Ryerson University MTH 330

Family Name:__________________________ Given Name:__________________________________

Ryerson Email:____

Student ID # (ONLY LAST 3 DIGITS):____

Signature:____

Date: Dec. 14, 2017

The test starts at 8:00 am (Time Allowed: 180 Minutes)

Final Exam

Instructions

1. **Tests written in pencil are not eligible for remarking.**
2. One letter size double-sided formula sheet is allowed. Other aids are **not** permitted.
3. Answer all questions in this booklet. If you need extra room, use the last three pages, clearly indicating where your answer continues.
4. There are no part marks for multiple choice questions and only the final answer will be marked. You can use the space on the last two pages to take up the MC questions.
5. In long-answer questions, you must show your work, presented clearly and in the correct order. **Unjustified answers will be given little or no credit.**
6. Cross out all irrelevant or incorrect work, as marks may be deducted for work which is misleading, irrelevant or incorrect.
7. Keep your Ryerson ID displayed on your desk.
8. **Do not separate the sheets.**
9. Make sure your test paper is complete; there are 9 questions on 11 pages (including this one). The last two pages are given for extra space and do NOT contain any questions.

To Students: DO NOT WRITE ANYTHING ON THIS PAGE

Question Number	Score
Q1-3	/6
Q4	/7
Q5	/6
Q6	/7
Q7	/7
Q8	/8
Q9	/9
Total	/50

Part One: Multiple Choice/True-False Questions (2 marks each)- Circle your answer

1. Which of the following sets is a subspace of R^3 with the usual addition and scalar multiplication?

(i) The set with a single vector 0 = (0, 0, 0).

(ii) The set of all vectors (a, b, c) with 7a + 10b + 4c = 0.

(iii) The set of all vectors (a, b, c) with 7a + 10b + 4c = 1.

(a) None of them.

(b) (i) and (ii) only.

(c) (i) and (iii) only.

(d) (ii) and (iii) only.

(e) (i), (ii), and (iii).

2. Consider the following two subsets of the vector space $F(-\infty,\infty)$ (i.e. functions defined on the real line with the usual addition and scalar multiplication). Which of these are closed under addition?

W = {f(x) such that f(2x) = 1+f(x)}

Y = {f(x) such that f(−x) = −f(x)}

(a) W only

(b) both W and Y

(c) Y only

(d) neither W nor Y

(e) I like cupcakes!

3. Which of the following sets form an orthogonal set of vectors, but NOT orthonormal?

(a) $\left\{\frac{1}{\sqrt{2}}(0,1,0,-1), \frac{1}{\sqrt{2}}(0,1,0,1), \frac{1}{\sqrt{2}}(2,1,2,0)\right\}$

(b) $\left\{\frac{1}{\sqrt{2}}(0,\sqrt{3},0,1), \frac{1}{\sqrt{2}}(0,1,0,\sqrt{3}), \frac{1}{\sqrt{2}}(2,1,2,0)\right\}$

(c) $\left\{\frac{1}{5}(0,3,4,0), \frac{1}{5}(4,0,0,3), \frac{1}{5}(-3,0,0,4)\right\}$

(d) $\left\{(1,0,0,0), \frac{1}{\sqrt{2}}(0,1,1,0), \frac{1}{\sqrt{2}}(0,0,1,1)\right\}$

(e) $\left\{\frac{1}{4}(0,3,4,0), \frac{1}{4}(4,0,0,3), \frac{1}{4}(3,0,0,4)\right\}$

Part Two: Show all your work.

7 Marks

4. Evaluate the line integral $\mathbf{F} = (ye^{xy} + \cos(x+y) + yz)\mathbf{i} + (xe^{xy} + \cos(x+y) + xz)\mathbf{j} + xy\,\mathbf{k}$ along the spiral $\mathbf{r}(t) = t\cos(t)\,\mathbf{i} + t\sin(t)\,\mathbf{j} + t\,\mathbf{k}$ where $0 \le t \le 2\pi$.

$\vec{F} = \langle ye^{xy} + \cos(x+y) + yz,\ xe^{xy} + \cos(x+y) + xz,\ xy \rangle$

Let $\nabla\phi = \vec{F}$

$\phi_x = ye^{xy} + \cos(x+y) + yz \Rightarrow \phi = e^{xy} + \sin(x+y) + xyz + h(y,z)$

$\frac{\partial\phi}{\partial y} = \phi_y = xe^{xy} + \cos(x+y) + xz + \frac{\partial h}{\partial y} = F_2 \Rightarrow \frac{\partial h}{\partial y} = 0 \Rightarrow h = g(z)$

$\frac{\partial\phi}{\partial z} = xy + g'(z) = xy \Rightarrow g'(z) = 0 \Rightarrow g(z) = K$

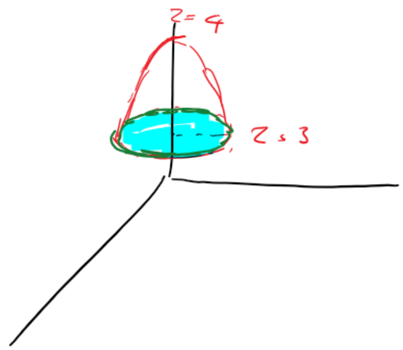
$\phi(x,y,z) = e^{xy} + \sin(x+y) + xyz + k$

$\int \vec{F}\cdot d\vec{r} = \phi(r(2\pi)) - \phi(r(0)) = \phi(2\pi, 0, 2\pi) - \phi(0,0,0) = 0$

6 Marks

5. Find the flux of the field $\mathbf{F} = (xy)\mathbf{i} + (yz)\mathbf{j} + xz\,\mathbf{k}$ through the region bounded by $z = 4 - x^2 - y^2$ and $z = 3$.

$F = \langle xy, yz, xz \rangle$



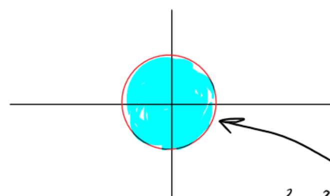
$\text{div} F = \nabla \cdot F$

$= y + z + x$

$$\iiint \text{div } F \, dv = \iiint (y+z+x)\, dv$$

r^2 $\quad 4-(x^2+y^2)$

$$\int_0^{2\pi}\int_0^1\int_3^{4-x^2-y^2} (r\sin\theta + z + r\cos\theta)\, r\, dz\, dr\, d\theta$$



$$\begin{cases} z = 4-x^2-y^2 \\ z = 3 \end{cases}$$

$4 - x^2 - y^2 = 3$

$x^2 + y^2 = 1$

$$\int_0^{2\pi}\int_0^1 \boxed{\int_3^{4-r^2} \left[r^2\sin\theta + zr + r^2\cos\theta\right] dz}\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left[r^2 z\sin\theta + \frac{z^2}{2} r + r^2 z\cos\theta\right]_3^{4-r^2} dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^2(\sin\theta + \cos\theta)(1-r^2) + \frac{r}{2}(4-r^2)^2 - 9 \quad dr\, d\theta$$

$$= \frac{5\pi}{3}$$

7 Marks 6. Let S be the surface given by $z = 1 - x^2$ with $0 \le x \le 1$ and $-2 \le y \le 2$ oriented upward. Find $\oint_C \boldsymbol{F}.d\boldsymbol{r}$ where C is the boundary of S.

$z = 1 - x^2 \quad 0 \le x \le 1 \quad -2 \le y \le 2$

$\vec{F} = xy\,i + yz\,j + xz\,k \qquad g(x,y,z) = x^2 + z - 1$

$f(x,y) = 1 - x^2$

$\text{curl } F = \langle -y, -z, -x \rangle$

$\oint_C F.dr = \iint_S \text{curl } F.\vec{n}\, ds \quad , \vec{n} = \frac{\nabla g}{|\nabla g|} = \frac{\langle 2x, 0, 1\rangle}{\sqrt{4x^2+1}}$

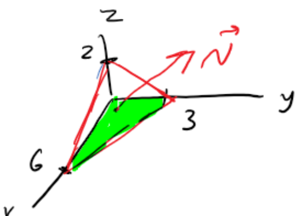
$ds = \sqrt{1 + f_x^2 + f_y^2}\, dA = \sqrt{1 + 4x^2}\, dA$

$= \iint \langle -y, -z, -x \rangle . \frac{\langle 2x, 0, 1 \rangle}{\cancel{\sqrt{4x^2+1}}} . \cancel{\sqrt{1+4x^2}}\, dA$

$= \int_0^1 \int_{-2}^{2} -2xy - x \; dy\, dx \quad = -2$

7 Marks

7. Find the flux of the vector field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ through the part of the surface $x + 2y + 3z = 6$ in the first octant.



Find flux of $F = x\,i + y\,j + z\,k$

through a part of $x + 2y + 3z = 6$ in 1st octant.

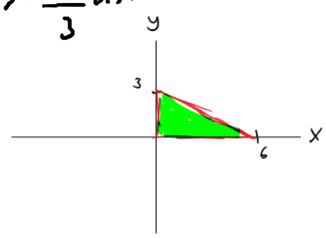
$$\iint \underbrace{\vec{F}\cdot\vec{n}}\, ds \qquad \vec{n} = \frac{\langle 1, 2, 3\rangle}{\sqrt{1+4+9}} = \frac{1}{\sqrt{14}}\langle 1, 2, 3\rangle$$

$$\vec{F} = \langle x, y, z\rangle \qquad ds = \sqrt{1 + f_x^2 + f_y^2}\, dA = \sqrt{1 + \frac{1}{9} + \frac{4}{9}}\, dA$$

$$f = 2 - \frac{x}{3} - \frac{2}{3}y \qquad = \frac{\sqrt{14}}{3}\, dA$$

$$\iint \langle x, y, z\rangle \cdot \frac{1}{\cancel{\sqrt{14}}}\langle 1, 2, 3\rangle \frac{\cancel{\sqrt{14}}}{3}\, dA$$

$$\iint \frac{x}{3} + \frac{2y}{3} + z\, dA$$



$x + 2y + 3z = 6$

$\rightarrow z = 2 - \frac{x}{3} - \frac{2}{3}y$

$$\iint \cancel{\frac{x}{3}} + \cancel{\frac{2y}{3}} + 2 - \cancel{\frac{x}{3}} - \cancel{\frac{2}{3}y}\, dA = \iint 2\, dA$$

$$= 2\iint dA = 2(\text{area of triangle})$$

$$= 2\left(\tfrac{1}{2}\right)(6)(3) = 18$$

8 Marks 8. (a) Use the Gram-Schmidt procedure to produce an orthonormal basis for the subspace spanned by
$\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$

Solution: $\vec{v}_1 = \begin{bmatrix}1\\2\\1\end{bmatrix}$, $\mathbb{S}_1 = \mathrm{Span}\{\vec{v}_1\}$.

$\mathrm{perp}_{\mathbb{S}_1}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix} - \frac{2}{6}\begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}-1/3\\1/3\\-1/3\end{bmatrix}$, choose $\vec{v}_2 = \begin{bmatrix}-1\\1\\-1\end{bmatrix}$, $\mathbb{S}_2 = \mathrm{Span}\{\vec{v}_1, \vec{v}_2\}$.

$\mathrm{perp}_{\mathbb{S}_2}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{1}{6}\begin{bmatrix}1\\2\\1\end{bmatrix} - \frac{-1}{3}\begin{bmatrix}-1\\1\\-1\end{bmatrix} = \begin{bmatrix}1/2\\0\\-1/2\end{bmatrix}$, choose $\vec{v}_3 = \begin{bmatrix}1\\0\\-1\end{bmatrix}$.

Then an orthonormal basis for the subspace is $\left\{\frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\1\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}-1\\1\\-1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$

b) Find the coordinated of the vector $\mathbf{v} = \begin{bmatrix}0\\4\\2\end{bmatrix}$ with respect to the orthonormal basis obtained in part (a).

Solution: $\vec{x} = \begin{bmatrix}0\\4\\2\end{bmatrix} = \left(\frac{10}{\sqrt{6}}\right)\frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\1\end{bmatrix} + \left(\frac{2}{\sqrt{3}}\right)\frac{1}{\sqrt{3}}\begin{bmatrix}-1\\1\\-1\end{bmatrix} + \left(\frac{-2}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}$ so

$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}10/\sqrt{6}\\2/\sqrt{3}\\-2/\sqrt{2}\end{bmatrix}$.

9 Marks

9- a) Find all eigenvalues and eigenvectors for the matrix $A = \begin{pmatrix} 4 & 8 \\ 2 & 4 \end{pmatrix}$.

$\det(A-\lambda I)=0$

$\det\begin{bmatrix} 4-\lambda & 8 \\ 2 & 4-\lambda \end{bmatrix}=0$

$(4-\lambda)^2-16=0$: $4-\lambda=4 \rightarrow \lambda_1=0$; $4-\lambda=-4 \rightarrow \lambda_2=8$

for $\lambda_1=0 \rightarrow \begin{bmatrix} 4 & 8 \\ 2 & 4 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}}_{RREF}$ $\quad x_1+2x_2=0 \rightarrow x_1=-2t$, $x_2=t$ $\rightarrow v_1=\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$

for $\lambda_2=8 \rightarrow \begin{bmatrix} -4 & 8 \\ 2 & -4 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}}_{RREF}$ $\quad x_1-2x_2=0 \rightarrow x_1=2t$, $x_2=t$ $\rightarrow v_2=\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$

b) Find the least square line for the points (0,-1), (1,3), (2,5), and (3,7).

We have $\mathbf{Y}^T = \begin{pmatrix} -1 & 3 & 5 & 7 \end{pmatrix}$ and $\mathbf{A}^T = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{pmatrix}$.

Now $\mathbf{A}^T\mathbf{A} = \begin{pmatrix} 14 & 6 \\ 6 & 4 \end{pmatrix}$ and $(\mathbf{A}^T\mathbf{A})^{-1} = \frac{1}{20}\begin{pmatrix} 4 & -6 \\ -6 & 14 \end{pmatrix}$

so $\mathbf{X} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{Y} = \begin{pmatrix} \frac{13}{5} \\ -\frac{2}{5} \end{pmatrix}$ and the least squares line is $y = 2.6x - 0.4$.