MAT1332 Assignment 5 Solutions

Total =20 points

1. **(6 points)** For the system of linear equations

$$\begin{array}{rcrcrcl} x & + & 3y & + & 9z & = & 3 \\ 2x & + & 7y & + & 23z & = & 2 \\ x & + & ay & + & a^2z & = & a \end{array}$$

   (a) determine the values of a for which the system has
      (i) no solution,
      (ii) infinitely many solutions,
      (iii) a unique solution.

   (b) In case (ii) above describe all solutions.

   (c) If $a = 1$ find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 7 & 2 \\ 1 & a & a^2 \end{bmatrix}$$

   The augmented matrix of the system is

$$A = \left[\begin{array}{ccc|c} 1 & 3 & 9 & 3 \\ 2 & 7 & 23 & 2 \\ 1 & a & a^2 & a \end{array}\right]$$

   We perform the following operations, where R_i is row i: $R_2 \rightsquigarrow R_2 - 2R_1$, $R_3 \rightsquigarrow R_3 - R_1$, $R_3 \rightsquigarrow R_3 - (a-3)R_2$, and obtain

$$A \sim \left[\begin{array}{ccc|c} 1 & 3 & 9 & 3 \\ 0 & 1 & 5 & -4 \\ 0 & a-3 & a^2-9 & a-3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & a^2-9-5a+15 & 5(a-3) \end{array}\right].$$

   Since $a^2 - 9 - 5a + 15 = (a-3)(a-2)$ we get:

$$\left[\begin{array}{ccc|c} 1 & 3 & 9 & 3 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & (a-3)(a-2) & 5(a-3) \end{array}\right]$$

   - If $a = 2$, then the last row of the matrix is is $\left[\begin{array}{ccc|c} 0 & 0 & 0 & -5 \end{array}\right]$. Hence the system is inconsistent.
   - If $a = 3$ then

$$M = \left[\begin{array}{ccc|c} 1 & 2 & 4 & 2 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -6 & 10 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

     Hence the system has infinitely many solutions.

- If $a \notin \{3, 2\}$, then $(a-3)(a-2) \neq 0$ and so the system is uniquely solvable

The answer to question (a) is therefore:

(i) The system in inconsistent if $a = 2$. **1 point**
(ii) The system has infinitely many solutions if $a = 3$. **1 point**
(iii) The system is uniquely solvable if $a \notin \{2, 3\}$. **1 point**

(b) The RREF of the matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & -6 & 15 \\ 0 & 1 & 5 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The corresponding linear system is

$$\begin{array}{ccccccc} x & & & - & 6z & = & 15 \\ & & y & + & 5z & = & -4 \end{array}$$

Thus z is a free variable. Putting $z = t$, the general solution is

$$(15 + 6t, -4 - 5t, t) \quad (t \text{ a free parameter})$$

1 point
(c) We set up the 3 by 6 matrix for the inverse.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 9 & 1 & 0 & 0 \\ 2 & 7 & 23 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right].$$

0.5 point for correct approach We use Gauss- Jordan elimination on this matrix $R_2 - 2R_1 \to R_2, R_3 - R_1 \to R_1$ $\left[\begin{array}{ccc|ccc} 1 & 3 & 9 & 1 & 0 & 0 \\ 0 & 1 & 5 & -2 & 1 & 0 \\ 0 & -2 & 8 & 0 & -1 & 1 \end{array}\right]$.
Next: $R_3 + 2R_2 \to R_2$ and then $R_3 \to 0.5R_3$: $\left[\begin{array}{ccc|ccc} 1 & 3 & 9 & 1 & 0 & 0 \\ 0 & 1 & 5 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2.5 & 1 & 0.5 \end{array}\right]$. The next step is $R_1 - 3R_2 \to R_1$ $\left[\begin{array}{ccc|ccc} 1 & 0 & -6 & 7 & -3 & 0 \\ 0 & 1 & 5 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2.5 & 1 & 0.5 \end{array}\right]$ and finally $R_1 + 6R_3 \to R_1, R_2 - 5R_3 \to R_3$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -8 & 3 & 3 \\ 0 & 1 & 0 & -2 & -4 & -2.5 \\ 0 & 0 & 1 & -2.5 & 1 & 0.5 \end{array}\right]$. **1 point for row operations, no points if the left hand side is not reduced to RREF.** So the inverse is

$$\left[\begin{array}{ccc} -8 & 3 & 3 \\ -2 & -4 & -2.5 \\ -2.5 & 1 & 0.5 \end{array}\right].$$

0.5 for correct answer which is consistent with previous work.

2. Your doctor has asked you to take 5 units of vitamin A, 13 units of vitamin B and 23 units of vitamin C every day. There are three brands available in your local pharmacy which contain the following units of vitamins A, B, C as indicated.

	vitamin A	vitamin B	vitamin C
Brand 1	1	2	4
Brand 2	1	1	3
Brand 3	0	1	1

(a) Find all combination of pills that provide you with the exact daily requirement (no partial pills!).

(b) If brands 1, 2, and 3 cost \$0.60, \$0.40, and \$1.00 per pill respectively, find the least expensive treatment and its cost.
(a) We set up linear equation, x_i is the number of pills of Brand i. For a moments, let's forget about the fact that no partial or negative pills are possible

$$\begin{aligned} x_1 + x_2 &= 5 \\ 2x_1 + x_2 + x_3 &= 13 \\ 4x_1 + 3x_2 + x_3 &= 23 \end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 2 & 1 & 1 & 13 \\ 4 & 3 & 1 & 23 \end{array}\right]$$

We reduce this to RREF: $R_2 - 2R_1 \to R_2, R_3 - 4R_1 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & -1 & 1 & 3 \\ 0 & -1 & 1 & 3 \end{array}\right]$$

and then $R_2 - R_3 \to R_2, R_1 + R_2 \to R_1$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 8 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

. Hence $x_1 + x_3 = 8, x_2 - x_3 = -3$ and x_3 can be any number. If $x_3 = s$, then the general solution is $(8 - s, -3 + s, s)$. Since the number of pills of each brand is a non-negative integer, we see that $3 \leq s \leq 8$ and $s \in \{3, 4, 5, 6, 7, 8\}$. Hence we get 6 combinations:

$$(5, 0, 3), (4, 1, 4), (3, 2, 5), (2, 3, 6), (1, 4, 7), (0, 5, 8).$$

The cost of a treatment is $\$\,(0.6(8 - s) + 0.4(-3 + s) + s) = \$(0.8s + 3.6)$. Hence the cheapest treatment is the treatment which s is minimal for; that is $s = 3$. The cheapest treatment option is 5 pills of brand 1, no pills of brand 2 and 2 pills of brand 3.

3. **(8 points)** Given the following matrices and vectors

$$A = \begin{bmatrix} 1 & -7 & -9 \\ 6 & 0 & 8 \\ 4 & 4 & 0 \end{bmatrix}, B = \begin{bmatrix} 3 & -7 \\ 6 & 4 \\ 0 & 1 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 7/4 \\ 0 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 1/2 \\ 3/4 \end{bmatrix}.$$

Compute the following if possible. If not possible, explain in one sentence why.
Marking: 1 point each

(a) $3\mathbf{v} + 2B^T\mathbf{w}$ is not defined since $B^T\mathbf{w}$ is 2×1 and $\mathbf{v}$ is 3×1.

(b) $\mathbf{w}\mathbf{v}^T = \begin{bmatrix} 0 & 0 & 0 \\ 1/2 & 7/8 & 0 \\ 3/4 & 21/6 & 0 \end{bmatrix}$.

(c) $\mathbf{v}^T\mathbf{w} = 1 * 0 + 7/4 * 1/2 + 0 * 3/4 = 14/8 = 7/4$.

(d) $A^TB + 2\mathbf{v}^T\mathbf{w}$ is not defined since A^TB is a 3×2 matrix and $\mathbf{v}^T\mathbf{w}$ is 1×1.

(e) $AB = \begin{bmatrix} -34 & 44 \\ 18 & -34 \\ 36 & -12 \end{bmatrix}$.

(f) $B\mathbf{u}$ is not defined since B has 2 columns and $\mathbf{u}$ has 3 rows.

(g) BA is not defined since B has 2 columns and A has 3 rows.

(h) $A^2 = \begin{bmatrix} -77 & -43 & -65 \\ 38 & -10 & -54 \\ 28 & -28 & -4 \end{bmatrix}$.

4. Determine the matrix A such that:

$$\left(3A^T - \begin{bmatrix} 1 & 3 & -2 \\ -4 & 5 & 1 \end{bmatrix}\right)^T = \begin{bmatrix} -4 & 3 \\ 2 & 4 \\ -2 & 6 \end{bmatrix} + 3\begin{bmatrix} 7 & -5 & 4 \\ 9 & 12 & 3 \end{bmatrix}^T.$$

We calculate the both sides of the equation

$$\left(3A^T - \begin{bmatrix} 1 & 3 & -2 \\ -4 & 5 & 1 \end{bmatrix}\right)^T = 3(A^T)^T - \begin{bmatrix} 1 & 3 & -2 \\ -4 & 5 & 1 \end{bmatrix}^T = 3A - \begin{bmatrix} 1 & -4 \\ 3 & 5 \\ -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 3 \\ 2 & 6 \\ -2 & 6 \end{bmatrix} + 3\begin{bmatrix} 7 & -5 & 4 \\ 9 & 12 & 3 \end{bmatrix}^T = \begin{bmatrix} -4 & 3 \\ 2 & 6 \\ -2 & 6 \end{bmatrix} + 3\begin{bmatrix} 7 & 9 \\ -5 & 12 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 3 \\ 2 & 6 \\ -2 & 6 \end{bmatrix} + \begin{bmatrix} 21 & 27 \\ -15 & 36 \\ 12 & 9 \end{bmatrix} = \begin{bmatrix} 17 & 30 \\ -13 & 42 \\ 10 & 15 \end{bmatrix}$$

Hence

$$3A - \begin{bmatrix} 1 & -4 \\ 3 & 5 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 30 \\ -13 & 42 \\ 10 & 15 \end{bmatrix}$$

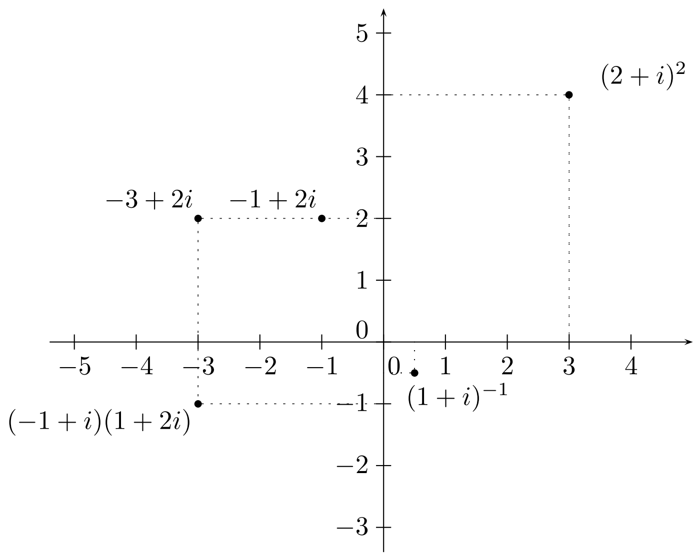

Figure 1: Points in the complex plane

Therefore

$$3A = \begin{bmatrix} 17 & 30 \\ -13 & 42 \\ 10 & 6+9 \end{bmatrix} + \begin{bmatrix} 1 & -4 \\ 3 & 5 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 26 \\ -10 & 47 \\ 8 & 16 \end{bmatrix}$$

$$A = \frac{1}{3}\begin{bmatrix} 18 & 26 \\ -10 & 47 \\ 8 & 16 \end{bmatrix} = \begin{bmatrix} 6 & \frac{26}{3} \\ \frac{1}{3}(-10) & \frac{1}{3}(47) \\ \frac{8}{3} & \frac{4}{3}(4) \end{bmatrix}.$$

5. **(6 points)** Draw the following complex numbers in the plane :

$$-1+2i, \quad -3+2i, \quad (-1+i)(1+2i), \quad (2+i)^2, \quad (1+i)^{-1}.$$

Solution:

$$(-1+i)(1+2i) = -3 - \mathrm{i} \quad (2+i)^2 = 3+4\mathrm{i}, \quad (1+i)^{-1} = \frac{1}{2} - \frac{1}{2}\mathrm{i}$$

See Figure 1.

3 points for sketch, 1 point for finding the correct cartesian coordinates of each of $(-1+i)(1+2i), \quad (2+i)^2, \quad (1+i)^{-1}$**.**