MAT 1332 Solutions Assignment 2

Marking
Q1(ii)b: 2 Q5: 2
Q4a: 2
Q4b: 3 Total: 9 pt

(i)

1. (a) $\int_1^3 \frac{1}{(t-3)^{4/3}}\,dt$ is an improper integral

since $f(t)=\frac{1}{(t-3)^{4/3}}$ is continuous on $[1,3)$ and not continuous on $[1,3]$.

(b) $\int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}\,dx$ is an improper integral since $g(x)=\frac{x}{\sqrt{1-x^2}}$ is continuous on $(-1,1)$ and discontinuous on $[-1,1]$.

(ii)

(a) $\int_1^x \frac{1}{(t-3)^{4/3}}\,dt = \int_1^x (t-3)^{-4/3}\,dx = \left[-3(t-3)^{-1/3}\right]_1^x$

$= -3(x-3)^{-1/3} - 3(-2)^{-1/3}$

$= -3(x-3)^{-1/3} - \frac{3}{\sqrt[3]{2}}$ for all $x \geq 1$.

We have

$\lim_{x\to 3^-}\left(-3(x-3)^{-1/3} + \frac{3}{\sqrt[3]{2}}\right) =$

$-3\left(\lim_{x\to 3^-} \frac{1}{\sqrt[3]{(x-3)}}\right) + \frac{3}{\sqrt[3]{2}} = \infty$

note $\lim_{x\to 3^-} \sqrt[3]{(x-3)} = 0^-$ (approaches 0 from below)

$\Rightarrow \lim_{x\to 3}\int_1^x \frac{1}{(t-3)^{4/3}}\,dt$ DNE $\Rightarrow$ the improper integral is divergent.

Marking 1(ii)b: 2 points

(b) $\displaystyle\int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}\,dx = \underbrace{\int_{-1}^{0} \frac{x}{\sqrt{1-x^2}}\,dx}_{①} + \underbrace{\int_{0}^{1} \frac{x}{\sqrt{1-x^2}}\,dx}_{②}$ 0.5

① $\displaystyle\int_{t}^{0} \frac{x}{\sqrt{1-x^2}}\,dx = \int_{1-t^2}^{0} \frac{-1}{2\sqrt{y}}\,dy$ $\qquad y = 1-x^2$, $\frac{dy}{dx} = -2x$ 0.5

$$= -1\left[\sqrt{y}\right]_{1-t^2}^{1} = -\left[\sqrt{1} - \sqrt{1-t^2}\right]$$

$$= -1 + \sqrt{1-t^2} \quad \forall\, t<0$$

and $\displaystyle\lim_{t\to -1} \int_{t}^{0} \frac{x}{\sqrt{1-x^2}}\,dx = \lim_{t\to -1} \left[-1+\sqrt{1-t^2}\right] = \underline{\underline{-1}}$

② $\displaystyle\int_{0}^{t} \frac{x}{\sqrt{1-x^2}}\,dx = \int_{1}^{1-t^2} \frac{-1}{2\sqrt{y}}\,dy = -1\left[\sqrt{y}\right]_{1}^{1-t^2} = -1\left(\sqrt{1-t^2} - 1\right)$ 0.5

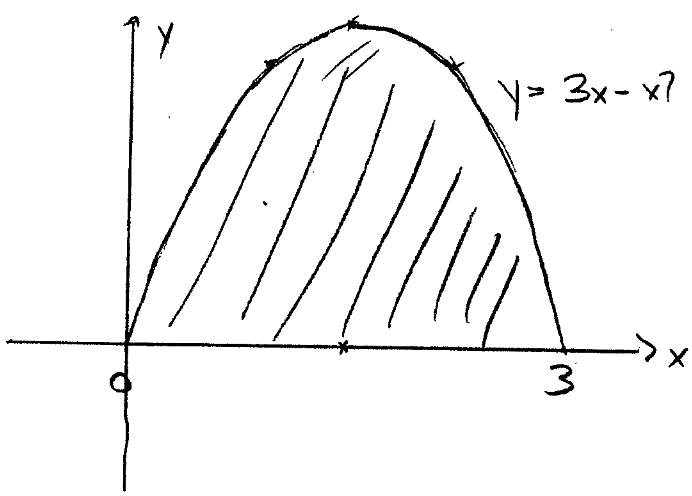
$$= 1 + \sqrt{1-t^2} \quad \forall\, t>0$$

and $\displaystyle\lim_{t\to 1} \int \frac{x}{\sqrt{1-x^2}}\,dx = \lim_{t\to 1} \left(1+\sqrt{1-t^2}\right) = \underline{\underline{1}}$

① and ② are convergent $\Rightarrow \displaystyle\int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}\,dx$ is 0.5
convergent and

$$\int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}\,dx = -1 + 1 = 0. \quad 0.5$$

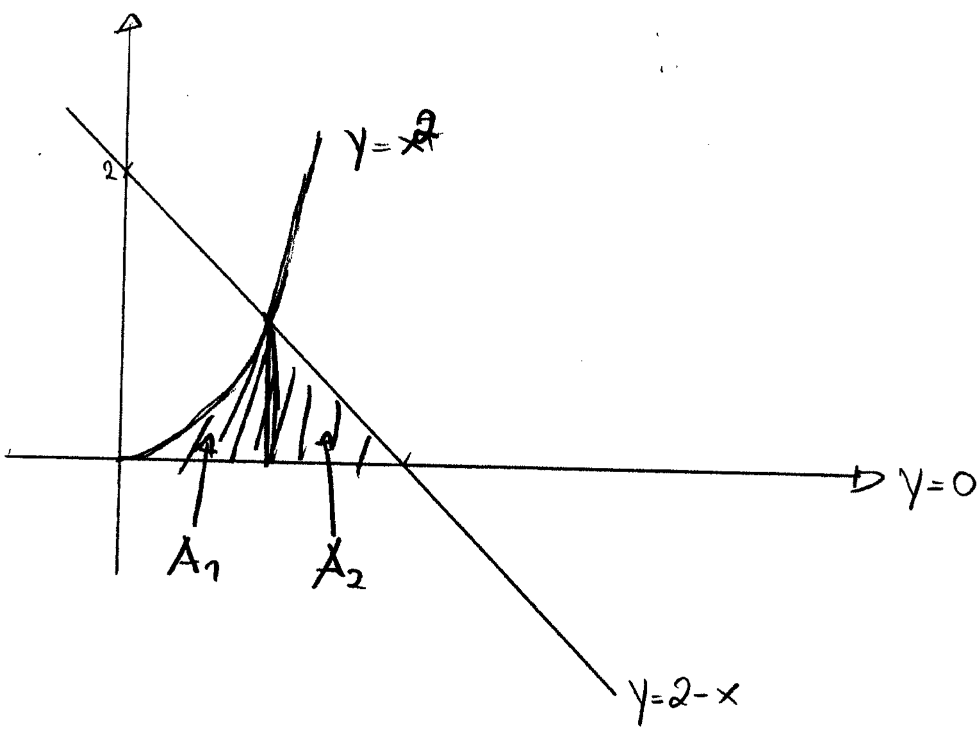
2.



$$\text{Area} = \int_0^3 3x - x^2 \, dx = \left[\frac{3}{2}x^2 - \frac{1}{3}x^3 \right]_0^3 = \frac{27}{2} - \frac{27}{3} = \frac{27 \cdot 3 - 27 \cdot 2}{6}$$

$$= \frac{27}{6} = \frac{9}{2}$$

3.



the curves $y = x^2$ and $y = 2 - x$ intersect at $x=1, y=1$.
Thus the area is

$$A = A_1 + A_2 \quad \text{where}$$

$$A_1 = \int_0^1 x^2\,dx = \left[\tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{3}$$

$$A_2 = \int_1^2 (2-x)\,dx = \left[2x - \tfrac{1}{2}x^2\right]_1^2 = \left[4 - \tfrac{1}{2}\cdot 4 - 2 + \tfrac{1}{2}\right]$$

$$= \left[\frac{8-4-4+1}{2}\right] = \frac{1}{2}$$

$$A = \frac{1}{3} + \frac{1}{2} = \frac{2+3}{6} = \frac{5}{6}$$

4.

(a) $\int \frac{x^3+x^2+5}{x^2+4}\,dx$

0.5 Step 1) Long division

$$\begin{array}{l} x^3+x^2+5 \\ \boxed{-x}\,(x^2+4) \\ \hline x^2-4x+5 \\ \boxed{-1}\,(x^2+4) \\ \hline \boxed{-4x+1} \end{array}$$

so $x^3+x^2+5 = (x+1)(x^2+4) - 4x + 1$

$$\frac{x^3+x^2+5}{x^2+4} = \frac{(x+1)(x^2+4) - 4x+1}{x^2+4}$$

0.5

$$= x+1 + \frac{1-4x}{x^2+4}$$

Step 2): Note that x^2+4 cannot be factored.

so $= (x+1) + \frac{1}{x^2+4} - \frac{4x}{x^2+4}$

is the partial fraction decomposition

1 $\int \frac{x^3+x^2+5}{x^2+4}\,dx \int x+1\,dx + \int \frac{1}{x^2+4}\,dx - \int \frac{4x}{x^2+4}\,dx$

$$= \frac{1}{2}x^2 + x + \frac{1}{2}\arctan\left(\frac{x}{2}\right) - 2\ln|x^2+4| + C$$

(b) $\displaystyle\int \frac{x^4 - 8x^2 - 10}{x^2 - 3x - 10}\,dx$

1) Long division: 0.5

$$\begin{array}{l} x^4 \quad -8x^2 \quad -10 \\ -x^2(x^2 - 3x - 10) \\ \quad 3x^3 + 2x^2 \quad -10 \\ \quad -3x(x^2 - 3x - 10) \\ \quad\quad 11x^2 + 30x - 10 \\ \quad\quad -11(x^2 - 3x - 10) \\ \quad\quad\quad 63x + 100 \end{array}$$

$x^4 - 8x^2 - 10 = (x^2 - 3x - 10)(x^2 + 3x + 11) + 63x + 100$

$$\boxed{\frac{x^4 - 8x - 10}{x^2 - 3x - 10} = x^2 + 3x + 11 + \frac{63x + 100}{x^2 - 3x - 10}}$$ 0.5

2) Factor: $x^2 - 3x - 10 = (x - 3/2)^2 - \frac{9}{4} - 10 = \left(x - \frac{3}{2}\right) - \frac{49}{4}$

hence the roots are $x_1 = -\sqrt{\frac{49}{4}} + \frac{3}{2} = \frac{-7+3}{2} = -2$

$x_2 = +\sqrt{\frac{49}{4}} + \frac{3}{2} = \frac{7+3}{2} = 5.$

and $x^3 - 3x - 10 = \underline{(x+2)(x-5)}$ 0.5

0.5 3) $\displaystyle\frac{63x + 100}{x^2 - 3x - 10} = \frac{A}{x+2} + \frac{B}{x-5} = \frac{A(x-5) + B(x+2)}{(x+2)(x-5)}$

so $(A + B)x - 5A + 2B$
$= 63x + 100$

$A + B = 63, \quad -5A + 2B = 100$

$B = 63 - A \quad -5A + 126 - 2A = 100$

$-7A = -26 \Rightarrow A = \frac{26}{7}$

$B = \frac{63 \cdot 7 - 26}{7} = \frac{415}{7}$

$A = \frac{26}{7}$, $B = \frac{415}{7}$

4) Thus

$$\int \frac{x^4 - 8x^2 - 10}{x^2 - 3x - 10}\,dx = \int x^2 + 3x + 11\,dx + \int \frac{26}{7(x+2)}\,dx + \int \frac{415}{7(x-5)}\,dx$$

0.5

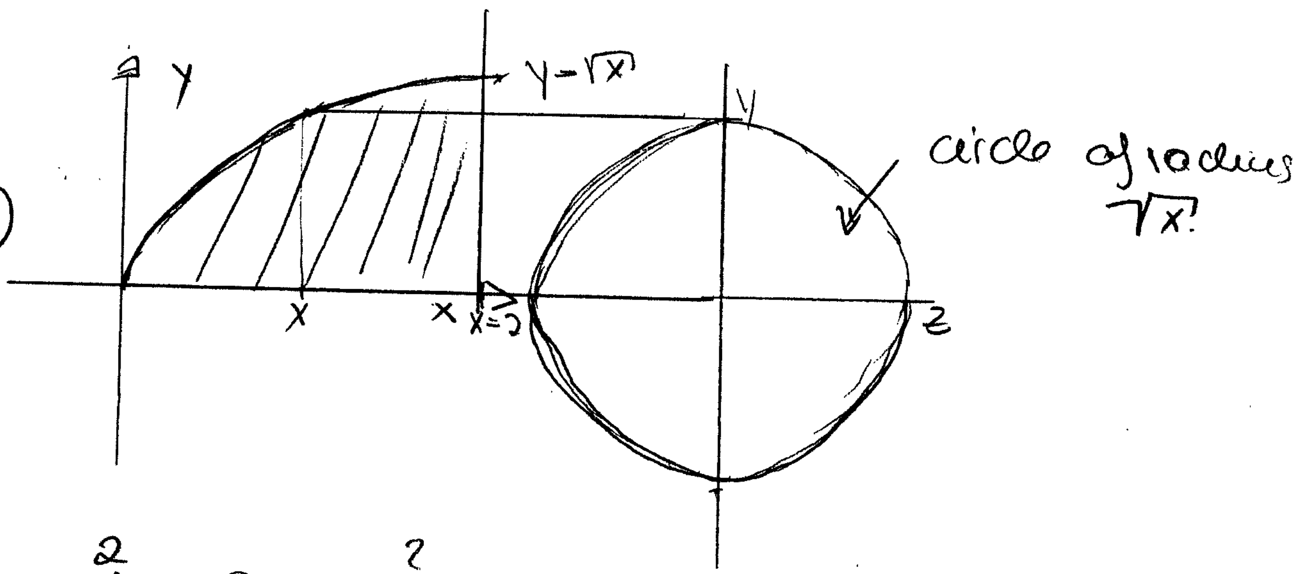
$$= \underbrace{\frac{1}{3}x^3 + \frac{3}{2}x^2 + 11x + \frac{26}{7}\ln|x+2| + \frac{415}{7}\ln|x-5 + C}_{0.5}$$

4 (a) 2points

4 (b) 3points

5.

1



2 $$V = \int_0^2 \pi \sqrt{x}^2 \, dx = \int_0^2 \pi x \, dx = \pi \left[\frac{1}{2} x^2 \right]_0^2 = \pi \cdot \left(\frac{1}{2} 4 - \frac{1}{2} \cdot 0 \right) = 2\pi$$

5 2points $\begin{cases} 1 \text{ for sketch} \\ 1 \text{ for calculation} \end{cases}$