Total=12 points.

1. (a) Find the equilibrium solutions of the following differential equations. You should find three

$$y' = y^3 - 6y^2 + 11y - 6.$$

Solution: **(2 points)** We find the roots of $f(y) = y^3 - 6y^2 + 11y - 6 = (y-1)(y-2)(y-3)$ to be $1, 2, 3$.

(b) Draw the phase line diagram.

Solution: **(2 points)** Since f is a polynomial we easily identify the intervals where f is increasing/decreasing. This allows us to graph the phase line diagram. See Figure 1

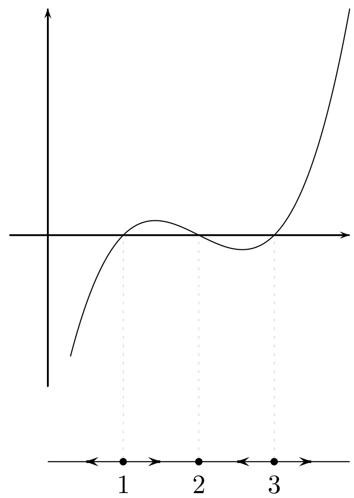

Figure 1: Phase-line diagram

Of course, we can also use the derivative criterion to decide about stability.

(c) Graph the equilibrium solutions and some solution curves

Solution: **(2 points)**You should clearly indicate the behaviour of the solution curves as $t \to \infty$. See Figure 2.

2. Suppose that size N if a populations satisfies the following differential equation:

$$\frac{dN}{dt} = \frac{5N^2}{1+N^2} - 2N.$$

(a) Find all equilibrium points.

(b) Use the derivative criterion to decide if the equilibria are stable or unstable.

(c) Draw the phase line diagram.

Solution: (a) **(2 points)** $f(N) = \frac{5N^2}{1+N^2} - 2N = 0 \iff 5N^2 = 2N(1+N^2) \iff N(2N^2 - 5N + 2) = 0 \iff 2N(N - \frac{1}{2})(N-2) = 0$. Hence the equilibria 0, 1/2 et 2.

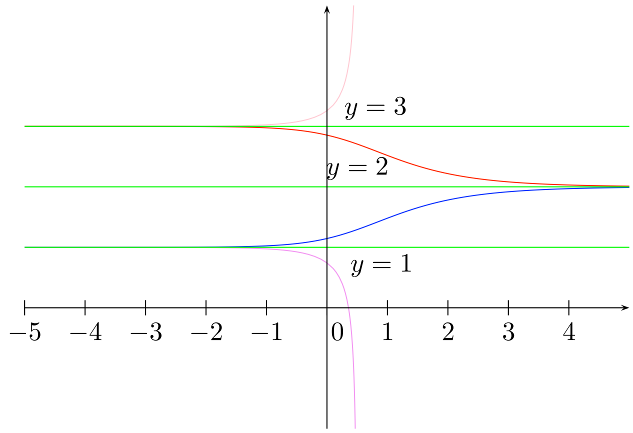

Figure 2: Some solution curves

(b) **(2 points)** The derivative of f is $f'(N) = \frac{10N}{(1+N^2)^2} - 2$, and we evaluate f' at the equilibria : $f'(0) = -2 < 0$, $f'(1/2) = 1.2 > 0$ and $f'(2) = -1.2 < 0$. Hence, 0 and 2 are stable and 1/2 is unstable.

(c) **(2 points)** See Figure 3.

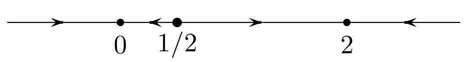

Figure 3: Phase-line diagram

3. The Zombies have invaded Campus again! The campus population is about 36,000. The Zombies are infecting humans at **a rate which is proportional to the product of the Zombie population and the portion of humans on campus** (where $\alpha > 0$ is the proportionality constant).
   At the same time, the Biology department has developed a method to turn Zombies back into humans by distributing a drug through the ventilation system. Zombies are transformed into humans at **a rate which is proportional the population of Zombies** (where $\beta > 0$ is the proportionality constant).

   (a) Denote by $z(t)$ the number of Zombies at time t. Find the differential equation which $z(t)$ satisfies according to the information given. Note that the total population is of size $T = 36,000 + z(0)$ and $z(0)$ is the number of invaders.

   (b) Find the equilibria of the differential equation in (a). These equilibria will depend on α and β.

   (c) Assume that $\alpha = 1/2$ and $\beta = 2/3$. Without solving the differential equation, what is the number of zombies in the long run (when $t \to \infty$)? What if $\alpha = 2/3$ and $\beta = 1/2$?

   (a) The Zombie population is given by $z(t)$ and the portion of humans by $(36000 + z(0) - z(t))/36000 + z(0))$. Write $P_{tot} = 36,000 + z(0)$. So,

$$z'(t) = \alpha z(t) \frac{(P_{tot} - z(t))}{P_{tot}} - \beta z(t).$$

(b) Solve $\alpha z(P_{tot} - z)/P_{tot} - \beta z = 0 \iff z(\alpha - \alpha z/P_{tot} - \beta) = 0$ and hence there are two equilibrium points 0 and $P_{tot}(1 - \frac{\beta}{\alpha})$. (c) If $\alpha = 1/2$ and $\beta = 2/3$, then the equilbrium points are 0 et $-P_{tot}/3$. Calculating the derivative of $1/2z(t)\frac{(P_{tot}-z(t))}{P_{tot}} - 2/3z(t)$ gives that 0 is a stable equilibrium. Hence the Zombie population approached zero and will eventually disappear.
If $\alpha = 2/3$ and $\beta = 1/2$, then the equilbrium points are 0 et $P_{tot}/4$. Calculating the derivative of $2/3z(t)\frac{(P_{tot}-z(t))}{P_{tot}} - 1/2z(t)$ gives that $P_{tot}/4$ is a stable equilibrium. Hence the Zombie population will make up a quarter of the Campus population in the long run; that is $9000 + z(0)/4$.

4. Find a differential equation whose phase line diagram is the following and graph the solution curve for $y(0) = 3$:

Solution : We know that the equilibrium points are given by -1, 1 et 2. We try the following differential equation $y' = (y+1)(y-1)(y-2)$. But since the derivative criterion reveals that the arrows in our first try point into the wrong direction, a correct solution would be $y' = -(y+1)(y-1)(y-2)$.The solution curve passing through $(0, 3)$ is given in Figure 4.

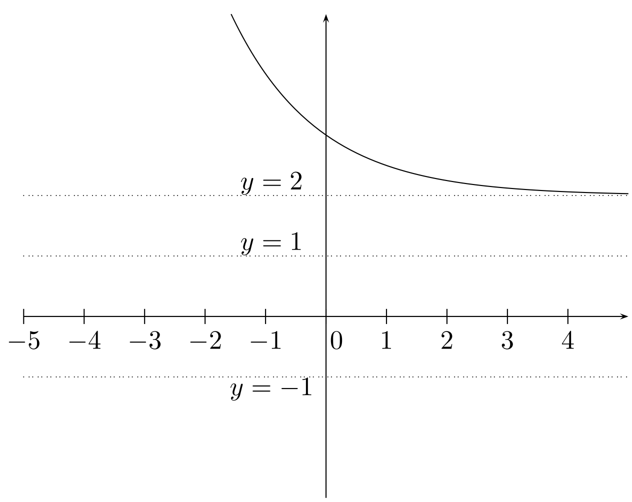

Figure 4: Solution curve