 7 questions worth a total of 70 marks.

1. Find the derivative of the following functions:

[3] a) $y = 3\sqrt{x} - \dfrac{4}{x^3} + e^2$.

Solution: $y' = \frac{3}{2\sqrt{x}} + \frac{12}{x^4}$.

[3] b) $y = e^x \sin(x)$.

Solution: $y' = e^x \sin(x) + e^x \cos(x)$.

[3] c) $y = \dfrac{\tan(x)}{\sqrt[5]{x}}$.

Solution: $y' = \dfrac{\sec^2(x)\sqrt[5]{x} - \tan(x)\frac{1}{5x^{4/5}}}{x^{2/5}}$.

[4] d) $y = \cos(x)x^2e^x$.

Solution: $y' = -\sin(x)x^2e^x + \cos(x)2xe^x + \cos(x)x^2e^x$.

2. An object is launched directly upward from a platform. If the height of the object with respect to the ground, in feet, after t seconds that it was launched is given by

$$h(t) = -2t^2 + 8t + 10.$$

[4] a) Find the formulas that describe the velocity and acceleration of the object at time t.

Solution: velocity at time t is $v(t) = h'(t) = -4t + 8$, acceleration at time is $a(t) = h''(t) = -4$.

[2] b) Find the height and velocity of the object after 2 seconds. Give units.

Solution: $v(2) = 0$ feet/sec.

[4] c) What is the speed of the object at the moment that hits the ground? Give the units.

Solution: The object hits the ground when $0 = h(t)$. Solving for t we get:

$$0 = -2t^2 + 8t + 10 = -2(t^2 - 4t - 5) = -2(t-5)(t+1)$$

Thus, the object hits the ground after 5 seconds of being launched. Hence, the speed of the object when it hits the ground is $\mid v(5) \mid = \mid -12 \mid =$ 12 feet/sec.

[8] 3. Find an equation of the tangent line to $f(t) = \pi\sin(t)$ at $\left(\dfrac{2\pi}{3}, f\left(\dfrac{2\pi}{3}\right)\right)$.

Solution: We have that $f\left(\frac{2\pi}{3}\right) = \pi\sin\left(\frac{2\pi}{3}\right) = \frac{\pi\sqrt{3}}{2}$. Also, since $f'(t) = \pi\cos(t)$, we have that $f'\left(\frac{2\pi}{3}\right) = \pi\cos\left(\frac{2\pi}{3}\right) = \frac{\pi}{2}$.

Hence, the equation we want is

$$y = f'\left(\frac{2\pi}{3}\right)\left(x - \frac{2\pi}{3}\right) + f\left(\frac{2\pi}{3}\right) = \frac{\pi}{2}\left(x - \frac{2\pi}{3}\right) + \frac{\pi\sqrt{3}}{2}.$$

4. Consider the function $f(x) = 4x^5 - 5x^4$.

[3] a) Find the x-intercepts and y-intercept of f.

[2] b) Find $f'(x)$ and $f''(x)$.

[6] c) Determine the intervals where f is increasing and where is decreasing. Conclude giving the first entry of the local maximums and local minimums of f.

[6] d) Determine the intervals where f is concave upward and where is concave downward. Conclude giving the first entry of the inflection points of f.

Solution:

(i) The y-intercept is $(0, f(0)) = (0, 0)$. The x-intercepts are determined by the solutions of

$$0 = f(x) = 4x^5 - 5x^4 = x^4(4x - 5).$$

Thus, the x-intercepts are $(0, 0)$ and $(5/4, 0)$.

(ii) The first derivative is $f'(x) = 20x^4 - 20x^3$, the second derivative is $f''(x) = 80x^3 - 60x^2$.

(iii) Candidates for local maximums/minimums are determined by the solutions of

$$0 = f'(x) = 20x^4 - 20x^3 = 20x^3(x - 1).$$

Thus, we apply the first derivative test on the intervals determined by $x = 0, 1$:

- On $(-\infty, 0)$, $f' > 0$, so f is increasing on $(-\infty, 0)$.
- On $(0, 1)$, $f' < 0$, so f is decreasing on $(0, 1)$.
- On $(1, \infty)$, $f' > 0$, so f is increasing on $(1, \infty)$.

Hence, f has a local maximum at $x = 0$ and a local minimum at $x = 1$.

(iv) Candidates for inflection points are determined by the solutions of

$$0 = f''(x) = 80x^3 - 60x^2 = 20x^2(4x - 3).$$

Thus, we apply the second derivative test on the intervals determined by $x = 0, 3/4$:

- On $(-\infty, 0)$, $f'' < 0$, so f is concave downward on $(-\infty, 0)$.
- On $(0, 3/4)$, $f'' < 0$, so f is concave downward on $(0, 3/4)$.
- On $(3/4, \infty)$, $f'' > 0$, so f is concave upward on $(3/4, \infty)$.

Hence, f has an inflection point at $x = 3/4$.

[**8**] 5. Consider the function $f(x) = \sqrt{2x}$. Using limits, compute $f'(x)$. Also, determine the domain of $f'(x)$.

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{2x+2h}-\sqrt{2x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{2x+2h}-\sqrt{2x}}{h}\left(\frac{\sqrt{2x+2h}+\sqrt{2x}}{\sqrt{2x+2h}+\sqrt{2x}}\right) \\
&= \lim_{h\to 0} \frac{\left(\sqrt{2x+2h}\right)^2-\left(\sqrt{2x}\right)^2}{h\left(\sqrt{2x+2h}+\sqrt{2x}\right)} \\
&= \lim_{h\to 0} \frac{2h}{h\left(\sqrt{2x+2h}+\sqrt{2x}\right)} \\
&= \frac{2}{\sqrt{2x}+\sqrt{2x}} \\
&= \frac{1}{\sqrt{2x}}.
\end{aligned}$$

Domain: all $x > 0$.

[**7**] 6. Find all the solutions of $\sin(2x) = 2\sin(x)$ on $[0, 2\pi]$.

Solution: Since $\sin(2x) = 2\sin(x)\cos(x)$, we have

$$\begin{aligned}
\sin(2x) &= 2\sin(x) \\
2\sin(x)\cos(x) &= 2\sin(x) \\
2\sin(x)\left(\cos(x)-1\right) &= 0
\end{aligned}$$

So we need to solve the equations $2\sin(x) = 0$ and $\cos(x) = 0$.

The first equation gives $x = 0, \pi, 2\pi$. The second equation gives $x = 0, 2\pi$.

Hence, the solution to the equation is $x = 0, \pi, 2\pi$.

[**7**] 7. Consider the function $f(x) = e^x$. Find all the points where the slope m of the tangent line to f is of the form $m = e^{x^2}e^{x-4}$.

Solution: Since $m = f'(x) = e^x$, we need to solve

$$e^x = e^{x^2}\,e^{x-4} = e^{x^2+x-4}.$$

Thus, we need to solve

$$\begin{aligned}
x &= x^2+x-4 \\
0 &= x^2-4 = (x-2)(x+2).
\end{aligned}$$

Hence, the slope m of the tangent to f is $m = e^{x^2}e^{x-4}$ at $x = -2, 2$.