Assignment 6 solutions

Total=5 points

1. Find the tangent plane to the graph of the function

$$f(x, y) = 2\pi - 3\cos 4x + 3\sin 2y$$

at the point $(0, \pi)$.

Solution: The tangent plane of f at a point (a, b) is given by the equation

$$z = f(a, b) + \text{grad} f(a, b) \begin{pmatrix} x - a \\ y - b \end{pmatrix}.$$

Here $(a, b) = (0, \pi)$. The x-partial derivative of f, $\frac{\partial f}{\partial x}$, is obtained by treating y as a constant and taking the derivative with respect to x: here it is given by $0 + 12\sin 4x + 0 = 12\sin 2x$. Similarly $\frac{\partial f}{\partial y} = 6\cos 2y$, and so

$$\text{grad} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (12\sin 2x,\ 6\cos 2y).$$

For the tangent plane (or linear approximation), we evaluate the gradient at the specified point $(0, \pi)$, that is, we plug in 0 for x and π for y:

$$\text{grad} f(a, b) = \text{grad} f(0, \pi) = (0, 6).$$

So the tangent plane is given by

$$\begin{aligned} z =& f(a, b) + \text{grad} f(a, b) \begin{pmatrix} x - a \\ y - b \end{pmatrix} \\ =& f(0, \pi) + (0, 6) \begin{pmatrix} x \\ y - \pi \end{pmatrix} \\ =& (2\pi - 3 * 1 + 3 * 0) + (0, 6) \begin{pmatrix} x \\ y - \pi \end{pmatrix} \\ =& (2\pi - 3) + 0 * x + (6)(y - \pi) \\ =& -4\pi - 3 + 6y. \end{aligned}$$

Alternatively we could have used the equivalent formula

$$z = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b)$$

for the tangent plane.

2. Find the (2,2)-entry of the Jacobian matrix of the function

$$F(x,y) = \begin{bmatrix} x^2e^y + 2x\sin(x^y) \\ \sin(x^2) - 3ye^{-x} \end{bmatrix}$$

at the point $(2,1)$.

Solution: The Jacobian of $F = \begin{pmatrix} f \\ g \end{pmatrix}$ is given by

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}.$$

The (2,2) entry means the second row, second column ($(i,j)^{\text{th}}$ entry is the entry in the i^{th} row and j^{th} column), that is, $\frac{\partial g}{\partial y}$.

Here g, the second row of F, is the function $\sin(x^2) - 3ye^{-x}$. Its y-partial derivative is

$$\frac{\partial g}{\partial y} = \frac{\partial}{\partial y}(\sin(x^2) - 3ye^{-x}) = 0 - 3e^{-x}.$$

We are asked to evaluate this at the point $(2,1)$; in other words, plug in 2 for x and 1 for y:

$$\frac{\partial g}{\partial y}(2,1) = -2 * 1^{-2} - 3e^{-2} = -2 - 3e^{-2}.$$

3. Consider the following system of linear differential equations:

$$\frac{dx}{dt} = -3x + y$$
$$\frac{dy}{dt} = 4x - 3y$$

(a) Find the eigenvalues and eigenvectors associated with the system.

(b) Write down the general solution formula for the system.

(c) Give the particular solution for the initial values $x(0) = 4, y(0) = 4$.

(d) Draw the x- and y-nullclines and the direction arrows in the phase plane.

(e) Sketch the solution curve for the initial condition in part (c) into the phase plane.

(f) Is the point (0,0) stable or unstable? Classify this equilibrium.

Solution: (a) The matrix associated to the system is the matrix of coefficients in the differential equations:

$$A = \begin{pmatrix} -3 & 1 \\ 4 & -3 \end{pmatrix}.$$

Its eigenvalues are the roots of

$$\det(A-\lambda I) = \det\begin{pmatrix}-3-\lambda & 1\\ 4 & -3-\lambda\end{pmatrix} = (-3-\lambda)(-3-\lambda)-1*4 = \lambda^2+6\lambda+5 = (\lambda+5)(\lambda+1),$$

namely $\lambda = -5, -1$. (We could use the quadratic formula instead of factoring directly; in general the quadratic formula will be necessary.)

We find the eigenvectors for each eigenvalue λ by solving $(A-\lambda I)\mathbf{v} = \mathbf{0}$ by row reduction. For $\lambda = -5$,

$$A - \lambda I = \begin{pmatrix}2 & 1\\ 4 & 2\end{pmatrix},$$

and the augmented matrix is immediately row-reduced to

$$\left(\begin{array}{cc|c}1 & 1/2 & 0\\ 0 & 0 & 0\end{array}\right).$$

The general solution to this (i.e. to $x_1 + 1/2x_2 = 0$) is

$$\mathbf{v} = \begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}-t/2\\ t\end{pmatrix} = t\begin{pmatrix}-1/2\\ 1\end{pmatrix},$$

and an eigenvector corresponding to $\lambda_1 = -5$ is

$$\mathbf{v}_1 = \begin{pmatrix}-1/2\\ 1\end{pmatrix}.$$

(We took $t = 1$ here. Alternatively we could take $t = 2$, which gives the eigenvector $\left(\begin{smallmatrix}-1\\ 2\end{smallmatrix}\right)$.)

Similarly we find that

$$\mathbf{v}_2 = \begin{pmatrix}1/2\\ 1\end{pmatrix}$$

is an eigenvector for the eigenvalue $\lambda_2 = -1$.

(b) The general solution is

$$\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} = C_1e^{\lambda_1 t}\mathbf{v}_1 + C_2e^{\lambda_2 t}\mathbf{v}_2 = C_1e^{-5t}\begin{pmatrix}-1/2\\ 1\end{pmatrix} + C_2e^{-t}\begin{pmatrix}1/2\\ 1\end{pmatrix} = \begin{pmatrix}-1/2C_1e^{-5t} + 1/2C_2e^{-t}\\ C_1e^{-5t} + C_2e^{-t}\end{pmatrix},$$

so $x(t) = -1/2C_1e^{-5t} + 1/2C_2e^{-t}$, $y(t) = C_1e^{-5t} + C_2e^{-t}$, with C_1 and C_2 arbitrary constants.

(c) At $t = 0$, $x = -1/2C_1 + 1/2C_2$ and $y = C_1 + C_2$, so we need to solve the linear system

$$\begin{array}{rcrcl}-1/2C_1 & + & 1/2C_2 & = & 4\\ C_1 & + & C_2 & = & 4\end{array}$$

We can use any method we like, such as row-reduction, to find that there is a single solution, with $C_1 = -2$, $C_2 = 6$. The particular solution is therefore

$$\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}e^{-5t} + 3e^{-t}\\ -2e^{-5t} + 6e^{-t}\end{pmatrix}.$$

(We should verify that this really is a solution of the system of differential equations and that it takes the specified initial values.)

(d) The x-nullcline is the solutions to $-3x + y = 0$, which is the line $y = 3x$. The y-nullcline is the solutions to $4x - 3y = 0$, which is the line $y = 4x/3$. See the figure.

(e) See the figure. The arrows are obtained by taking various points (x, y) along the nullclines (or the initial condition point $(4, 4)$), and plotting the vector $(-3x + y, 4x - 3y)$ originating from that point. Note that vectors are not drawn to scale.

The curve is obtained by taking various values for t ($t = 0, 0.5, 1, \ldots$) and plotting the $(x(t), y(t))$ pairs, where x and y are given by the functions in part (c).

(f) Note that $(0, 0)$ is the only equilibrium point. This will be true for any **linear** system of differential equations. Since both of the eigenvalues (-5 and -1) have negative real part, this equilibrium is unstable.

We can further see that the system is stable since the particular solution in part (c) heads off towards (0,0), as time t gets larger and larger.

4. Consider a disease that propogates according to the system

$$\begin{aligned} \frac{dx}{dt} &= 16 - 0.2xy - 0.4x \\ \frac{dy}{dt} &= 0.1xy - 8y \end{aligned}$$

where x represents susceptible individuals, y represents infected individuals.

(a) Find all biologically meaningful steady states.

(b) Show that the Jacobian matrix of this system is given by

$$\begin{bmatrix} -0.4 - 0.2y & -0.4x \\ 0.1y & 0.1x - 8 \end{bmatrix}$$

(c) For the biologically meaningful steady states from (a), find the eigenvalues of the Jacobian matrix.

(d) Determine the stability of the biologically meaningful steady states.

Solution: (a) Biologically meaningful here simply means that the numbers are not negative. The steady states (= equilibrium points) are the places where both $16 - 0.2xy - 0.4x = 0$ and $0.1xy - 8y = 0$. The second equation is easier (since we can factor it) so we deal with it first: $y(0.1x - 8) = 0$ when $y = 0$ or when $x = 8/0.1 = 80$. For each of these cases we plug the given value into the first equation (which must also hold).

If $y = 0$, then the first equation says that $16 - 0.4x = 0$, so $x = 16/0.4 = 40$. Therefore $(40, 0)$ is one equilibrium. **(1 point)**

The only other case is when $x = 80$. Here, the first equation says that $16 - 0.2(80)y - 0.4(80) = 0$, so $16y = -16$ and $y = -1$. Therefore $(60, -1)$ is another equilibrium, and there are no others. This equilibrium point is not biologically meaningful since its second coordinate is negative. **(1 point for ruling out this equilibrium with an explanation)**

(b) The Jacobian of

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 16 - 0.2xy - 0.4x \\ 0.1xy - 8y \end{pmatrix}$$

is given by

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}.$$

We just have to confirm four partial derivatives were given correctly. So, for example,

$$\frac{\partial}{\partial x}(16 - 0.2xy - 0.4x) = -0.4 - 0.2y.$$

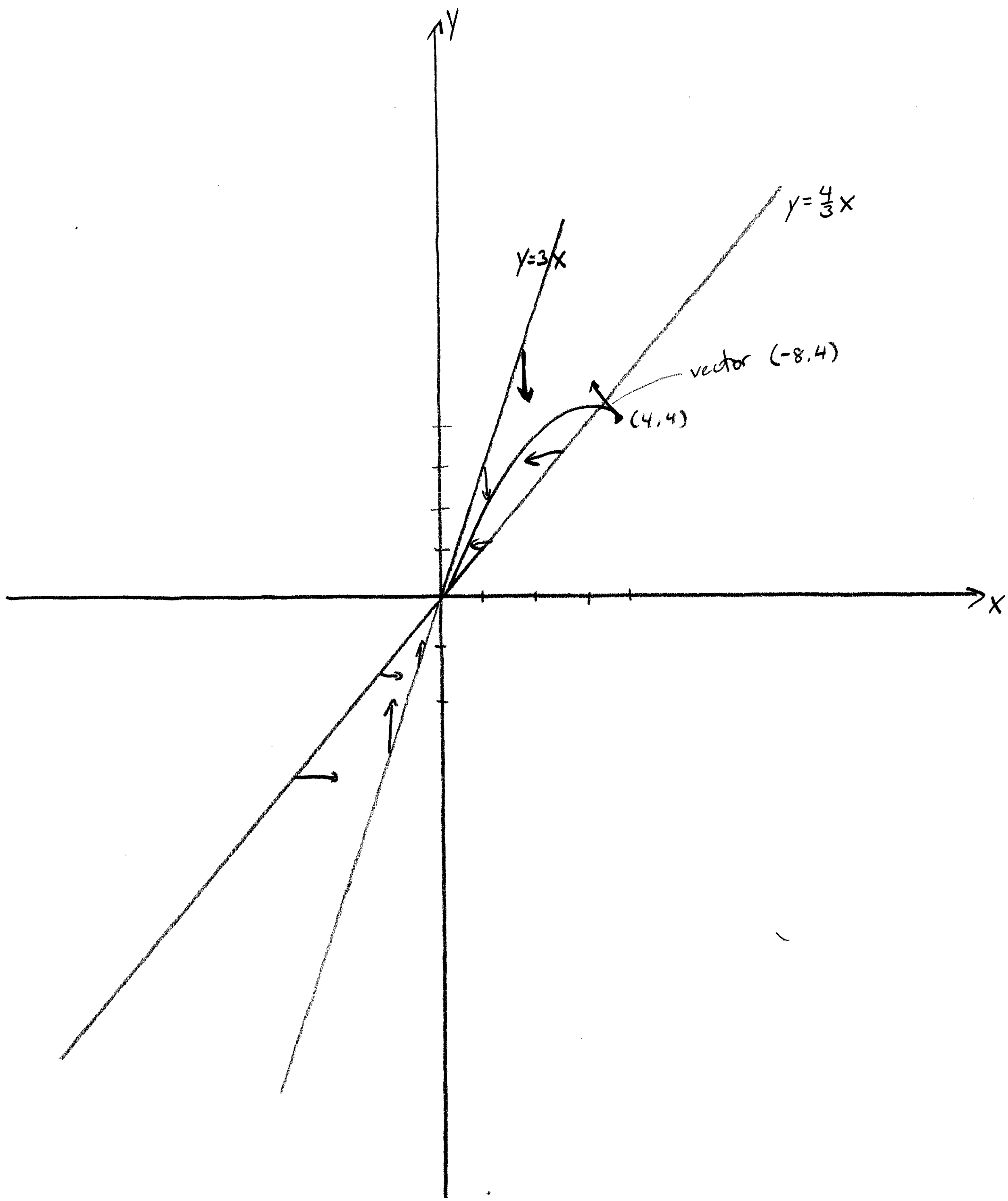
(c) We have one biologically meaningful steady states: (40,0). We plug $x = 40$, $y = 0$ into the formula given in part (b) for J:

$$J(40, 0) = \begin{pmatrix} -0.4 & -8 \\ 0 & -4 \end{pmatrix}.$$

This matrix is upper-triangular (since the only entry below the main diagonal is zero), so its eigenvalues are its diagonal entries: -0.4 and -4. **(2 points)** You should compute the characteristic polynomial to verify that these are in fact the eigenvalues.

(d) Since the eigenvalues of the Jacobian matrix at the equilibrium have negative real part (in fact, are negative real numbers), we can conclude that this equilibrium is stable. **(1 point)** In fact, it is a stable sink.

What this means in concrete terms is that starting from any population with any infection rate, after enough time the end result will be that x is very close to 40 and y is very close to 0; in particular the disease will be wiped out in time.



y
x
y=3x
y = 4/3 x
vector (-8,4)
(4,4)