

1. For each of the following improper integrals, determine whether it converges, and evaluate it if it does.

$$(a) \int_0^{\infty} \frac{t^2}{4+t^6} dt \qquad (b) \int_1^{\infty} e^{-t} \sin(2t) dt$$

Solution: (a) We express the improper integral as a limit, and use two substitutions, $u = t^3$ ($du = 3t^2 dt$), and then $v = u/2$ ($dv = du/2$):

$$\begin{aligned} \int_0^{\infty} \frac{t^2}{4+t^6} dt &= \lim_{T \rightarrow \infty} \int_0^T \frac{t^2}{4+t^6} dt = \lim_{T \rightarrow \infty} \int_0^{T^3} \frac{du/3}{4+u^2} \\ &= \lim_{T \rightarrow \infty} \frac{1}{3 \cdot 4} \int_0^{T^3} \frac{du}{1+(u/2)^2} \\ &= \lim_{T \rightarrow \infty} \frac{1}{12} \int_0^{T^3/2} \frac{2dv}{1+(v)^2} \\ &= \lim_{T \rightarrow \infty} \frac{1}{6} \arctan(v) \Big|_0^{T^3/2} = \lim_{T \rightarrow \infty} \frac{1}{6} (\arctan(T^3/2) - 0). \end{aligned}$$

As $T \rightarrow \infty$, $T^3/2$ also $\rightarrow \infty$, and $\arctan(T^3/2) \rightarrow \pi/2$. So the final answer is

$$\frac{1}{6}(\pi/2) = \frac{\pi}{12}.$$

(b) Let's first evaluate the indefinite integral, as this is a little tricky: Write I for $\int e^{-t} \sin(2t) dt$. Then

$$\begin{aligned} I &= \int e^{-t} \sin(2t) dt \\ &= -e^{-t} \sin(2t) + 2 \int e^{-t} \cos(2t) dt \quad \left(\text{by parts, with } u = \sin(2t), \quad dv = e^{-t} dt \right) \\ &= -e^{-t} \sin(2t) - 2e^{-t} \cos(2t) - 4 \int e^{-t} \sin(2t) dt \\ &\qquad \qquad \qquad \left(\text{by parts, with } u = \cos(2t), \quad dv = e^{-t} dt \right) \\ &= -e^{-t} \sin(2t) - 2e^{-t} \cos(2t) - 4I, \end{aligned}$$

Notes: (1) Really the equation should be $I = -e^{-t} \sin(2t) - 2e^{-t} \cos(2t) - 4I + C$, C any constant. The $+C$ is because of the inherent ambiguity in an indefinite integral: it isn't a particular function, rather it is a function plus an arbitrary constant. You don't really need to worry about this, as long as your final answer for the indefinite integral does include $+C$.

(2) In order for this to give a non-trivial answer, we need to take $dv = e^{-t} dt$ both

times, or $u = e^{-t}$ both times. If we did integration by parts twice with any other choices, we'd only end up with the result $I = I$, which is useless.

We can now use this equation, $I = -e^{-t} \sin(2t) - 2e^{-t} \cos(2t) - 4I$ to solve for I :

$$\begin{aligned} 5I &= -e^{-t} \sin(2t) - 2e^{-t} \cos(2t) \\ I &= -\frac{1}{5}e^{-t} \sin(2t) - \frac{2}{5}e^{-t} \cos(2t), \end{aligned}$$

so

$$\int e^{-t} \sin(2t) dt = -\frac{1}{5}e^{-t} \sin(2t) - \frac{2}{5}e^{-t} \cos(2t) + C.$$

Now we compute the improper integral:

$$\begin{aligned} \int_1^{\infty} e^{-t} \sin(2t) dt &= \lim_{T \rightarrow \infty} - \int_1^T e^{-t} \sin(2t) dt \\ &= \lim_{T \rightarrow \infty} - \left(\frac{1}{5}e^{-t} \sin(2t) + \frac{2}{5}e^{-t} \cos(2t) \right) \Big|_1^T \\ &= \lim_{T \rightarrow \infty} - \left(\frac{1}{5}e^{-T} \sin(2T) + \frac{2}{5}e^{-T} \cos(2T) \right) + \left(\frac{1}{5}e^{-1} \sin(2 * 1) + \frac{2}{5}e^{-1} \cos(2 * 1) \right) \\ &= \lim_{T \rightarrow \infty} - \left(\frac{1}{5}e^{-T} \sin(2T) + \frac{2}{5}e^{-T} \cos(2T) \right) + \frac{1}{5e} (\sin(2) + 2 \cos(2)). \end{aligned}$$

As $T \rightarrow \infty$, both $\sin(2T)$ and $\cos(2T)$ will always be between -1 and 1 , while the exponential decay of e^{-T} will approach 0 , so that the expression $\frac{1}{5}e^{-T} \sin(2T) + \frac{2}{5}e^{-T} \cos(2T)$ will approach 0 , and the limit will be

$$\int_1^{\infty} e^{-t} \sin(2t) dt = \frac{1}{5e} (\sin(2) + 2 \cos(2)) \sim 0.00567.$$

2. The concentration of toxin in a cell is increasing at a rate of $50e^{-2t}$ $\mu\text{mol/L/s}$, starting from a concentration of 10 $\mu\text{mol/L}$. If the cell is poisoned when the concentration exceeds 30 $\mu\text{mol/L}$, will the cell survive? Hint: Express the limiting concentration as an improper integral.

Solution: Let $c(t)$ represent the concentration of toxin at time t , measured in $\mu\text{mol/L}$. The formula for c , given that the concentration at time 0 is $c(0) = 10$, is

$$c(x) = c(0) + \int_0^x 50e^{-2t} dt = 10 + \int_0^x 50e^{-2t} dt.$$

The limiting value is the limit as time goes to ∞ , $10 + \int_0^{\infty} 50e^{-2t} dt$. Let's evaluate this:

$$\begin{aligned} 10 + \int_0^{\infty} 50e^{-2t} dt &= 10 + \lim_{T \rightarrow \infty} \int_0^T 50e^{-2t} dt \\ &= 10 + \lim_{T \rightarrow \infty} (-25e^{-2t}) \Big|_0^T \\ &= 10 + \lim_{T \rightarrow \infty} (-25e^{-2T} + 25) \\ &= 10 + 25 = 35. \end{aligned}$$

So the limiting concentration is $35 \mu\text{mol/L}$; since this is higher than the tolerance of the cell, the cell will be poisoned and die.

3. Find the solution to the autonomous differential equation

$$\frac{dy}{dt} = y^2 + 1.$$

satisfying the initial condition $y(0) = 1$.

Solution: First we put all the y 's on the left and all the t 's (including the dt) on the right, then integrate, to get

$$\int \frac{dy}{y^2 + 1} = \int dt$$

Evaluating, we get $\arctan(y) = t + C$, with C a constant to be determined. Solving for y gives $y = \tan(t + C)$.

Now, when $t = 0$, $y = 1$ (since $y(0) = 1$), so $1 = \tan(0 + C)$. In other words, $\tan(C) = 1$, so we can take $C = \pi/4$, and the final answer is $y = \tan(t + \pi/4)$.

4. Solve the separable differential equation

$$\frac{dy}{dt} = \frac{e^t + 2t}{y^2}$$

Solution: First we separate and integrate, to get

$$\int y^2 dy = \int (e^t + 2t) dt$$

Evaluating the integrals gives

$$y^3/3 = e^t + t^2 + C,$$

with C any constant. Solving for y gives $y = (3e^t + 3t^2 + C)^{1/3}$, C any constant, as the general solution. (Notice that we wrote C instead of $3C$, because 3 times an arbitrary constant is an arbitrary constant. However, leaving the expression as $3C$ is just as correct.)

5. Find the solution to the autonomous differential equation

$$\frac{dy}{dt} = 0.5(100 - y).$$

satisfying the initial condition $y(0) = 20$. What is the limiting value of y as $t \rightarrow \infty$?

Solution: Again we separate the t 's and y 's and integrate:

$$\int \frac{dy}{y - 100} = \int -0.5 dt,$$

so

$$\begin{aligned}\ln |y - 100| &= -0.5t + C \\ |y - 100| &= e^{-0.5t+C} = e^C e^{-0.5t} \\ y - 100 &= \pm e^C e^{-0.5t} \\ y &= 100 \pm e^C e^{-0.5t},\end{aligned}$$

with C any constant. Writing K for $\pm e^C$, we have that K can be any constant. The general solution is then

$$y = 100 + K e^{-0.5t},$$

with K any constant.

We use the initial condition to solve for K : $20 = 100 + K e^{-0.5 \cdot 0}$, so $K = 20 - 100 = -80$, and the solution is $y = 100 - 80 e^{-0.5t}$.

As $t \rightarrow \infty$, $e^{-0.5t} \rightarrow 0$ and so $y \rightarrow 100$.

Note: In the first step of this solution we divided through by $y - 100$. This is not allowed if $y = 100$, so we should really have independently verified that $y = 100$ is a solution to the equation. This fits into the final answer since it corresponds to $K = 0$.